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Pointwise Convergence of Solutions to the Schrödinger Equation on Manifolds

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Abstract. Let (M^n, g) be a Riemannian manifold without boundary. We study the amount of initial regularity required so that the solution to a free Schrödinger equation converges pointwise to its initial data. Assume the initial data is in $H^\alpha(M)$. For hyperbolic space, the standard sphere, and the two-dimensional torus, we prove that $\alpha > \frac{1}{2}$ is enough. For general compact manifolds, due to the lack of a local smoothing effect, it is hard to improve on the bound $\alpha > 1$ from interpolation. We managed to go below 1 for dimension ≤ 3. The more interesting thing is that, for a one-dimensional compact manifold, $\alpha > \frac{1}{3}$ is sufficient.

1 Introduction

In the Euclidean setting, L. Carleson [8] proposed a question regarding the amount of regularity required on the initial data f, so that

$$e^{-it\triangle} f(x) = \int_{\mathbb{R}^n} e^{i(\langle x,\xi\rangle + t|\xi|^2)} \widehat{f}(\xi) d\xi \to f(x), \quad \text{a.e. } x \in \mathbb{R}^n,$$

as t goes to 0. Here $e^{-it\triangle} f(x)$ is the solution to the free Schrödinger equation

$$\begin{cases} i\partial_t u - \Delta u = 0 & (t,x) \in \mathbb{R} \times \mathbb{R}^n, \\ u(0,x) = f(x) & x \in \mathbb{R}^n. \end{cases}$$

The problem has been treated by many authors. When $n = 1$, Carleson himself proved that $f \in H^{\frac{1}{4}}(\mathbb{R})$ is sufficient. B. E. J. Dahlberg and C. E. Kenig [13] proved this is necessary for all dimensions. For higher dimensions, M. Cowling [9] studied the problem for a general class of self-adjoint operators and obtained $\alpha > 1$ for the Schrödinger operator. Later on, P. Sjölin [20] proved a local smoothing effect and thus improved the bound to $\alpha > \frac{1}{2}$, which was also independently proved by L. Vega [25].

For n=2, J. Bourgain [2] showed that we can go below $\frac{1}{2}$ a little bit. Furthermore, continuous improvement has been made by Moyua, Vargas, Vega [18], Tao, Vargas [23, 24] and Lee [16]. Recently, Du, Guth, and Li [12] proved that $s > 1/3$ is sufficient and sharp, thus completely solving the problem.

For $n \geq 3$, Bourgain[5] beat the bound $\alpha > \frac{1}{2}$ and showed that the solution to a free Schrödinger equation converges pointwise to its initial data f, provided $f \in H^\alpha(\mathbb{R}^n)$,

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where $\alpha > \frac{1}{2} - \frac{1}{4n}$. Furthermore, he also showed that when $n > 4$, $\alpha > \frac{1}{2} - \frac{1}{n}$ is necessary. Lucá and Rogers [17] refined the necessary condition to $\alpha > \frac{1}{2} - \frac{1}{n+2}$.

In this paper, we deal with a similar problem in the manifold setting. We always take (M^n, g) to be a complete manifold endowed with a C^∞ metric g. Denote by Δ the Laplace–Beltrami operator associated with g.

The Schrödinger equation on (M^n, g) is given by

$$\begin{cases} i\partial_t u - \Delta u = 0 & (t,x) \in \mathbb{R} \times M, \\ u(0,x) = f(x) & x \in M. \end{cases} \tag{1.1}$$

Specifically, if M is compact, it has a discrete spectrum, and there exists an orthonormal basis $\{e_j\}$ of eigenfunctions such that

$$\Delta e_j(x) = \lambda_j^2 e_j, \quad e_j \in C^\infty(M), \quad \int_{\mathcal{M}} e_j e_k \, dV_g = \delta_{jk}.$$

The eigenvalues $0 = \lambda_0 < \lambda_1 \le \lambda_2 \le \lambda_3, \le \cdots$ are listed in ascending order and counted by multiplicity.

We can then write the solution of (1.1) as

$$e^{-it\triangle} f(x) = \sum_{j=0}^{\infty} e^{it\lambda_j^2} \widehat{f}_j e_j(x), \tag{1.2}$$

where $\widehat{f}_j$ is the j-th Fourier coefficient given by $\widehat{f}_j = \int_M f(x) e_j(x) dV_g$.

When $M = \mathbb{H}^n, \mathbb{S}^n$ or $\mathbb{T}^2$, we are able to obtain the same result as in P. Sjölin's work for $\mathbb{R}^n$ [20].

Theorem 1.1 *The solution $u(t,x)$ to equation (1.1) converges pointwise to the initial data f, whenever $f \in H^\alpha(M)$, $\alpha > \frac{1}{2}$. Here $M = \mathbb{H}^n, \mathbb{S}^n$ or $\mathbb{T}^2$ endowed with standard metric.*

Remark 1.2 These three cases are actually proved via three different methods. We derive the hyperbolic space by showing a local smoothing effect. For the standard sphere, we take full advantage of the spectrum concentration. For the two-dimensional torus, we apply our argument for the general manifold and combine it with Strichartz estimates on the two-dimensional torus obtained by Bourgain [3].

For a general compact manifold, if $\alpha > 1$, there is a way to prove the pointwise convergence quickly using a method from [26]. In fact, by (1.2) and Parseval's formula, we easily have

$$\|e^{-it\Delta} f(x)\|_{L^2(M,L^2(0,1])} \le \|f\|_{L^2(M)},$$
$$\|e^{-it\Delta} f(x)\|_{L^2(M,H^1(0,1])} = \|(e^{-it\Delta} - i\Delta e^{-it\Delta}) f(x)\|_{L^2(M,L^2(0,1])} \le \|f\|_{H^2(M)}.$$

Interpolating between the two, yields

$$\|e^{-it\Delta} f(x)\|_{L^2(M,H^s(0,1])} \le \|f\|_{H^{2s}(M)},$$

which, when combined with the Sobolev imbedding, further leads to

$$\|\sup_{0<t\le 1} |e^{-it\Delta} f(x)|\|_{L^2(M)} \lesssim \|f\|_{H^{2s}(M)}, \ s > \frac{1}{2}. \tag{1.3}$$

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The pointwise convergence for $e^{-it\Delta} f(x)$ always comes from the boundedness of the maximal operator (see [20, p. 714] for a detailed argument).

Henceforth, we will use the notation $A \lesssim B$ to mean that there is some constant C independent of all essential variables, such that $A \le CB$. For convenience, we denote $T^* f(x) = \sup_{0<t\le 1} |e^{-it\Delta} f(x)|$.

Inequality (1.3) says that the maximal Schrödinger operator T^* is bounded from $H^\alpha(M)$, $\alpha > 1$ to $L^2(M)$. Thus the pointwise convergence follows. From Theorem 1.1, it is reasonable to conjecture that $\alpha > \frac{1}{2}$ should be sufficient, but it is hard to break the bound $\alpha > 1$. This is due to the absence of a local smoothing effect [10, 11], and we no longer have scaling invariances as in Euclidean spaces. Fortunately, by utilizing the Strichartz estimate [6], we manage to overcome these difficulties and break the bound $\alpha > 1$ in lower dimensions.

Theorem 1.3 *Let (M, g) be a connected, compact manifold without boundary of dimension n. The solution $u(t,x)$ to equation (1.1) converges pointwise to the initial data f, whenever $f \in H^\alpha(M)$, $\alpha > \frac{1}{3}$ for $n = 1$, $\alpha > \frac{3}{4}$ for $n = 2$, or $\alpha > \frac{9}{10}$ for $n = 3$.*

We now give a brief outline of what follows. In Section 2, we provide some basic facts about hyperbolic spaces and spheres. In Section 3, we prove the hyperbolic space case in Theorem 1.1. In Section 4, we prove the standard sphere case that exhibits some difference between the hyperbolic space and the Euclidean case. In Section 5, we prove Theorem 1.3 for $n = 2, 3$. In Section 6, we apply the previous results to the torus case, and several other examples.

2 Preliminaries

In this section, we provide some basic facts we will need in the later context.

Definition 2.1 We define the *hyperbolic space* as given by the polar parametrization:

$$\mathbb{H}^n = \{(t,x) \in \mathbb{R}^{n+1}, (t,x) = (\cosh r, \sinh r\omega), r > 0, \omega \in \mathbb{S}^{n-1}\}.$$

The metric tensor is given by $g = dr^2 + \sinh r d\omega^2$, where $d\omega^2$ is the standard metric on the sphere. The volume element is $dV = \sinh r^{n-1} dr d\omega$. The Laplace–Beltrami operator is

$$\Delta_{\mathbb{H}^n} = \partial_r^2 + (n-1)\frac{\cosh r}{\sinh r}\partial_r + \frac{1}{\sinh^2 r}\Delta_{\mathbb{S}^{n-1}}.$$

The Sobolev space is defined by $H^s(\mathbb{H}^n) = \{f | (1 - \Delta_{\mathbb{H}^n})^{s/2} f \in L^2(\mathbb{H}^n)\}$. When $s = m$ is a positive integer, this is equivalent to the usual definition.

Let f be the function on $\mathbb{H}^n$ given by

$$F = \sqrt{1 + r^2}, \tag{2.1}$$

written in polar coordinates. Noticing that $F \in C^\infty(\mathbb{H}^n)$, with some direct calculation, one has the following.

Proposition 2.2 *Let F be as above.*

(i) $\mathrm{Hess}_F = \frac{1}{\sqrt{1+r^2}} dr \otimes dr + \frac{r}{\sqrt{1+r^2}} \mathrm{Hess}_r \geq A(r)g$, *where*

$$A(r) = \min\Big(\frac{1}{\sqrt{1+r^2}}, \frac{r}{\sqrt{1+r^2}} \frac{\cosh r}{\sinh r}\Big).$$

(ii) $\|\nabla^k F\| \in L^\infty(\mathbb{H}^n)$ *for any integer* $k > 0$.

Now we describe the spectrum concentration of the sphere.

Proposition 2.3 *Denote μ_k to be the k-th eigenvalue on the standard sphere without counting multiplicity. Let E_k be the corresponding eigenspace. Then $\mu_k = k(k+n-1)$ and* $\dim E_k \approx k^{n-1}$.

In the general compact manifold case, we will use Hörmander's oscillatory integral estimates (see [21, Theorem 2.2.1] for a detailed reference). Consider oscillatory integrals of the form $T_h f(z) = \int e^{\frac{i}{h}\phi(z,y)} a(z,y) f(y)\, dy$. Here $a \in C_0^\infty(\mathbb{R}^{n+1} \times \mathbb{R}^n)$, ϕ is real and C^∞ in a neighborhood of $\operatorname{supp} a$. Then the canonical relation associated with ϕ is defined as

$$C_\phi = \{(z, \phi_z(z,y)), y - \phi_y(z,y))\} \subset T^*\mathbb{R}^{n+1} \times T^*\mathbb{R}^n.$$

Lemma 2.4 *Let $\Pi_{T^*\mathbb{R}^n} \colon C_\phi \to T^*\mathbb{R}^n$ be the natural projection, and similarly $\Pi_{T^*\mathbb{R}^{n+1}}$. Assume that*

(i) $\operatorname{rank} d\Pi_{T^*\mathbb{R}^n} \equiv 2n$;

(ii) $S_{z_0} = \Pi_{T^*_{z_0}\mathbb{R}^{n+1}} C_\phi$ *has everywhere non-vanishing Gaussian curvature for any* $z_0 \in \operatorname{supp}_z\ a$.

Then $\|T_h f\|_{L^p(\mathbb{R}^{n+1})} \lesssim h^{(n+1)/q} \|f\|_{L^p(\mathbb{R}^n)}$ *if* $q = \frac{n+2}{n} p'$ *and* $1 \leq p \leq 2$ *for* $n \geq 2$, *or* $1 \leq p < 4$ *for* $n = 1$.

3 A Solution on a Hyperbolic Space

We start with the following local smoothing lemma.

Lemma 3.1 *Let $u(t) = u(x,t)$ be the solution of (1.1) with $M = \mathbb{H}^n$. Let B_R denote the geodesic ball centered at the origin. Then there exists a constant $C = C(n,R) > 0$ such that*

$$\|u\|_{L^2([0,1]\times B_R)} \leq C\|f\|_{H^{-\frac{1}{2}}(\mathbb{H}^n)}. \tag{3.1}$$

Remark 3.2 The smoothing effect of the Schrödinger evolution group has been intensively studied. Here we refer readers to [10, 11]. The proof of Lemma 3.1 follows from [10] with some modifications.

Proof of Lemma 3.1 In this section, we will denote $\Delta_{\mathbb{H}^n} = \Delta$.

Choose $\phi \in C^\infty(\mathbb{H}^n)$ to be a cutoff function such that $\phi \equiv 1$ in B_R and $\phi \equiv 0$ outside B_{2R}, $0 \leq \phi \leq 1$ and $|\nabla\phi| \lesssim \frac{1}{R}$. Let f be the function defined by (2.1). Consider the self-adjoint linear differential operator

$$X = \frac{\nabla F}{i} + \Big(\frac{\nabla F}{i}\Big) * = \frac{2\nabla F}{i} + \frac{\Delta F}{i}.$$

In local coordinates, $\nabla F = g^{ij}\frac{\partial F}{\partial x^i}\frac{\partial}{\partial x^j}$. Let $N = 1 - \Delta$. Then $P = N^{-1/4}XN^{-1/4}$ is a pseudodifferential operator of order 0. Since $\|\nabla^k F\| \in L^\infty(\mathbb{H}^n)$, one can see that $X\colon H^s(\mathbb{H}^n) \to H^{s-1}(\mathbb{H}^n)$ is continuous. Thus, P is bounded on $L^2(\mathbb{H}^n)$.

$$\begin{aligned}(3.2)\quad \frac{d}{dt}(-Pu(t), u(t)) &= (iP\Delta u(t), u(t)) + (Pu(t), i\Delta u(t))\\ &= 2\Re(iP\Delta u(t), u(t)) = 2\Re(i\Delta N^{-1/4}u(t), XN^{-1/4}u(t))\\ &= \int_{\mathbb{H}^n}(4\operatorname{Hess}_F(\nabla v(t), \overline{\nabla v(t)}) - \Delta^2 F|v(t)|^2)\,dV\\ &\geq A_{2R}\int_{B_{2R}}\phi|\nabla v(t)|^2 - \int_{\mathbb{H}^n}\Delta^2 F|v(t)|^2\,dV.\end{aligned}$$

Here $v(t) = N^{-1/4}u(t)$ and $A_{2R} = 4\inf_{0\leq r\leq 2R}A(r)$. Integrating (3.2) from 0 to 1, we have

$$\begin{aligned}(3.3)\quad \int_{[0,1]\times B_{2R}}\phi|\nabla v(t)|^2\,dVdt &\lesssim \|u(0)\|_{L^2(\mathbb{H}^n)} + \|u(1)\|_{L^2(\mathbb{H}^n)} + \|v(t)\|_{L^2([0,1]\times\mathbb{H}^n)}\\ &\lesssim \|f\|_{L^2(\mathbb{H}^n)}\end{aligned}$$

with a constant depending on R.

Notice that $v(t) = N^{-1/4}u(t)$ solves the Schrödinger equation

$$\begin{cases} i\partial_t v - \Delta v = 0 & (t,x) \in \mathbb{R}\times\mathbb{H}^n,\\ v(0,x) = N^{-1/4}f(x) & x \in \mathbb{H}^n.\end{cases}$$

Applying (3.3), one obtains

$$\int_{[0,1]\times B_{2R}}\phi|\nabla N^{-1/2}u(t)|^2\,dVdt \lesssim \|f\|_{H^{-1/2}(\mathbb{H}^n)}.$$

Since

$$\begin{aligned}(\phi\nabla N^{-1/2}u, \nabla N^{-1/2}u) &= (\phi(-\Delta)N^{-1/2}u, N^{-1/2}u) + (\nabla\cdot\phi\nabla(N^{-1/2}u), N^{-1/2}u)\\ &= (N^{1/2}u, \phi N^{-1/2}u) - (\phi N^{-1/2}u, N^{-1//2}u)\\ &\quad + (\nabla\phi\cdot\nabla(N^{-1/2}u), N^{-1/2}u)\\ &\geq (u, N^{1/2}(\phi N^{-1/2}u)) - C_1\|u\|_{H^{-1/2}(\mathbb{H}^n)}.\end{aligned}$$

By the sharp Gårding inequality [14], we have

$$(u, N^{1/2}(\phi N^{-1/2}u)) \geq (\phi u, u) - C_2\|u\|_{H^{-1/2}(\mathbb{H}^n)}.$$

Here C_1, C_2 are constants depending on R.

Combining the estimates above, we finish the proof of the lemma. ∎

As a corollary, we have the following.

Corollary 3.3 *The following estimate holds:*

$$(3.4)\qquad \|\Delta u\|_{L^2([0,1]\times B_R)} \leq C(R)\|f\|_{H^{3/2}(\mathbb{H}^n)}.$$

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Proof of Theorem 1.1 for $\mathbb{H}^n$ The proof is similar to the argument for (1.3) in the introduction. We finish the proof by interpolation between (3.1) and (3.4). ■

Remark 3.4 The argument above also applies to other manifolds where the local smoothing effect holds. For example, if M is the complement of a compact, smooth and non-trapping obstacle in $\mathbb{R}^n$, Burq, Gérard, and Tzvetkov [7] proved a local smoothing effect with a gain of "1/2". Thus by our method here, one gets exactly the same theorem as Theorem 1.1 (with the same index $\alpha > 1/2$). One can also extend the results to the variable coefficient context considered by Doi [10] as long as one has the local smoothing theorem. Besides, if we perturb the the standard metric of Euclidean or hyperbolic space in a finite domain such that there are no trapped geodesics, then the local smoothing effect still holds (see [10, 11] for more examples). Thus we also have similar pointwise theorems.

4 A Solution on the Sphere

By Proposition 2.3, we know that the k-th eigenvalue of $-\Delta_{\mathbb{S}^n}$ is

$$\mu_k = k(k+n-1),$$

and that the eigenfunctions attached to μ_k are the sphere harmonics of degree k that form a linear space of dimension $d_k \approx k^{n-1}$. Take $e_{j_1}(x),\ e_{j_2}(x),\dots,e_{j_{d_k}}(x)$ to be an L^2 normalized base of this linear space. Then for each $f \in C^\infty(\mathbb{S}^n)$, we have

$$e^{-it\Delta_{\mathbb{S}^n}} f(x) = \sum_{k=0}^{+\infty}\sum_{l=1}^{d_k} e^{-it\mu_k}\widehat{f}_{k_l} e_{k_l}(x). \tag{4.1}$$

We wish to prove $\big\|\sup_{0<t\le 1}|e^{-it\Delta_{\mathbb{S}^n}} f(x)|\big\|_{L^2(\mathbb{S}^n))} \lesssim \|f\|_{H^\alpha(\mathbb{S}^n)}$, $\alpha > \frac{1}{2}$. It suffices to bound $e^{-it(x)\Delta_{\mathbb{S}^n}} f$, and for this, we have, by (4.1),

$$\begin{aligned}
\|e^{-it(x)\Delta_{\mathbb{S}^n}} f(x)\|_{L^2} &= \Big\|\sum_{k=0}^{+\infty} e^{-it(x)\mu_k}\sum_{l=1}^{d_k}\widehat{f}_{k_l}e_{k_l}(x)\Big\|_{L^2} \lesssim \sum_{k=0}^{+\infty}\Big\|\sum_{l=1}^{d_k}\widehat{f}_{k_l}e_{k_l}(x)\Big\|_{L^2}\\
&\lesssim \sum_{k=0}^{+\infty}\Big(\sum_{l=1}^{d_k}|\widehat{f}_{k_l}|^2\Big)^{1/2} \lesssim \sum_{k=0}^{+\infty}(1+\mu_k)^{-\alpha/2}\Big(\sum_{l=1}^{d_k}(1+\mu_k)^{\alpha}|\widehat{f}_{k_l}|^2\Big)^{1/2}\\
&\lesssim \Big(\sum_{k=0}^{+\infty}(1+\mu_k)^{-\alpha}\Big)^{1/2}\Big(\sum_{k=0}^{+\infty}\sum_{l=1}^{d_k}(1+\mu_k)^{\alpha}|\widehat{f}_{k_l}|^2\Big)^{1/2}\\
&= \Big(\sum_{k=0}^{+\infty}(1+k(k+n-1))^{-\alpha}\Big)^{1/2}\|f\|_{H^\alpha(\mathbb{S}^n)}\\
&\le C_\alpha\|f\|_{H^\alpha(\mathbb{S}^n)}.
\end{aligned}$$

5 The General Case

To prove Theorem 1.3, we first do a spectrum decomposition. Take $\widetilde{\psi} \in C_0^\infty(\mathbb{R})$ and $\psi \in C_0^\infty(\mathbb{R}\backslash\{0\})$ such that $\widetilde{\psi}(\lambda^2) + \sum_{k=1}^{+\infty}\psi(2^{-2k}\lambda^2) = 1$ for all λ. Then we have the decomposition $f = \widetilde{\psi}(\Delta^2)f + \sum k = 1^{+\infty}\psi(2^{-2k}\Delta^2)f$, for any $f \in C^\infty(M)$, and furthermore,

$$T^*f \lesssim T^*(\widetilde{\psi}(\Delta^2)f) + \sum_{k=1}^{+\infty} T^*(\psi(2^{-2k}\Delta^2)f). \tag{5.1}$$

As mentioned in the introduction, we only need to show

$$\|T^*f\|_{L^p(M)} \lesssim \|f\|_{H^\alpha(M)}, \quad p = 2(n+2)/n,$$

for $\alpha > 3/4$, $n = 2$, or $\alpha > 9/10$, $n = 3$.

The low frequency part in (5.1) is easy to control, and we can prove

$$\|T^*(\widetilde{\psi}(\Delta)f)\|_{L^q(M)} \lesssim \|f\|_{L^2(M)},$$

for any $q \geq 2$. In fact, by making t into a function $t(x)$, we only need to show

$$\|e^{-it(x)\Delta}\widetilde{\psi}(\Delta)f\|_{L^q(M)} \lesssim \|f\|_{L^2(M)}.$$

By the compact support of $\widetilde{\psi}$, we have

$$\begin{aligned}
\|e^{-it(x)\Delta}\widetilde{\psi}(\Delta)f\|_{L^q} &= \Big\| \sum_{\lambda_j \leq c_0} e^{-it(x)\lambda_j^2}\widetilde{\psi}(\lambda_j^2)\widehat{f}_j e_j(x) \Big\|_{L^q} \\
&\lesssim \sum_{\lambda_j \leq c_0} \|e^{-it(x)\lambda_j^2}\widetilde{\psi}(\lambda_j^2)\widehat{f}_j e_j(x)\|_{L^q} \lesssim \sum_{\lambda_j \leq c_0} |\widehat{f}_j| \|e_j(x)\|_{L^q} \\
&\lesssim \sum_{\lambda_j \leq c_0} \lambda_j^{\delta(q)} |\widehat{f}_j| \|e_j(x)\|_{L^2},
\end{aligned}$$

for some positive $\delta(q)$. In the last step, we applied the L^q estimate for eigenfunctions of $-\Delta$ (see [21, Theorem 5.1.1] or [22]). If we take all the eigenfunctions to be L^2 normalized, the last term above is clearly bounded by $\|f\|_{L^2}$, after using Schwartz's inequality.

To handle the rest of the terms in (5.1), we prove that, for $0 < h \leq 1$,

$$\|T^*(\psi(h^2\Delta)f)\|_{L^p(M)} \lesssim h^{-\alpha}\|\psi(h^2\Delta)f\|_{L^2(M)}, \tag{5.2}$$

where $\alpha = 3/4$ if $n = 2$ or $\alpha = 9/10$ if $n = 3$. If we prove (5.2), then

$$\begin{aligned}
\sum_{k=1}^{+\infty} \|T^*(\psi(2^{-2k}\Delta)f)\|_{L^p} &\lesssim \sum_{k=1}^{+\infty} 2^{\alpha k}\|\psi(2^{-2k}\Delta)f\|_{L^2} \\
&= \sum_{k=1}^{+\infty} 2^{-\epsilon k}\|2^{(\alpha+\epsilon)k}\psi(2^{-2k}\Delta)f\|_{L^2} \\
&\lesssim \Big(\sum_{k=1}^{+\infty} 2^{-2\epsilon k}\Big)^{1/2} \Big(\sum_{k=1}^{+\infty} \|(I-\Delta)^{(\alpha+\epsilon)/2}\psi(2^{-2k}\Delta)f\|_{L^2}^2\Big)^{1/2} \\
&\leq C_\epsilon \|(I-\Delta)^{(\alpha+\epsilon)/2} f\|_{L^2} \\
&= C_\epsilon \|f\|_{H^{\alpha+\epsilon}}.
\end{aligned}$$

Now we are left to prove (5.2). To do this, we need the following Strichartz estimate.

Lemma 5.1 *Let $0 < h \leq 1$ and $p = \frac{2(n+2)}{n}$. Then*

$$\|e^{-it\Delta}(\psi(h^2\Delta)f)\|_{L^p((0,1]\times M)} \lesssim h^{-1/p}\|\psi(h^2\Delta)f\|_{L^2(M)}.$$

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Proof This lemma can be inferred from the general Strichartz estimate [6, Theorem 1], which was proved by applying Keel and Tao's theorem [15] after they constructed a parametrix for the frequency localized Schrödinger equation in local coordinates and proved a very short time version of the dispersion estimate. Here, we would like to present another proof that applies Hörmander's oscillatory integral estimates. First we state the frequency localized parametrix [6, Lemma 2.7].

Parametrix. Let U_1 be an open ball in $\mathbb{R}^n$ endowed with a Riemannian metric g. Take U_2 to be an open ball in U_1, $\chi_0 \in C_0^\infty(U_2)$, and $\phi \in C_0^\infty(\mathbb{R}^n)$. Then for every $h \in (0,1]$ and $w_0 \in C_0^\infty(U_1)$ there exists an $\alpha > 0$ and $\widetilde{w}(s,x) \in C_0^\infty([-\gamma,\gamma] \times U_2)$ that solves

$$\begin{cases} ih\partial_t \widetilde{w} - h^2 \Delta_g \widetilde{w} = r, \\ \widetilde{w}(0,x) = \chi_0(x)\phi(hD)w_0(x) \end{cases}$$

with $r(s,x) \in C_0^\infty([-\gamma,\gamma] \times U_2)$ satisfying

$$\|r(s,x)\|_{L^\infty([-\gamma,\gamma],L^p(U_2))} \lesssim C_N h^N \|w_0\|_{L^2(U_1)}, \quad \text{for all } N. \tag{5.3}$$

Furthermore, we have

$$\|\widetilde{w}(s,x)\|_{L^p([-\gamma,\gamma]\times U_2)} \lesssim h^{\frac{n+1}{p}-\frac{n}{2}} \|w_0\|_{L^2(U_1)}. \tag{5.4}$$

Here we sketch the construction of $\widetilde{w}$ from [6]. Consider

$$\widetilde{w}(s,x) = (2\pi h)^{-n} \int_{\mathbb{R}^n} e^{\frac{i}{h}\Phi(s,x,\xi)} a(s,x,\xi,h) \widehat{w}_0(\xi/h)\, d\xi, \tag{5.5}$$

where $a(s,x,\xi,h) = \sum_{j=0}^N h^j a_j(s,x,\xi)$. Here N is to be chosen large enough, $a_j \in C_0^\infty([-\gamma,\gamma] \times U_2 \times \mathbb{R}^n)$, with initial constraints

$$a_0(0,x,\xi) = \chi_0(x)\phi(\xi), \quad a_j(0,x,\xi) = 0,\ j \geq 1,$$

and $\Phi \in C^\infty([-t_0,t_0] \times U_2 \times B)$, where B is a ball containing the support of ϕ, with initial constraint $\Phi(0,x,\xi) = x\cdot\xi$. Then the equations for ϕ and a_j are given by the eikonal equation $\partial_s \Phi + \sum_{1\leq i,j\leq n} g^{ij}\partial_i\Phi\partial_j\Phi = 0$ and the transport equations

$$\partial_s a_0 + 2g(\nabla_g\Phi, \nabla_g a_0) + \Delta_g(\Phi)a_0 = 0,$$
$$\partial_s a_j + 2g(\nabla_g\Phi, \nabla_g a_j) + \Delta_g(\Phi)a_j = -\Delta_g(a_{j-1}),\ j \geq 1.$$

By the proof of Lemma 2.7 in [6], we also know that

$$r(s,x) = h^{N+2}(2\pi h)^{-n} \int_{\mathbb{R}^n} e^{\frac{i}{h}\Phi(s,x,\xi)} b(s,x,\xi,h)\widehat{w}_0(\xi/h)\, d\xi,$$

for some $b \in C_0^\infty([-\gamma,\gamma] \times U_2 \times B)$. This easily yields (5.3).

Next we apply Lemma 2.4 to (5.5). The existence of phase function Φ on a small interval for s is guaranteed by Hamilton–Jacoby theory. It is easy to see that the two conditions are satisfied when $s = 0$. Actually, when $s = 0$, for each x, $S(0,x)$ is a parabola. Then by continuity and compactness, the two conditions are satisfied for $s < \delta$ for some fixed $\delta = \delta(M) > 0$. Thus we get

$$\|\widetilde{w}(s,x)\|_{L^p([-\gamma,\gamma]\times U_2)} \lesssim h^{\frac{n+1}{p}-n} \|\widehat{w}_0(\xi/h)\|_{L^2(\mathbb{R}^n)} \lesssim h^{\frac{n+1}{p}-\frac{n}{2}} \|w_0(x)\|_{L^2(U_1)}.$$

Now let us continue to prove Lemma 5.1. Denote

$$\widetilde{w}_l(s,x) = e^{-ihs\Delta_g}(\chi_0\phi(hD)w_0)(s),$$

which is the solution to linear equation

$$\begin{cases} ih\partial_t \widetilde{w} - h^2\Delta\widetilde{w} = 0, \\ \widetilde{w}(0,x) = \chi_0(x)\phi(hD)w_0(x). \end{cases}$$

Therefore, $\widetilde{w}(s,x) - \widetilde{w}_l(s,x) = \int_0^s e^{-ih(s-\tau)\Delta_g} r(\tau,x)\,d\tau$. So by (5.3), we have

$$\|\widetilde{w}(s,x) - \widetilde{w}_l(s,x)\|_{L^p([-\gamma,\gamma]\times U_2)} \lesssim h^N \|w_0\|_{L^2(U_1)}.$$

Thus (5.4) also holds for $\widetilde{w}_l(s,x)$. Note $\phi(hD)$ is a cutoff on frequencies in $\mathbb{R}^n$. But in order to prove the lemma, we need to show

$$\|e^{-ihs\Delta}(\psi(h^2\Delta)f)\|_{L^p([-\gamma,\gamma]\times M)} \lesssim h^{\frac{n+1}{p}-\frac{n}{2}}\|f\|_{L^2(M)}, \tag{5.6}$$

where the cutoff $\psi(h^2\Delta)$ is made on frequencies λ_j. To treat this difference, we apply [6, Corollary 2.4], which actually says that there is a pseudodifferential operator $\Psi(D)$ of order 0 on M, such that, in local coordinates, $\Psi(\xi)$ is compactly supported and

$$\|(I-\Psi(hD))\psi(h^2\Delta)f\|_{H^\sigma(M)} \lesssim C_{\sigma,N}h^N\|f\|_{L^2(M)}$$

holds for all $h \in (0,1]$, $\sigma > 0$, $N > 0$, and $f \in C^\infty(M)$. Combining this with (5.4) for $\widetilde{w}_l$ and the boundedness of $e^{-ihs\Delta}$ on $H^\sigma(M)$, we then reach (5.6) by constructing partitions of unity.

Finally, let us see how (5.6) implies Lemma 5.1. With a change of variable $hs \to t$, since $p = \frac{2(n+2)}{n}$, (5.6) implies

$$\|e^{-it\Delta}(\psi(h^2\Delta)f)\|_{L^p([-\gamma h,\gamma h]\times M)} \lesssim h^{\frac{n+2}{p}-\frac{n}{2}}\|f\|_{L^2(M)} = \|f\|_{L^2(M)}.$$

It is also easy to see that we can replace f in the above L^2 norm by $\psi(h^2\Delta)f$ so that

$$\|e^{-it\Delta}(\psi(h^2\Delta)f)\|_{L^p([-\gamma h,\gamma h]\times M)} \lesssim \|\psi(h^2\Delta)f\|_{L^2(M)}.$$

Set $I_k = [(k-1)\gamma h, k\gamma h]$. Then

$$\begin{aligned}\|e^{-it\Delta}(\psi(h^2\Delta)f\|^p_{L^p((0,1]\times M)} &= \sum_{k=1}^{(\gamma h)^{-1}} \|e^{-it\Delta}\psi(h^2\Delta)f\|^p_{L^p(I_k\times M)} \\ &\lesssim \sum_{k=1}^{(\gamma h)^{-1}} \|e^{-ik\gamma h\Delta}\psi(h^2\Delta)f\|^p_{L^2(M)} \\ &\lesssim \sum_{k=1}^{(\gamma h)^{-1}} \|\psi(h^2\Delta)f\|^p_{L^2(M)} \\ &\lesssim h^{-1}\|\psi(h^2\Delta)f\|^p_{L^2(M)},\end{aligned}$$

which proves Lemma 5.1. ∎

Lemma 5.2 *Let $q \geq 2$. Suppose we have the following Strichartz estimate:*

$$\|e^{-it\Delta}(\psi(h^2\Delta)f)\|_{L^q((0,1]\times M)} \lesssim h^{-\beta}\|\psi(h^2\Delta)f\|_{L^2(M)}.$$

Then for the maximal Schrödinger operator T^*, the following estimate holds:

$$\|T^*(\psi(h^2\Delta)f\|_{L^q(M)} \lesssim h^{-2/q-\beta}\|\psi(h^2\Delta)f\|_{L^2(M)} + \|\psi(h^2\Delta)f\|_{L^q(M)}.$$

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Proof We will need an inequality from [16], which states

$$\sup_{t\in[a,b]} |g(t)| \le C_p(|g(a)| + \mu^{1/q-1}\|g'(t)\|_{L^q[a,b]} + \mu^{1/q}\|g\|_{L^q[a,b]}) \tag{5.7}$$

for any smooth $g(t)$ on $[a,b]$, $\mu > 0$, and $q \ge 1$.

Take $[a,b] = [0,1]$ and $g(t) = e^{-it\delta}\psi(h^2\Delta)f$. By (5.7) and Lemma 5.1,

$$\begin{aligned}\|T^*(\psi(h^2\Delta)f\|_{L^q(M)} &\lesssim \|\psi(h^2\Delta)f\|_{L^q(M)} + \mu^{1/q}\|e^{-it\Delta}\psi(h^2\Delta)f\|_{L^q((0,1]\times M)}\\ &\qquad + \mu^{1/q-1}\|e^{-it\Delta}(-i\Delta)\psi(h^2\Delta)f\|_{L^q((0,1]\times M)}\\ &\lesssim \|\psi(h^2\Delta)f\|_{L^q(M)} + \mu^{1/q}h^{-\beta}\|\psi(h^2\Delta)f\|_{L^2}\\ &\qquad + \mu^{1/q-1}h^{-\beta}\|(-i\Delta)\psi(h^2\Delta)f\|_{L^2}\\ &\lesssim \|\psi(h^2\Delta)f\|_{L^q} + \mu^{1/q}(h^{-\beta} + \mu^{-1}h^{-2-\beta})\|\psi(h^2\Delta)f\|_{L^2}.\end{aligned}$$

By taking $\mu = h^{-2}$, we finish the proof of Lemma 5.2. ∎

Now we are in a position to finish the proof of inequality (5.2), and hence the main theorem. Let us combine Lemmas 5.1 and 5.2 to get

$$\|T^*(\psi(h^2\Delta)f)\|_{L^p(M)} \lesssim h^{-3/p}\|\psi(h^2\Delta)f\|_{L^2(M)} + \|\psi(h^2\Delta)f\|_{L^p(M)}.$$

By the Sobolev imbedding, the last term above is no larger than

$$\|\psi(h^2\Delta)f\|_{H^{n/2-n/p}(M)} \simeq h^{-(n/2-n/p)}\|\psi(h^2\Delta)f\|_{L^2(M)}.$$

Note $p = \frac{2(n+2)}{n}$. So $\frac{n}{2} - \frac{n}{p} = \frac{n}{n+2}$, $\frac{3}{p} = \frac{3n}{2(n+2)}$. A simple calculation then yields

$$\|T^*(\psi(h^2\Delta)f)\|_{L^p(M)} \lesssim h^{-3/4}\|\psi(h^2\Delta)f\|_{L^2(M)},$$

when $n = 2$ and $\|T^*(\psi(h^2\Delta)f)\|_{L^p(M)} \lesssim h^{-9/10}\|\psi(h^2\Delta)f\|_{L^2(M)}$, when $n = 3$. The case of $n = 1$ will be dealt with in the next section.

6 A Solution on the Flat Torus and Other Special Manifolds

We may be able to improve our result if we could get a better Strichartz estimate than in Lemma 5.1. The argument in Section 5 comes from a trial to improve the Strichartz estimates on general manifolds [6]. Although for general manifolds we still get the same index and same loss, in some special manifolds we do have more precise Strichartz type inequalities that enable us to get improved theorems.

To continue with the two-dimensional flat Torus case, we will need the following Strichartz estimate on $\mathbb{T}^n$. It can be inferred from Bourgain [3, Proposition 3.6].

Lemma 6.1 *For $n \ge 2$, the following Strichartz estimate holds:*

$$\|e^{-it\Delta}f\|_{L^4((0,1]\times\mathbb{T}^n)} \lesssim \|f\|_{H^s(\mathbb{T}^n)},\ s > \frac{n}{4} - \frac{1}{2}.$$

Thus, by Lemma 5.2, we have

$$\|T^*(\psi(h^2\Delta)f)\|_{L^4(\mathbb{T}^2)} \lesssim h^{-1/2-s}\|\psi(h^2\Delta)f\|_{L^2(\mathbb{T}^2)} + \|\psi(h^2\Delta)f\|_{L^4(\mathbb{T}^2)}.$$

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By the Sobolev embedding

$$\|\psi(h^2\Delta)f\|_{L^4(\mathbb{T}^2)} \lesssim h^{-1/2}\|\psi(h^2\Delta)f\|_{L^2(\mathbb{T}^2)}.$$

Then $\|T^*(\psi(h^2\Delta)f)\|_{L^4(\mathbb{T}^2)} \lesssim h^{-1/2-s}\|\psi(h^2\Delta)f\|_{L^2(\mathbb{T}^2)}$, for any $s > 0$. By a similar argument as in Section 5, we finish the proof of the two-dimensional flat Torus case.

For the $n = 1$ case of Theorem 1.3, we first notice that all connected, compact one-dimensional manifolds are isometric to circles. So we only need to consider $\mathbb{T}^1$. We have the following Strichartz estimate [3, Proposition 2.36].

Lemma 6.2 *The following Strichartz estimate holds:*

$$\|e^{-it\Delta}f\|_{L^6((0,1]\times\mathbb{T}^1)} \lesssim \|f\|_{H^s(\mathbb{T}^1)},\ s > 0.$$

As above, we conclude $\|T^*(\psi(h^2\Delta)f)\|_{L^6(\mathbb{T}^1)} \lesssim h^{-1/3-s}\|\psi(h^2\Delta)f\|_{L^2(\mathbb{T}^1)}$. Thus we finish the proof of Theorem 1.3 for $n = 1$.

Now let us consider the higher dimensional flat torus $\mathbb{T}^n$, $n \geq 3$. By applying the stronger Strichartz estimate in the following lemma and the argument above, we can reduce the amount of regularity requirement to some number less than 1 for flat tori of all dimensions.

Lemma 6.3 *The following Strichartz estimate holds:*

$$\|e^{-it\Delta}f\|_{L^q((0,1]\times\mathbb{T}^n)} \lesssim \|f\|_{H^s(\mathbb{T}^n)},\ s > 0,\ q \leq \frac{2(n+1)}{n}.$$

The proof of this lemma can be found in [4]. As a consequence, we have the following theorem.

Theorem 6.4 *Let $e^{-it\Delta}$ be the Schrödinger operator defined on $\mathbb{T}^n$. Then $e^{-it\Delta}f$ converges pointwise to f if $f \in H^\alpha(\mathbb{T}^n)$, where $\alpha > \frac{n}{n+1}$ and $n \geq 3$.*

Finally let us consider one type of manifold whose geodesics are closed with a common period. For the geometric properties of such manifolds, see [1]. Here we only apply the Strichartz estimate [6, Theorem 4] for the Schrödinger operator on such manifolds,

$$\|e^{-it\Delta}f\|_{L^4((0,1]\times M)} \lesssim \|f\|_{H^s(M)},\ s > s_0(n),$$

where $s_0(2) = 1/8$, $s_0(n) = n/4 - 1/2$ for $n \geq 3$. Note the n sphere $\mathbb{S}^n$ is one of the above manifolds. Furthermore, the loss $s_0(n)$ has been proved to be sharp for $\mathbb{S}^n$. Similarly, we have the following.

Theorem 6.5 *Let M be the manifold described above. Then $e^{-it\Delta}f$ converges pointwise to f if*

(i) *$n = 2$ and $f \in H^\alpha(\mathbb{M}^2)$, $\alpha > 5/8$, or*
(ii) *$n = 3$ and $f \in H^\alpha(\mathbb{M}^3)$, $\alpha > 3/4$.*

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