A ROOM DESIGN OF ORDER 14

C.D. O'Shaughnessy

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1. A Room design of order 2n, where n is a positive integer, is an arrangement of 2n objects in a square array of side 2n - 1, such that each of the $(2n - 1)^2$ cells of the array is either empty or contains exactly two distinct objects; each of the 2n objects appears exactly once in each row and column; and each (unordered) pair of objects occurs in exactly one cell. A Room design of order 2n is said to be cyclic if the entries in the (i + 1)th row are obtained by moving the entries in the i th row one column to the right (with entries in the (2n - 1)th column being moved to the first column), and increasing the entries in each occupied cell by 1(mod 2n - 1), except that the digit 0 remains unchanged.

It has been shown that Room designs of orders 4 and 6 do not exist [1], but that Room designs of orders 2 and 8[1], 2^{2m+1}[2], 8, 12, 20 and 24 [3], and 10 [5] do exist. No previous mention of a Room design of order 14 has been made.

2. Bruck [4] defines a Room pair of quasigroups as a pair (G, r), (G, c) of commutative, idempotent quasigroups satisfying the orthogonality conditions: (01). if p is in G and if x, y are elements of G such that x r y = x c y = p, then x = y = p, and (02) if p, q are distinct elements of G, there is at most one unordered pair (x, y) of elements of G such that x r y = p and x c y = q. It is to be noted that if such a Room pair of quasigroups of order 2n - 1 exists, then a Room design of order 2n may be defined from them by letting x r y and x c y be, respectively, the row and column of the Room design in which the unordered pair (x, y) appears. (NOTE: an additional element must be added to G to make 2n elements, and this new element appears with the other elements in the leading diagonal of the array). It is further noted in [4] that a Steiner Triple System G of order 2n - 1 gives rise to an idempotent, totally symmetric quasigroup (G, o) of the same order by defining, for any two distinct elements a, b of G, a o b = c where a, b, c is the triple of the system containing a and b, and a o a = a for all a in G.

Using these results it is obvious that if one can find a pair of Steiner Triple Systems of order 2n - 1 for a given set G such that the two systems have no triples in common, and such that if two pairs

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of elements appear with the same third element in one system then they appear with distinct third elements in the other system, then one can define a Room pair of (totally symmetric) quasigroups of order $2n - 1$ and hence a Room design of order $2n$.

3. Since Steiner Triple Systems exist only if the order is congruent to 1 or 3 mod 6, we need consider only such cases. For $2n - 1$ congruent to 1 mod 6, it is relatively easy to find the required pair of Steiner Triple Systems.

For $2n = 8$, let $G = \{1, 2, 3, 4, 5, 6, 7\}$ and two systems of triples are

$$S_1 = \{(1\,2\,4), (2\,3\,5), (3\,4\,6), (4\,5\,7), (5\,6\,1), (6\,7\,2), (7\,1\,2)\}$$

and $$S_2 = \{(1\,2\,6), (2\,3\,7), (3\,4\,1), (4\,5\,2), (5\,6\,3), (6\,7\,4), (7\,1\,5)\}.$$

We then define the Room design of order 8 as follows:- if $(x\,y\,z)$ is a triple from S_1, then (x, y) appear together in row z, (x, z) in row y, and (y, z) in row x. The same may be said with regard to columns if $(x\,y\,z)$ is a triple from S_2. The pair $(0, i)$ will appear in the i-th row, i-th column. The above pair S_1 and S_2 of systems yields the Room design of order 8 given in [3], which is cyclic with first row:

$(0, 1)$ $(3, 7)$ $(5, 6)$ ϕ $(2, 4)$ ϕ ϕ

For $2n = 14$, let $G = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13\}$ and two systems of triples are

$$\begin{aligned} S_3 = \{&(1\,4\,5), (2\,5\,6), (3\,6\,7), (4\,7\,8), (5\,8\,9), (6\,9\,10), (7\,10\,11),\\ &(8\,11\,12), (9\,12\,13), (10\,13\,1), (11\,1\,2), (12\,2\,3), (13\,3\,4),\\ &(1\,6\,12), (2\,7\,13), (3\,8\,1), (4\,9\,2), (5\,10\,3), (6\,11\,4), (7\,12\,5),\\ &(8\,13\,6), (9\,1\,7), (10\,2\,8), (11\,3\,9), (12\,4\,10), (13\,5\,11)\}\end{aligned}$$

and $$\begin{aligned} S_4 = \{&(1\,2\,5), (2\,3\,6), (3\,4\,7), (4\,5\,8), (5\,6\,9), (6\,7\,10), (7\,8\,11),\\ &(8\,9\,12), (9\,10\,13), (10\,11\,1), (11\,12\,2), (12\,13\,3), (13\,1\,4),\\ &(1\,7\,12), (2\,8\,13), (3\,9\,1), (4\,10\,2), (5\,11\,3), (6\,12\,4), (7\,13\,5),\\ &(8\,1\,6), (9\,2\,7), (10\,3\,8), (11\,4\,9), (12\,5\,10), (13\,6\,11)\}.\end{aligned}$$

These systems yield the following cyclic Room design of order 14.

0, 1	7, 9		6, 12				4, 5	10, 13	3, 8		2, 11	
	0, 2	8, 10		7, 13				5, 6	1, 11	4, 9		3, 12
4, 13		0, 3	9, 11		1, 8				6, 7	2, 12	5, 10	
	1, 5		0, 4	10, 12		2, 9				7, 8	3, 13	6, 11
7, 12		2, 6		0, 5	11, 13		3, 10				8, 9	1, 4
2, 5	8, 13		3, 7		0, 6	1, 12		4, 11				9, 10
10, 11	3, 6	1, 9		4, 8		0, 7	2, 13		5, 12			
	11, 12	4, 7	2, 10		5, 9		0, 8	1, 3		6, 13		
		12, 13	5, 8	3, 11		6, 10		0, 9	2, 4		1, 7	
			1, 13	6, 9	4, 12		7, 11		0, 10	3, 5		2, 8
3, 9				1, 2	7, 10	5, 13		8, 12		0, 11	4, 6	
	4, 10				2, 3	8, 11	1, 6		9, 13		0, 12	5, 7
6, 8		5, 11				3, 4	9, 12	2, 7		1, 10		0, 13

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The Steiner Triple Systems S_1, S_2, S_3 and S_4 have in effect been generated by the triples (1 2 4), (1 2 6), (1 4 5) and (1 6 12), and (1 2 5) and (1 7 12) respectively. In the same manner, for $2n = 20$ and G the set of the first 19 positive integers, the 57 triples of S_5 may be generated by (1 2 7), (1 3 11) and (1 4 8), and those of S_6 by (1 2 16), (1 3 9) and (1 4 11). These lead to the cyclic Room design of order 20 with first row

(0, 1) (9, 18) ϕ (13, 16) (3, 11) (2, 7) ϕ ϕ
(4, 8) (14, 15) ϕ ϕ ϕ (5, 17) ϕ ϕ
(6, 19) (10, 12) ϕ

4. Conjecture. The above procedure will generate Room designs of any order congruent to 2 mod 6. However, it does not appear to work for orders congruent to 4 mod 6.

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University of Windsor

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