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AN ADDITION THEOREM AND SOME PRODUCT FORMULAS FOR q-BESSEL FUNCTIONS

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1. Introduction. The most familiar series representation of the Bessel function is

$$(1.1) \qquad J_\nu(x) = \frac{(x/2)^\nu}{\Gamma(\nu + 1)} \sum_{n=0}^{\infty} \frac{(-x^2/4)^n}{n!(\nu + 1)_n}.$$

Jackson [12] gave the following q-analogues:

$$(1.2) \qquad J_\nu^{(1)}(x; q) = \frac{\left[\frac{x}{2(1 - q)}\right]^\nu}{\Gamma_q(\nu + 1)} \sum_{n=0}^{\infty} \frac{\left(-\frac{x^2}{4}\right)^n}{(q, q^{\nu+1}; q)_n},$$

$$(1.3) \qquad J_\nu^{(2)}(x; q) = \frac{\left[\frac{x}{2(1 - q)}\right]^\nu}{\Gamma_q(\nu + 1)} \sum_{n=0}^{\infty} \frac{\left(-\frac{x^2}{4}\right)^n}{(q, q^{\nu+1}; q)_n} q^{n(\nu+n)},$$

where $0 < q < 1$, the q-shifted factorials are defined by

$$(a_1, a_2, \dots, a_k; q)_n = \prod_{j=1}^{k} (a_j; q)_n$$

$$(1.4) \qquad (a_j; q)_n = \begin{cases} 1 & \text{if } n = 0, \\ (1 - a_j)(1 - a_j q) \dots (1 - a_j q^{n-1}) & \text{if } n = 1, 2, \dots, \end{cases}$$

$$(a; q)_\infty = \prod_{n=1}^{\infty} (1 - aq^{n-1}),$$

and the q-gamma function is given by

$$(1.5) \qquad \Gamma_q(x) = \frac{(q; q)_\infty}{(q^x; q)_\infty}(1 - q)^{1-x}, \quad \lim_{q \to 1^-} \Gamma_q(x) = \Gamma(x).$$

It is easy to see that

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$$(1.6)\qquad \lim_{q\to 1^-} J_\nu^{(1)}((1-q)x;q) = \lim_{q\to 1^-} J_\nu^{(2)}((1-q)x;q) = J_\nu(x).$$

However, the infinite series in (1.2) is convergent only for $|x| < 2$ while the one in (1.3) converges for all x because of the q^{n^2} factor. For this reason, $J_\nu^{(2)}(x;q)$ would seem to be a more useful analogue, but it turns out that $J_\nu^{(1)}(x;q)$ is almost just as useful. Hahn **[9]** found that for $|x| < 2$ they are connected by the formula

$$(1.7)\qquad \frac{J_\nu^{(2)}(x;q)}{\left(-\frac{x^2}{4};q\right)_\infty} = J_\nu^{(1)}(x;q).$$

Ismail **[11]** derived a number of properties of these q-analogues and studied the q-Lommel polynomials associated with them. He **[10]** also gave some integral representations of the modified q-Bessel functions. The notations $J_\nu^{(k)}(x;q)$, $k = 1, 2$, used here is due to Ismail.

Recently the author **[15]** found a Poisson-type integral representation of $J_\nu^{(2)}(x;q)$ and used it to obtain a number of different series representations of the q-Bessel functions as well as a q-analogue of a degenerate addition formula for Bessel functions. In a later investigation **[14]** the author used one of Ramanujan's integrals to compute some infinite integrals of the q-Bessel functions. It is in this latter work that the author found it convenient to use a slightly modified definition of $J_\nu^{(2)}(x;q)$:

$$(1.8)\qquad J_\nu(x|q) = \frac{\left[\frac{x}{(1-q)(1+q^{1/2})}\right]^\nu}{\Gamma_q(\nu+1)} \sum_{n=0}^\infty \frac{(-1)^n q^{n(\nu+n)}}{(q, q^{\nu+1};q)_n}\left(\frac{x}{1+q^{1/2}}\right)^{2n}.$$

It is obvious that

$$(1.9)\qquad J_\nu(x|q) = J_\nu^{(2)}\left(2x/(1+q^{1/2});q\right)$$

and that

$$\lim_{q\to 1^-} J_\nu((1-q)x|q) = J_\nu(x).$$

We shall assume throughout the paper that $0 < q < 1$ and we shall continue to use (1.8) as the defining formula for the q-Bessel functions.

The main objective of this paper is to derive the following addition formula:

$$(1.10)\qquad (-(a(1-q^{1/2})x)^2;q)_\infty \times {}_2\phi_1\left[\begin{matrix}\frac{b}{a}q^{(\nu+1)/2}e^{i\psi}, \frac{b}{a}q^{(\nu+1)/2}e^{-i\psi} \\ q^{\nu+1}\end{matrix};q, -a^2(1-q^{1/2})^2x^2\right]$$

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$$= \Gamma_q(\nu + 1)\left[\frac{(1 + q^{1/2})^2}{abx^2}\right]^\nu \sum_{n=0}^{\infty} \frac{1 - q^{\nu+n}}{1 - q^\nu} q^{\binom{n}{2}+((\nu+1)/2)n}$$

$$\times J_{\nu+n}(a(1 - q)x|q)J_{\nu+n}(b(1 - q)x|q)C_n(\cos\psi;\ q^\nu|q),$$

where Re $\nu > 0$, $0 < b < a$, $0 \leqq \psi \leqq \pi$ and $C_n(z;\ \beta|q)$ is Rogers' q-ultraspherical polynomial defined by

$$(1.11)\quad C_n(z;\ \beta|q) = \sum_{k=0}^{n} \frac{(\beta;\ q)_k(\beta;\ q)_{n-k}}{(q;\ q)_k(q;\ q)_{n-k}} \cos(n - 2k)\,\psi.$$

$$= \frac{(\beta;\ q)_n}{(q;\ q)_n} e^{in\psi} {}_2\phi_1\left[\begin{matrix} q^{-n}, \beta \\ \beta^{-1}q^{1-n} \end{matrix};\ q,\ q\beta^{-1}e^{-2i\psi}\right]$$

where $z = \cos\psi$, see [2]. In (1.10) and (1.11), the ${}_2\phi_1$ series are special cases of a basic hypergeometric series defined by

$$(1.12)\quad {}_r\phi_{r+s-1}\left[\begin{matrix} a_1, a_2, \dots, a_r \\ b_1, b_2, \dots, b_{r+s-1} \end{matrix};\ q;\ x\right]$$

$$= \sum_{n=0}^{\infty} \frac{(a_1, a_2, \dots, a_r;\ q)_n}{(q, b_1, \dots, b_{r+s-1};\ q)_n} q^{s\binom{n}{2}}(-1)^{sn}x^n,$$

$r = 0, 1, \dots;\ s = 0, 1, \dots$. When $s = 0$ and $r \geqq 1$ the series is called *balanced* if $x = q$ and $b_1b_2 \dots b_{r-1} = qa_1a_2 \dots a_r$; it is called *well-poised* if $b_i = a_1q/a_i$, $i = 2, 3, \dots, r$ and *very-well-poised* if, in addition, $a_2 = qa_1^{1/2}$, $a_3 = -qa_1^{1/2}$. Note that the ${}_2\phi_1$ series in (1.11) is well-poised.

It follows from the q-binomial formula

$$(1.13)\quad \sum_{n=0}^{\infty} \frac{(a;\ q)_n}{(q;\ q)_n} x^n = \frac{(ax;\ q)_\infty}{(x;\ q)_\infty}, \quad |x| < 1,$$

that

$$(1.14)\quad (x;\ q)_\infty^{-1} = \sum_{n=0}^{\infty} \frac{x^n}{(q;\ q)_n},$$

and

$$(-x;\ q)_\infty = \sum_{n=0}^{\infty} \frac{x^n}{(q;\ q)_n} q^{\binom{n}{2}}.$$

Clearly,

$$(1.16)\quad e^x = \lim_{q\to 1^-} (x(1 - q);\ q)_\infty^{-1} = \lim_{q\to 1^-} (-x(1 - q);\ q)_\infty,$$

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and

(1.17) $$\lim_{q\to 1^-} (-(1-q^{1/2})^2x^2;q)_\infty = 1.$$

In the limit $q \to 1^-$ formula (1.10) then gives Gegenbauer's addition formula [20, p. 363]

(1.18) $$\left(\frac{\omega}{2}\right)^{-\nu} J_\nu(\omega) = 2^{2\nu}\Gamma(\nu+1) \cdot \sum_{n=0}^{\infty} \frac{\nu+n}{\nu} x^{-\nu} J_{\nu+n}(x) y^{-\nu} J_{\nu+n}(y) C_n^\nu(\cos\psi),$$

where $\omega = (x^2+y^2-2xy\cos\psi)^{1/2}$ and $C_n^\nu(z)$ is Gegenbauer's ultraspherical polynomial. In order to prove (1.10) we shall first derive the product formula

(1.19) $$J_\nu(a(1-q)x|q)J_\nu(b(1-q)x|q)/(-(a(1-q^{1/2})x)^2;q)_\infty$$
$$= \frac{\Gamma_q(\nu)}{2\pi\Gamma_q(\nu+1)\Gamma_q(2\nu)}\left(\frac{abx^2}{(1+q^{1/2})^2}\right)^\nu \cdot \int_0^\pi \frac{(e^{2i\psi}, e^{-2i\psi};q)_\infty}{(q^\nu e^{ei\psi}, q^\nu e^{-2i\psi};q)_\infty} \cdot {}_2\phi_1\left[\begin{matrix}\frac{b}{a}q^{(\nu+1)/2}e^{i\psi}, \frac{b}{a}q^{(\nu+1)/2}e^{-i\psi} \\ q^{\nu+1}\end{matrix}; q, -a^2(1-q^{1/2})^2x^2\right] d\psi,$$

where $\operatorname{Re}\nu > 0$ and $0 < b < a$.

In section 2 we shall first derive a general product formula for the q-Bessel functions from which we shall deduce that

(1.20) $$J_\mu((1-q)x|q)J_\nu((1-q)x|q)/(-((1-q^{1/2})x)^2;q)_\infty$$
$$= \frac{\left(\frac{x}{1+q^{1/2}}\right)^{\mu+\nu}}{\Gamma_q(\mu+1)\Gamma_q(\nu+1)} \cdot {}_4\phi_3\left[\begin{matrix}q^{(\mu+\nu+1)/2}, q^{(\mu+\nu+2)/2}, -q^{(\mu+\nu+1)/2}, -q^{(\mu+\nu+2)/2} \\ q^{\mu+1}, q^{\nu+1}, q^{\mu+\nu+1}\end{matrix}; q, -(1-q^{1/2})^2x^2\right]$$

which is a q-analogue of [6, (49), p. 11]

(1.21) $J_\mu(x)J_\nu(x)$

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$$= \frac{\left(\frac{x}{2}\right)^{\mu+\nu}}{\Gamma(\mu+1)\Gamma(\nu+1)} {}_2F_3\left[\begin{matrix} \frac{\mu+\nu+1}{2}, \frac{\mu+\nu+2}{2} \\ \mu+1, \quad \nu+1, \mu+\nu+1 \end{matrix}; -x^2\right].$$

We shall use (1.20) in Section 7 to prove that

$$(1.22)\quad J_\nu^2((1-q)x|q) - J_{\nu+1}((1-q)x|q)J_{\nu-1}((1-q)x|q)$$
$$= \frac{q^\nu(1-q)}{1-q^{\nu+1}} J_\nu^2((1-q)x|q) + \frac{q^\nu(1-q)(1+q^{\nu+1})}{(1-q^{\nu+2})}$$
$$\cdot \sum_{n=0}^{\infty} \frac{(q^\nu; q)_n (1-q^{2\nu+2+2n})}{(q^{\nu+3}; q)_n (1-q^{2\nu+2})} q^{\binom{n+1}{2}+(\nu+1)n}$$
$$\cdot J_{\nu+n}^2((1-q)x|q),$$

for $\nu \geqq 0$, which gives the Turán-type inequality

$$(1.23)\quad J_\nu^2((1-q)x|q) > J_{\nu+1}((1-q)x|q)J_{\nu-1}((1-q)x|q).$$

Also, by using (1.19) we will show in Section 6 that

$$(1.24)\quad \int_0^\infty x^{\lambda-1} \frac{J_\nu(a(1-q)x|q)J_\nu(b(1-q)x|q)}{(-(a(1-q^{1/2}x)^2; q)_\infty} dx$$
$$= \frac{a^{-\lambda}\Gamma\left(\nu+\frac{\lambda}{2}\right)\Gamma\left(1-\nu-\frac{\lambda}{2}\right)}{2\Gamma_q\left(1-\frac{\lambda}{2}\right)\Gamma_q\left(1-\nu-\frac{\lambda}{2}\right)\Gamma_q\left(\nu+1-\frac{\lambda}{2}\right)} \left(\frac{1+q^{1/2}}{1-q^{1/2}}\right)^{\lambda/2}$$
$$\cdot \frac{\left(\frac{b^2}{a^2} q^{1-(\lambda/2)}; q\right)_\infty}{\left(\frac{b^2}{a^2} q^{1-\lambda}; q\right)_\infty} \left(\frac{b}{a}\right)^\nu {}_2\phi_1\left[\begin{matrix} \frac{b^2}{a^2}, q^{\lambda/2} \\ \frac{b^2}{a^2} q^{1-(\lambda/2)} \end{matrix}; q, q^{\nu+1-(\lambda/2)}\right]$$

where $\mathrm{Re}(\lambda + 2\nu) > 0$, $\mathrm{Re}\,\lambda < 2$ and $0 < b \leqq a$.

2. A general product formula. Let us assume that a, b, x are real and that $0 < b \leqq a$ and $x \geqq 0$. Then, for $\mathrm{Re}(\mu, \nu) > -1$ we get, by (1.7)-(1.9).

$$(2.1)\quad J_\nu(a(1-q)x|q)J_\nu(b(1-q)x|q)/(-(a(1-q^{1/2})x)^2)_\infty$$
$$= \frac{\left(\frac{x}{1+q^{1/2}}\right)^{\mu+\nu} a^\mu b^\nu}{\Gamma_q(\mu+1)\Gamma_q(\nu+1)} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{a^{2n} b^{2m} q^{m(\nu+m)}}{(q, q^{\mu+1}; q)_n (q, q^{\nu+1}; q)_m}$$
$$\cdot (-x^2(1-q^{1/2})^2)^{m+n}$$

$$= \frac{\left(\frac{x}{1+q^{1/2}}\right)^{\mu+\nu} a^{\mu}b^{\nu}}{\Gamma_q(\mu+1)\Gamma_q(\nu+1)} \sum_{n=0}^{\infty} \frac{(-a^2(1-q^{1/2})^2x^2)^n}{(q, q^{\mu+1}; q)_n}$$

$$\cdot {}_2\phi_1\left[\begin{matrix} q^{-n}, q^{-\mu-n} \\ q^{\nu+1} \end{matrix}; q, \frac{b^2}{a^2} q^{\mu+\nu+1+2n}\right].$$

This is a q-analogue of the formula [**6**, (47), p. 11]

$$(2.2) \quad J_{\mu}(ax)J_{\nu}(bx)$$

$$= \frac{\left(\frac{x}{2}\right)^{\mu+\nu} a^{\mu}b^{\nu}}{\Gamma(\mu+1)\Gamma(\nu+1)} \sum_{n=0}^{\infty} \frac{(-a^2x^2/4)^n}{n!(\mu+1)_n} {}_2F_1\left[\begin{matrix} -n, -\mu-n \\ \nu+1 \end{matrix}; \frac{b^2}{a^2}\right].$$

When $a = b$, the ${}_2\phi_1$ series in (2.1) can be summed by the q-Vandermonde formula [**19**, (IV. 3), p. 247]. Since

$$(2.3) \quad (aq^m; q)_n = (a; q)_{m+n}/(a; q)_m$$

and

$$(2.4) \quad (a; q)_{2n} = (a, aq; q^2)_n = (a^{1/2}, -a^{1/2}, a^{1/2}q^{1/2}, -a^{1/2}q^{1/2}; q)_n$$

we find that

$$(2.5) \quad {}_2\phi_1\left[\begin{matrix} q^{-n}, q^{-\mu-n} \\ q^{\nu+1} \end{matrix}; q, q^{\mu+\nu+1+2n}\right] = \frac{(q^{\mu+\nu+1+n}; q)_n}{(q\nu+1; q)_n}$$

$$= \frac{(q^{(\mu+\nu+1)/2}, q^{(\mu+\nu+2)/2}, -q^{(\mu+\nu+1)/2}, -q^{(\mu+\nu+2)/2}; q)_n}{(q^{\nu+1}, q^{\mu+\nu+1}; q)_n},$$

which immediately leads to the product formula (1.20).

3. A quadratic transformation of (2.1) **when** $\mu = \nu$. The ${}_2\phi_1$ series in (2.1) is a terminating well-poised series in the case $\mu = \nu$ and hence can be transformed to a balanced ${}_4\phi_3$ series by a special case of Sears-Carlitz formula [**17**, (4.1)], [**5**, (2.4)], see also [**7**]. Thus

$$(3.1) \quad {}_2\phi_1\left[\begin{matrix} q^{-n}, q^{-\nu-n} \\ q^{\nu+1} \end{matrix}; q, c^2q^{2\nu+1+2n}\right]$$

$$= \frac{(-c^2q^{\nu+(n+1)/2}; q)_{\infty}}{(-c^2q^{\nu+(1/2)+(3n)/2}; q)_{\infty}}$$

$$\cdot {}_4\phi_3\left[\begin{matrix} q^{-(n/2)}, q^{(1-n)/2}, -q^{-(n/2)}, & -q^{\nu+(1/2)+(n/2)} \\ q^{\nu+1}, \quad -c^2q^{\nu+(1/2)+(n/2)}, & -c^{-2}q^{1/2-\nu-(3n/2)} \end{matrix}; q, q\right].$$

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By a series of transformations on the balanced and terminating ${}_4\phi_3$ series above we can show that

$$(3.2)\quad {}_2\phi_1\left[\begin{matrix} q^{-n}, q^{-\nu-n} \\ q^{\nu+1} \end{matrix}; q, c^2q^{2\nu+1+2n}\right]$$
$$= \frac{(cq^{\nu+(1/2)}, -cq^{\nu+(1/2)}, -q^{\nu+1}; q)_n}{(q^{2\nu+1}; q)_n}$$
$$\cdot {}_4\phi_3\left[\begin{matrix} q^{-n}, -q^{\nu}, & c, & -c, \\ & cq^{\nu+(1/2)}, -cq^{\nu+(1/2)}, & -q^{-\nu-n} \end{matrix}; q, q\right].$$

It can also be proved directly by using formulas (3.2) and (3.10) of Askey and Ismail [**2**], namely

$$(3.3)\quad C_n\left(\frac{z+z^{-1}}{2}; \beta|q\right)$$
$$= \frac{(\beta; q)_n}{(q; q)_n} z^{-n} {}_2\phi_1\left[\begin{matrix} q^{-n}, \beta \\ q^{1-n}\beta^{-1} \end{matrix}; q, q\beta^{-1}z^2\right]$$
$$= \frac{(\beta^2; q)_n}{(q; q)_n} \beta^{-(n/2)} {}_4\phi_3\left[\begin{matrix} q^{-n}, \beta^2q^n, & \beta^{1/2}z, & \beta^{1/2}z^{-1} \\ & \beta q^{1/2}, -\beta q^{1/2}, -\beta \end{matrix}; q, q\right].$$

Replacing β and z by $q^{-\nu-n}$ and $cq^{(\nu+n)/2}$, respectively, we find that

$$(3.4)\quad {}_2\phi_1\left[\begin{matrix} q^{-n}, q^{-\nu-n} \\ q^{\nu+1} \end{matrix}; q, c^2q^{2\nu+1+2n}\right]$$
$$= \frac{(q^{-2\nu-2n}; q)_n}{(q^{-\nu-n}; q)_n}(cq^{\nu+n})^n$$
$$\cdot {}_4\phi_3\left[\begin{matrix} q^{-n}, q^{-2\nu-n}, & c, & c^{-1}q^{-\nu-n} \\ & -q^{-\nu-n}, q^{(1/2)-\nu-n}, & -q^{(1/2)-\nu-n} \end{matrix}; q, q\right]$$
$$= \frac{(q^{-2\nu-2n}, cq^{\nu+(1/2)}, -cq^{\nu+(1/2)}; q)_n}{(q^{-\nu-n}, q^{(1/2)-\nu-n}, -q^{(1/2)-\nu-n}; q)_n}(-q^{-\nu})^n$$
$$\cdot {}_4\phi_3\left[\begin{matrix} q^{-n}, -q^{\nu}, & c, & -c \\ & cq^{\nu+(1/2)}, -cq^{\nu+(1/2)}, & -q^{-\nu-n} \end{matrix}; q, q\right]$$
$$= \frac{(q^{2\nu+1}; q)_{2n}(cq^{\nu+(1/2)}, -cq^{\nu+(1/2)}; q)_n}{(q^{2\nu+1}, q^{\nu+1}, q^{\nu+(1/2)}, -q^{\nu+(1/2)}; q)_n}$$
$$\cdot {}_4\phi_3\left[\begin{matrix} q^{-n}, -q^{\nu}, & c, & -c \\ & cq^{\nu+(1/2)}, -cq^{\nu+(1/2)}, & -q^{-\nu-n} \end{matrix}; q, q\right],$$

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which gives (3.2), by (2.4). In obtaining the second last expression on the right side of (3.4) we used Sears' transformation formula [**18**, (8.3)]

(3.5) $$ {}_4\Phi_3\left[\begin{matrix} q^{-n}, a, b, c \\ d, e, f \end{matrix}; q, q\right] = \frac{(aq^{1-n}/e, aq^{1-n}/f; q)_n}{(e, f; q)_n}\left(\frac{bc}{d}\right)^n \cdot {}_4\Phi_3\left[\begin{matrix} q^{-n}, a, d/b, \quad d/c \\ d, aq^{1-n}/e, aq^{1-n}/f \end{matrix}; q, q\right], $$

provided $def = abcq^{1-n}$.

From (2.1) and (3.2) we then obtain the product formula

(3.6) $$ J_\nu(a(1 - q)x|q)J_\nu(b(1 - q)x|q)/(-(a(1 - q^{1/2})x)^2; q)_\infty = \frac{\left(\frac{x}{1 + q^{1/2}}\right)^{2\nu}(ab)^\nu}{\Gamma_q^2(\nu + 1)} \cdot \sum_{n=0}^{\infty} \frac{\left(\frac{b}{a}q^{\nu+(1/2)}, -\frac{b}{a}q^{\nu+(1/2)}, -q^{\nu+1}; q\right)_n}{(q, q^{\nu+1}, q^{2\nu+1}; q)_n}(-a^2(1 - q^{1/2})^2x^2)^n \cdot {}_4\Phi_3\left[\begin{matrix} q^{-n}, -q^\nu, \quad \frac{b}{a}, \quad -\frac{b}{a} \\ \frac{b}{a}q^{\nu+(1/2)}, -\frac{b}{a}q^{\nu+(1/2)}, -q^{-\nu-n} \end{matrix}; q, q\right]. $$

4. Proof of (1.19). Askey and Wilson [**4**] showed that

(4.1) $$ \int_{-1}^{1} \frac{h(z; 1, -1, q^{1/2}, -q^{1/2}|q)}{h(z; a, b, c, d|q)} \frac{dz}{\sqrt{1 - z^2}} = \frac{2\pi(abcd; q)_\infty}{(q, ab, ac, ad, bc, bd, cd; q)_\infty}, $$

where $\max(|q|, |a|, |b|, |c|, |d|) < 1$, and

(4.2) $$ h(z; a_1, a_2, a_3, a_4|q) = \prod_{i=1}^{4} h(z; a_i|q), $$

with

$$(4.3)\quad h(z; a|q) = \prod_{k=0}^{\infty} (1 - 2azq^k + a^2q^{2k}) = (ae^{i\psi}, ae^{-i\psi}; q)_\infty,$$

where $z = \cos\psi$. Since

$$(4.4)\quad {}_3\phi_2\left[\begin{matrix} q^{-n}, q^{\nu/2}e^{i\psi}, \quad q^{\nu/2}e^{-i\psi} \\ \frac{b}{a}q^{\nu+(1/2)}, \frac{a}{b}q^{1/2-n} \end{matrix}; q, q\right]$$

$$= \frac{\left(\frac{b}{a}q^{(\nu+1)/2}e^{i\psi}, \frac{b}{a}q^{(\nu+1)/2}e^{-i\psi}; q\right)_n}{\left(\frac{b}{a}q^{\nu+(1/2)}, \frac{bq^{1/2}}{a}; q\right)_n}$$

by the q-Saalschütz formula [**19**, (IV. 4), p. 247], we find that, for Re $\nu > 0$, $0 < q < 1$,

$$(4.5)\quad \int_{-1}^{1} \frac{h(z; 1, -1, q^{1/2}, -q^{1/2}|q)}{h(z; q^{\nu/2}, -q^{\nu/2}, q^{(\nu+1)/2}, -q^{(\nu+1)/2}|q)}$$

$$\cdot \left(\frac{b}{a}q^{(\nu+1)/2}e^{i\psi}, \frac{b}{a}q^{(\nu+1)/2}e^{-i\psi}; q\right)_n \frac{dz}{\sqrt{1-z^2}}$$

$$= \sum_{k=0}^{n} \frac{(q^{-n}; q)_k \left(\frac{b}{a}q^{\nu+(1/2)}, \frac{b}{a}q^{1/2}; q\right)_n}{\left(q, \frac{b}{a}q^{\nu+(1/2)}, \frac{a}{b}q^{(1/2)-n}; q\right)_k} q^k$$

$$\cdot \int_{-1}^{1} \frac{h(z:1, -1, q^{1/2}, -q^{1/2}|q)}{h(z; q^{(\nu/2)+k}, -q^{\nu/2}, q^{(\nu+1)/2}, -q^{(\nu+1)/2}|q)} \frac{dz}{\sqrt{1-z^2}}$$

$$= L_\nu \left(\frac{b}{a}q^{\nu+(1/2)}, \frac{b}{a}q^{1/2}; q\right)_n$$

$$\cdot {}_4\phi_3\left[\begin{matrix} q^{-n}, -q^{\nu}, \quad q^{\nu+(1/2)}, -q^{\nu+(1/2)} \\ \frac{b}{a}q^{\nu+(1/2)}, q^{2\nu+1}, \quad \frac{a}{b}q^{(1/2)-n} \end{matrix}; q, q\right]$$

$$= L_\nu \frac{\left(\frac{b}{a}q^{\nu+(1/2)}, -\frac{b}{a}q^{\nu+(1/2)}, -q^{\nu+1}; q\right)_n}{(q^{2\nu+1}; q)_n}$$

$$\cdot\, {}_4\Phi_3\left[\begin{matrix} q^{-n}, & -q^{\nu}, & \frac{b}{a}, & -\frac{b}{a} \\ & \frac{b}{a}q^{\nu+(1/2)}, & -\frac{b}{a}q^{\nu+(1/2)}, & -\frac{a}{b}q^{-\nu-n} \end{matrix}; q, q\right]$$

by (4.1) and (3.5), where

$$(4.6)\qquad L_\nu = \frac{2\pi(q^{2\nu+1}; q)_\infty}{(q, -q^\nu, q^{\nu+(1/2)}, -q^{\nu+(1/2)}, q^{\nu+(1/2)}, -q^{\nu+(1/2)}, -q^{\nu+1}; q)_\infty}$$

$$= \frac{2\pi(q^\nu, q^{\nu+1}; q)_\infty}{(q, q^{2\nu}; q)_\infty} = \frac{2\pi\Gamma_q(2\nu)}{\Gamma_q(\nu)\Gamma_q(\nu+1)}, \text{by (1.5)}.$$

The integral representation (1.19) then follows by observing that

$$(4.7)\qquad \frac{h(\cos\psi; 1, -1, q^{1/2}, -q^{1/2}|q)}{h(\cos\psi; q^{\nu/2}, -q^{\nu/2}, q^{(\nu+1)/2}, -q^{(\nu+1)/2}|q)} = \frac{(e^{2i\psi}, e^{-2i\psi}; q)_\infty}{(q^\nu e^{2i\psi}, q^\nu e^{-2i\psi}; q)_\infty}.$$

5. Proof of the addition formula (1.10). Askey and Ismail [**2**] proved the following orthogonality relation for the q-ultraspherical polynomials:

$$(5.1)\qquad \int_0^\pi \frac{(e^{2i\psi}, e^{-2i\psi}; q)_\infty}{(q^\nu e^{2i\psi}, q^\nu e^{-2i\psi}; q)_\infty} C_m(\cos\psi; q^\nu|q) C_n(\cos\psi; q^\nu|q) d\psi$$

$$= \frac{2\pi\Gamma_q(2\nu)}{\Gamma_q(\nu)\Gamma_q(\nu+1)} \frac{1-q^\nu}{1-q^{\nu+n}} \frac{(q^{2\nu}; q)_n}{(q; q)_n} \delta_{m,n},$$

where $\operatorname{Re}\nu > 0$. This formula is also valid when $\nu = 0$, but one needs to use the limit

$$(5.2)\qquad \lim_{\nu\to 0^+} \frac{1-q^n}{2(1-q^\nu)} C_n(z; q^\nu|q) = T_n(z)$$

where $T_n(z) = \cos n\psi$, $z = \cos\psi$, is the Tchebichef polynomial of the first kind. In view of (1.19) and (5.1) we seek an expansion of the form

$$(5.3)\qquad {}_2\Phi_1\left[\begin{matrix} \frac{b}{a}q^{(\nu+1)/2}e^{i\psi}, \frac{b}{a}q^{(\nu+1)/2}e^{-i\psi} \\ q^{\nu+1} \end{matrix}; q, -a^2(1-q^{1/2})^2x^2\right]$$

$$= \sum_{n=0}^{\infty} A_n(x) C_n(\cos\psi; q^\nu|q).$$

Use of (5.1) then gives the Fourier coefficients $A_n(x)$ as an integral:

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$$(5.4)\quad A_n(x) = L_\nu^{-1}\frac{1-q^{\nu+n}}{1-q^\nu}\frac{(q;q)_n}{(q^{2\nu};q)_n}$$
$$\cdot \int_0^\pi \frac{(e^{2i\psi}, e^{-2i\psi}; q)_\infty}{(q^\nu e^{2i\psi}, q^\nu e^{-2i\psi}; q)_\infty} C_n(\cos\psi; q^\nu|q)$$
$$\cdot {}_2\phi_1\left[\begin{matrix}\frac{b}{a}q^{(\nu+1)/2}e^{i\psi}, \frac{b}{a}q^{(\nu+1)/2}e^{-i\psi} \\ q^{\nu+1}\end{matrix}; q, -a^2(1-q^{1/2})^2x^2\right]d\psi.$$

For fixed non-negative integers k and n, let us first compute the integral

$$(5.5)\quad \int_0^\pi \frac{(e^{2i\psi}, e^{-2i\psi}; q)_\infty}{(q^\nu e^{2i\psi}, q^\nu e^{-2i\psi}; q)_\infty}$$
$$\cdot {}_4\phi_3\left[\begin{matrix}q^{-n}, q^{2\nu+n}, & q^{\nu/2}e^{i\psi}, & q^{\nu/2}e^{-i\psi} \\ q^{\nu+(1/2)}, & -q^{\nu+(1/2)}, & -q^\nu\end{matrix}; q, q\right]$$
$$\cdot (cq^{(\nu+1)/2}e^{i\psi}, ce^{(\nu+1)/2}e^{-i\psi}; q)_k d\psi = Q_{n,k}, \text{ say.}$$

Since, by [**19**, (IV. 4), p. 247]

$$(5.6)\quad (cq^{(\nu+1)/2}e^{i\psi}, cq^{(\nu+1)/2}e^{-i\psi}; q)_k$$
$$= (cq^{\nu+1}, c; q)_k\, {}_3\phi_2\left[\begin{matrix}q^{-k}, q^{(\nu+1)/2}e^{i\psi}, q^{(\nu+1)/2}e^{-i\psi} \\ cq^{\nu+1}, \quad c^{-1}q^{1-k}\end{matrix}; q, q\right],$$

we find that

$$(5.7)\quad Q_{n,k} = \sum_{m=0}^n \frac{(q^{-n}, q^{2\nu+n}; q)_m q^m}{(q, q^{\nu+(1/2)}, -q^{\nu+(1/2)}, -q^\nu; q)_m}$$
$$\cdot \sum_{j=0}^k \frac{(q^{-k}; q)_j (cq^{\nu+1}, c; q)_k q^j}{(q, cq^{\nu+1}, c^{-1}q^{1-k}; q)_j}$$
$$\cdot \int_0^\pi \frac{h(\cos\psi; 1, -1, q^{1/2}, -q^{1/2}|q)}{h(\cos\psi; q^{(\nu/2)+m}, -q^{\nu/2}, q^{(\nu+1)/2+j}, -q^{(\nu+1)/2}|q)} d\psi$$
$$= L_\nu(cq^{\nu+1}, c; q)_k \sum_{j=0}^k \frac{(q^{-k}, q^{\nu+(1/2)}, -q^{\nu+(1/2)}, -q^{\nu+1}; q)_j}{(q, cq^{\nu+1}, q^{2\nu+1}, c^{-1}q^{1-k}; q)_j} q^j$$
$$\cdot {}_3\phi_2\left[\begin{matrix}q^{-n}, q^{2\nu+n}, & q^{\nu+(1/2)+j} \\ q^{\nu+(1/2)}, q^{2\nu+1+j}\end{matrix}; q, q\right],$$

by (4.1) and (4.6). However, by [**19**, (IV. 4), p. 247],

$$(5.8)\quad {}_3\phi_2\left[\begin{matrix} q^{-n}, q^{2\nu+n}, & q^{\nu+(1/2)+j} \\ q^{\nu+(1/2)}, q^{2\nu+1+j} \end{matrix}; q, q\right] = \frac{(q^{-j}; q)_n}{(q^{2\nu+1+j}; q)_n} q^{(\nu+(1/2)+j)n}$$

which vanishes unless $j \geqq n$. This implies that $Q_{n,k} = 0$ unless $k \geqq n$. Thus we obtain

$$(5.9)\quad Q_{n,k} = L_\nu \frac{(cq^{\nu+1}, c; q)_k (q^{-k}; q)_n}{(q^{\nu+1}, cq^{\nu+1}, c^{-1}q^{1-k}; q)_n} q^{\binom{n+1}{2}+(\nu+(1/2))n}(-1)^n \cdot {}_4\phi_3\left[\begin{matrix} q^{n-k}, q^{\nu+n+(1/2)}, -q^{\nu+n+(1/2)}, -q^{\nu+n+1} \\ q^{2\nu+2n+1}, cq^{\nu+n+1}, c^{-1}q^{1-k+n} \end{matrix}; q, q\right].$$

Using this in (5.4) and simplifying, we find that

$$(5.10)\quad A_n(x) = \frac{1-q^{\nu+n}}{1-q^\nu} \frac{q^{\binom{n}{2}+(\nu+1)/2}}{(q^{\nu+1}; q)_n^2} (ab(1-q^{1/2})^2x^2)^n \cdot \sum_{k=0}^{\infty} \frac{\left(\frac{b}{a}, \frac{b}{a}q^{\nu+n+1}; q\right)_k (-a^2(1-q^{1/2})^2x^2)^k}{(q, q^{\nu+n+1}; q)_k} \cdot {}_4\phi_3\left[\begin{matrix} q^{-k}, q^{\nu+n+(1/2)}, -q^{\nu+n+(1/2)}, -q^{\nu+n+1} \\ q^{2\nu+2n+1}, \frac{b}{a}q^{\nu+n+1}, \frac{a}{b}q^{1-k} \end{matrix}; q, q\right].$$

Applying the transformation formula (3.5) twice it can be shown that

$$(5.11)\quad {}_4\phi_3\left[\begin{matrix} q^{-k}, q^{\nu+n+(1/2)}, -q^{\nu+n+(1/2)}, -q^{\nu+n+1} \\ \frac{b}{a}q^{\nu+n+1}, q^{2\nu+2n+1}, \frac{a}{b}q^{1-k} \end{matrix}; q, q\right] = \frac{\left(\frac{b}{a}q^{\nu+n+(1/2)}, -\frac{b}{a}q^{\nu+n+(1/2)}, -q^{\nu+n+1}; q\right)_k}{\left(q^{2\nu+2n+1}, \frac{b}{a}, \frac{b}{a}q^{\nu+n+1}; q\right)_k} \cdot {}_4\phi_3\left[\begin{matrix} q^{-k}, -q^{\nu+n}, \frac{b}{a}, -\frac{b}{a} \\ \frac{b}{a}q^{\nu+n+(1/2)}, -\frac{b}{a}q^{\nu+n+(1/2)}, -q^{-\nu-n-k} \end{matrix}; q, q\right]$$

and hence

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$$(5.12)\quad A_n(x) = \frac{1-q^{\nu+n}}{1-q^\nu}\frac{q^{\binom{n}{2}+((\nu+1)/2)n}}{(q^{\nu+1};q)_n^2}\left(ab(1-q^{1/2})^2x^2\right)^n$$
$$\cdot\sum_{k=0}^{\infty}\frac{\left(\frac{b}{a}q^{\nu+n+(1/2)}, -\frac{b}{a}q^{\nu+n+(1/2)}, -q^{\nu+n+1}; q\right)_k}{(q, q^{\nu+n+1}, q^{2\nu+2n+1}; q)_k}$$
$$\cdot(-a^2(1-q^{1/2})^2x^2)^k$$
$$\cdot {}_4\phi_3\left[\begin{matrix} q^{-k}, -q^{\nu+n}, \frac{b}{a}, & -\frac{b}{a} \\ \frac{b}{a}q^{\nu+n+(1/2)}, -\frac{b}{a}q^{\nu+n+(1/2)}, -q^{-\nu-n-k} & \end{matrix}; q, q\right].$$

But the series over k in (5.12) is exactly the same as the series in (3.6) with ν replaced by $\nu + n$. It follows that

$$(5.13)\quad A_n(x)$$
$$= \frac{\Gamma_q^2(\nu+1)}{(-(a(1-q^{1/2})x)^2;q)_\infty}\left[\frac{(1-q^{1/2})^2}{abx^2}\right]^\nu \frac{1-q^{\nu+n}}{1-q^\nu}q^{\binom{n}{2}+n(\nu+1)/2}$$
$$\cdot J_{\nu+n}(a(1-q)x|q)J_{\nu+n}(b(1-q)x|q).$$

This completes the proof of (1.10).

6. An application of the product formula (1.19). We shall now evaluate the integral

$$(6.1)\qquad K_{\lambda,\nu}(a,b) = \int_0^\infty x^{\lambda-1}\frac{J_\nu(a(1-q)x|q)J_\nu(b(1-q)x|q)}{(-a^2(1-q^{1/2})^2x^2;q)_\infty}dx$$

by using the product formula (1.14) and Ramanujan's integral

$$(6.2)\qquad \int_0^\infty x^{c-1}\frac{(-ax;q)_\infty}{(-x;q)_\infty}dx = \frac{\pi}{\sin\pi c}\frac{(q^{1-c}, a; q)_\infty}{(q, aq^{-c}; q)_\infty},$$

see [**16**], [**1**] and [**3**], where $\mathrm{Re}\, c > 0$ and $|aq^{-c}| < 1$.

The ${}_2\phi_1$ series inside the integral of (1.19) is convergent only for

$$|ax| < (1-q^{1/2})^{-1},$$

however, Jackson's [**8**] transformation formula

$$(6.3)\qquad {}_2\phi_1\left[\begin{matrix} a, b \\ c \end{matrix}; q, z\right] = \frac{(az;q)_\infty}{(z;q)_\infty}{}_2\phi_2\left[\begin{matrix} a, c/b \\ c, cz \end{matrix}; q, bz\right]$$

enables us to express it in terms of a ${}_2\phi_1$ series which is convergent for all values of its argument. We then need only to compute the integral

(6.4) $$\int_0^\infty x^{\lambda+2\nu+2n-1}\frac{(-abq^{(\nu+1)/2+n}(1-q^{1/2})^2e^{i\psi}x^2;q)_\infty}{(-a^2(1-q^{1/2})^2x^2;q)_\infty}dx$$

$$= \frac{1}{2}(a(1-q^{1/2}))^{-(\lambda+2\nu+2n)}$$

$$\cdot \int_0^\infty t^{(\lambda/2)+\nu+n-1}\frac{\left(-\frac{b}{a}q^{(\nu+1)/2+n}e^{i\psi}t;q\right)}{(-t;q)_\infty}dt$$

$$\equiv M_n, \text{ say,}$$

for $n = 0, 1, 2, \ldots, 0 < b \leqq a$ and $\mathrm{Re}\,\nu > -1$, $\mathrm{Re}((\lambda/2) + \nu) > 0$. Assuming for the time being that

$$|ba^{-1}q^{(1-\lambda-\nu)/2}| < 1,$$

we find by using (6.2) that

(6.5) $$M_n = \frac{\Gamma\left(\nu+\frac{1}{2}\right)\Gamma\left(1-\nu-\frac{\lambda}{2}\right)}{2\Gamma_q\left(1-\nu-\frac{\lambda}{2}\right)}$$

$$\cdot \left[\frac{a^2(1-q^{1/2})}{1+q^{1/2}}\right]^{-\nu-(\lambda/2)}\frac{\left(\frac{b}{a}q^{(\nu+1)/2}e^{i\psi};q\right)_\infty}{\left(\frac{b}{a}q^{(1-\lambda-\nu)/2}e^{i\psi};q\right)_\infty}$$

$$\cdot \frac{(q^{\nu+(\lambda/2)};q)_n}{\left(\frac{b}{a}q^{(\nu+1)/2}e^{i\psi};q\right)_n}(a(1-q^{1/2}))^{-2n}q^{-\binom{n}{2}-(\nu+(\lambda/2))n}.$$

Thus we obtain

(6.6) $$K_{\lambda,\nu}(a,b) = \frac{\Gamma_q(\nu)\Gamma\left(\nu+\frac{\lambda}{2}\right)\Gamma\left(1-\nu-\frac{\lambda}{2}\right)}{4\pi\Gamma_q(\nu+1)\Gamma_q(2\nu)\Gamma_q\left(1-\nu-\frac{\lambda}{2}\right)}$$

$$\cdot \left[\frac{1+q^{1/2}}{a^2(1-q^{1/2})}\right]^{\nu+(\lambda/2)}\left[\frac{ab}{(1+q^{1/2})^2}\right]^\nu$$

$$\cdot \int_0^\pi \frac{(e^{2i\psi}, e^{-2i\psi};q)_\infty\left(\frac{b}{a}q^{(\nu+1)/2}e^{i\psi};q\right)_\infty}{(q^\nu e^{2i\psi}, q^\nu e^{-2i\psi};q)_\infty\left(\frac{b}{a}q^{(1-\lambda-\nu)/2}e^{i\psi};q\right)_\infty}$$

$$\cdot {}_2\phi_1\left[\begin{matrix} q^{\nu+(\lambda/2)}, \frac{a}{b}q^{(\nu+1)/2}e^{i\psi} \\ q^{\nu+1} \end{matrix}; q, \frac{b}{a}q^{(1-\lambda-\nu)/2}e^{-i\psi}\right]d\psi.$$

By the q-Gauss' formula [**19**, (IV. 2), p. 247], the ${}_2\phi_1$ series in (6.6) can be summed. This leads to the formula

(6.7) $K_{\lambda,\nu}(a, b)$

$$= \frac{\Gamma_q(\nu)\Gamma\left(\nu + \frac{\lambda}{2}\right)\Gamma\left(1 - \nu - \frac{\lambda}{2}\right)}{4\pi\Gamma_q(2\nu)\Gamma_q\left(1 - \nu - \frac{\lambda}{2}\right)\Gamma_q\left(1 - \frac{\lambda}{2}\right)}(1 + q^{1/2})^{\lambda}a^{-\lambda-\nu}b^{\nu}$$

$$\cdot \int_0^{\pi} \frac{(e^{2i\psi}, e^{-2i\psi}; q)_\infty\left(\frac{b}{a}q^{(\nu+1)/2}e^{i\psi}, \frac{b}{a}q^{(\nu+1)/2}e^{-i\psi}; q\right)_\infty}{(q^{\nu}e^{2i\psi}, q^{\nu}e^{-2i\psi}; q)_\infty\left(\frac{b}{a}q^{(1-\lambda-\nu)/2}e^{i\psi}, \frac{b}{a}q^{(1-\lambda-\nu)/2}e^{-i\psi}; q\right)_\infty}d\psi.$$

For

$$\mathrm{Re}\ \nu > 0 \text{ and } \left|\frac{b}{a}q^{(1-\lambda-\nu)/2}\right| < 1$$

the integral on the right side of (6.7) can be expressed as a multiple of a very-well-poised ${}_8\phi_7$ series by [**13**, (3.6)]. Thus we find that

(6.8) $K_{\lambda,\nu}(a, b)$

$$= \frac{\Gamma\left(\nu + \frac{\lambda}{2}\right)\Gamma\left(1 - \nu - \frac{\lambda}{2}\right)}{2\Gamma_q(\nu + 1)\Gamma_q\left(1 - \nu - \frac{\lambda}{2}\right)\Gamma_q\left(1 - \frac{\lambda}{2}\right)}$$

$$\cdot \frac{\left(\frac{b^2}{a^2}q^{2\nu_1}, \frac{b^2}{a^2}q^{1-(\lambda/2)}, q^{\nu+1-(\lambda/2)}; q\right)_\infty}{\left(q^{2\nu+1}, \frac{b^2}{a^2}q^{\nu+1}, \frac{b^2}{a^2}q^{1-\lambda}; q\right)_\infty}\left(\frac{1 + q^{1/2}}{a}\right)^{\lambda}\left(\frac{b}{a}\right)^{\nu}$$

$$\cdot {}_8\phi_7\left[\begin{matrix} \frac{b^2}{a^2}q^{\nu}, \frac{b}{a}q^{(\nu/2)+1}, & -\frac{b}{a}q^{(\nu/2)+1}, \frac{b}{a}, & -\frac{b}{a}, \\ \frac{b}{a}q^{\nu/2}, & -\frac{b}{a}q^{\nu/2}, & \frac{bq^{\nu+1}}{a}, -\frac{bq^{\nu+1}}{a}, \end{matrix}\right.$$

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$$\left[\begin{matrix} \frac{bq^{1/2}}{a}, & -\frac{bq^{1/2}}{a}, & q^{\nu+(\lambda/2)} \\ \frac{bq^{\nu+(1/2)}}{a}, & -\frac{bq^{\nu+(1/2)}}{a}, & \frac{b^2}{a^2}q^{1-(\lambda/2)} \end{matrix} ; q, q^{\nu+1-(\lambda/2)}\right],$$

where it is assumed that $\mathrm{Re}(\nu + 1 - (\lambda/2)) > 0$ so that the ${}_8\phi_7$ series converges. By using a known quadratic transformation formula, see for example, [**8**], the ${}_8\phi_7$ series can be transformed to a well-poised ${}_2\phi_1$ series:

$$(6.9)\qquad {}_8\phi_7[\ ] = \frac{\left(q^{2\nu+1}, \frac{b^2}{a^2}q^{\nu+1}; q\right)_\infty}{\left(q^{\nu+1}, \frac{b^2}{a^2}q^{2\nu+1}; q\right)_\infty} {}_2\phi_1\left[\begin{matrix} \frac{b^2}{a^2}, q^{\lambda/2} \\ \frac{b^2}{a^2}q^{1-(\lambda/2)} \end{matrix} ; q, q^{\nu+1-(\lambda/2)}\right].$$

Using (6.9) in (6.8) we finally obtain the formula

$$(6.10)\qquad \int_0^\infty x^{\lambda-1}\frac{J_\nu(a(1-q)x|q)J_\nu(b(1-q)x|q)}{(-a^2(1-q^{1/2})^2x^2;q)_\infty}dx$$

$$= \frac{\Gamma\left(\nu+\frac{\lambda}{2}\right)\Gamma\left(1-\nu-\frac{\lambda}{2}\right)\left(\frac{b^2}{a^2}q^{1-(\lambda/2)};q\right)_\infty}{2\Gamma_q\left(1-\nu-\frac{\lambda}{2}\right)\Gamma_q\left(1-\frac{\lambda}{2}\right)\Gamma_q\left(\nu+1-\frac{\lambda}{2}\right)\left(\frac{b^2}{a^2}q^{1-\lambda};q\right)_\infty}$$

$$\cdot (a^2(1-q^{1/2}))^{-(\lambda/2)}(1+q^{1/2})^{\lambda/2}\left(\frac{b}{a}\right)^\lambda$$

$$\cdot {}_2\phi_1\left[\begin{matrix} \frac{b^2}{a^2}, q^{\lambda/2} \\ \frac{b^2}{a^2}q^{1-(\lambda/2)} \end{matrix} ; q, q^{\nu+1-(\lambda/2)}\right],$$

where $\mathrm{Re}(\nu + (\lambda/2)) > 0$, $\mathrm{Re}(\nu + 1 - (\lambda/2)) > 0$ and $0 < b \leqq a$, which is the same as (1.24). In particular, for $\lambda = 0$ and $\mathrm{Re}\,\nu > 0$ we get for $0 < b \leqq a$

$$(6.11)\qquad \int_0^\infty x^{-1}\frac{J_\nu(a(1-q)x|q)J_\nu(b(1-q)x|q)}{(-a^2(1-q^{1/2})^2x^2;q)_\infty}dx$$

$$= \frac{\Gamma(\nu)\Gamma(1-\nu)}{2\Gamma_q(\nu+1)\Gamma_q(1-\nu)}\left(\frac{b}{a}\right)^\nu$$

and, for $\mathrm{Re}(2\nu + \lambda) > 0$, $\mathrm{Re}\,\lambda < 1$, we get

(6.12) $$\int_0^\infty x^{\lambda-1}\frac{J_\nu^2((1-q)x|q)}{(-(1-q^{1/2})^2x^2;q_\infty)}dx$$

$$= \frac{\Gamma\left(\nu+\frac{\lambda}{2}\right)\Gamma\left(1-\nu-\frac{\lambda}{2}\right)\Gamma_q(1-\lambda)}{2\Gamma_q^2\left(1-\frac{\lambda}{2}\right)\Gamma_q\left(\nu+1-\frac{\lambda}{2}\right)\Gamma_q\left(1-\nu-\frac{\lambda}{2}\right)}(1+q^{1/2})^\lambda.$$

7. Proof of (1.22). We shall first show that, for Re $\nu > -1$

(7.1) $$x^{2\nu} = \frac{(1+q^{1/2})^{2\nu}\Gamma_q^2(\nu+1)}{(-(1-q^{1/2})^2x^2;q)_\infty}$$

$$\cdot \sum_{m=0}^\infty \frac{(q^{2\nu};q)_m(1-q^{2\nu+2n})}{(q;q)_m(1-q^{2\nu})}q^{\binom{m}{2}}J_{m+\nu}^2((1-q)x|q).$$

Assuming for the time being that $|x| < (1-q^{1/2})^{-1}$ we may use (1.20) to show that the infinite series on the right side of (7.1) equals

(7.2) $$\frac{\left(\frac{x}{1+q^{1/2}}\right)^{2\nu}}{\Gamma^2(\nu+1)}$$

$$\cdot \sum_{n=0}^\infty \frac{(q^{2\nu+1};q)_{2n}}{(q,q^{2\nu+1};q)_n}(-(1-q^{1/2})^2x^2)^n(-(1-q^{1/2})^2x^2;q)_\infty$$

$$\cdot {}_4\phi_3\left[\begin{matrix}q^{2\nu}, q^{\nu+1}, -q^{\nu+1}, q^{-n} \\ q^\nu, \quad -q^\nu, \quad q^{2\nu+1+n}\end{matrix}; q, q^n\right]$$

$$= \frac{x^{2\nu}(-(1-q^{1/2})^2x^2;q)_\infty}{(1+q^{1/2})^{2\nu}\Gamma_q^2(\nu+1)},$$

since the very-well-poised ${}_4\phi_3$ series is $\delta_{n,0}$, by a special case of ${}_6\phi_5$ summation formula [**19**, (IV. 9), p. 247]. By analytic continuation, the restriction on x can clearly be removed.

Use of (1.20) once again enables us to show that

(7.3) $$J_\nu^2((1-q)x|q) - J_{\nu+1}((1-q)x|q)J_{\nu-1}((1-q)x|q)$$

$$= \frac{(-(1-q^{1/2})^2x^2;q)_\infty}{(q;q)_\infty^2}(x(1-q^{1/2}))^{2\nu}$$

$$\cdot \sum_{n=0}^\infty \frac{(q^{2\nu+1};q)_{2n}}{(q,q^{2\nu+1};q)_n}(-(1-q^{1/2})^2x^2)^n$$

$$\cdot \{ (q^{\nu+1+n}; q)_\infty^2 - (q^{\nu+n}, q^{\nu+2+n}; q)_\infty \}$$

$$= \frac{\left(\frac{xq^{1/2}}{1 + q^{1/2}}\right)^{2\nu}}{\Gamma_q(\nu + 1)\Gamma_q(\nu + 2)} (-(1 - q^{1/2})^2 x^2; q)_\infty$$

$$\cdot {}_3\phi_2\left[\begin{matrix} q^{\nu+(1/2)}, -q^{\nu+(1/2)}, -q^{\nu+1} \\ q^{\nu+2}, \quad q^{2\nu+1} \end{matrix}; q, -q(1 - q^{1/2})^2 x^2\right].$$

Use of (7.1) now gives

(7.4)
$$J_\nu^2((1 - q)x|q) - J_{\nu+1}((1 - q)x|q)J_{\nu-1}((1 - q)x|q)$$

$$= \frac{q^\nu \Gamma_q(\nu + 1)}{\Gamma_q(\nu + 2)} \sum_{n=0}^\infty \frac{(q^{2\nu+1}; q)_{2n}(q^{\nu+1}; q)_n}{(q; q^{2\nu+1}, q^{\nu+2}; q)_n} (-q)^n$$

$$\cdot \sum_{m=0}^\infty \frac{(q^{2\nu+2n}; q)_m (1 - q^{2\nu+2n+2m})}{(q; q)_m (1 - q^{2\nu+2n})} q^{\binom{m}{2}} J_{m+n+\nu}^2((1 - q)x|q)$$

$$= \frac{q^\nu(1 - q)}{1 - q^{\nu+1}} \sum_{n=0}^\infty \frac{(q^{2\nu}; q)_n (1 - q^{2\nu+2n})}{(q, q)_n (1 - q^{2\nu})} q^{\binom{n}{2}} J_{\nu+n}^2((1 - q)x|q)$$

$$\cdot {}_3\phi_2\left[\begin{matrix} q^{-n}, q^{n+2\nu}, q^{\nu+1} \\ q^{2\nu+1}, q^{\nu+2} \end{matrix}; q, q^2\right].$$

The ${}_3\phi_2$ series is not balanced, but it can be summed. Use of a standard transformation formula, see [**8**], for such a ${}_3\phi_2$ series gives

(7.5)
$${}_3\phi_2\left[\begin{matrix} q^{-n}, q^{n+2\nu}, q^{\nu+1} \\ q^{2\nu+1}, q^{\nu+2} \end{matrix}; q, q^2\right]$$

$$= \frac{(q; q)_n}{(q^{\nu+2}; q)_n} {}_3\phi_2\left[\begin{matrix} q^{-n}, q^{1-n}, q^{\nu+1} \\ q^{2\nu+1}, q^{-n} \end{matrix}; q, q\right]$$

which equals 1 when $n = 0$; when $n = 1, 2, \ldots,$

(7.6)
$${}_3\phi_2\left[\begin{matrix} q^{-n}, q^{1-n}, q^{\nu+1} \\ q^{2\nu+1}, q^{-n} \end{matrix}; q, q\right]$$

$$= {}_2\phi_1\left[\begin{matrix} q^{1-n}, q^{\nu+1} \\ q^{2\nu+1} \end{matrix}; q, q\right] = \frac{(q^\nu; q)_{n-1}}{(q^{2\nu+1}; q)_{n-1}} q^{(\nu+1)(n-1)}$$

by [**19**, (IV. 1), p. 247]. This completes the proof of (1.22).

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