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Large deviation principles of 2D stochastic Navier–Stokes equations with Lévy noises

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Taking the consideration of two-dimensional stochastic Navier–Stokes equations with multiplicative Lévy noises, where the noises intensities are related to the viscosity, a large deviation principle is established by using the weak convergence method skillfully, when the viscosity converges to 0. Due to the appearance of the jumps, it is difficult to close the energy estimates and obtain the desired convergence. Hence, one cannot simply use the weak convergence approach. To overcome the difficulty, one introduces special norms for new arguments and more careful analysis.

Keywords: stochastic Navier–Stokes equations; Euler equations; viscosity coefficient; Lévy noises; large deviation principles

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1. Introduction

The two-dimensional Navier–Stokes equations in D can be written as:

$$\mathrm{d}u(t) - \nu\Delta u(t)\,\mathrm{d}t + (u(t)\cdot\nabla)u(t)\,\mathrm{d}t + \nabla p(t)\,\mathrm{d}t = h(t)\,\mathrm{d}t, \tag{1.1}$$

where $D \subset \mathbb{R}^2$ is an open bounded domain with smooth boundary ∂D, u, $\nu > 0$, and p are the fluid velocity, viscosity, and the pressure, respectively. h is a deterministic external force. We add the incompressible condition:

$$\nabla\cdot u(t,x) = 0, \quad t\in[0,T],\ x\in D, \tag{1.2}$$

and the boundary condition (see [**8**, **61**]):

$$u(t,x)\cdot\mathbf{n} = 0 \text{ and } \operatorname{curl} u(t,x) = 0, \quad x\in\partial D,\ t\in[0,T], \tag{1.3}$$

where $\mathbf{n}$ is the unit outward normal and the initial data:

$$u(0,x) = \varsigma(x), \quad \forall x\in D. \tag{1.4}$$

Now, we can rewrite (1.1) in the following form:

$$\mathrm{d}u(t) + \nu Au(t)\,\mathrm{d}t + B(u(t),u(t))\,\mathrm{d}t = h(t)\,\mathrm{d}t, \quad t\in[0,T] \tag{1.5}$$

with the initial data (1.4). See § 2.1 for the definition of operators A and B.

However, the real world is more complex, and models that allow jumping, regardless of size, are desirable. For example, when particles are interfered or adsorbed by other media, the phenomenon of escape, vibration or other extreme activities can be understood as a stochastic process of jumping. This jump can be described by Lévy noises. Therefore, it is necessary to study the mathematical problems of stochastic Navier–Stokes equations with Lévy noises such as the large deviation principle. Taking the random external forces into account, we consider stochastic Navier–Stokes equations with the multiplicative Lévy noises, that is, the following stochastic Navier–Stokes equations:

$$\begin{cases} \mathrm{d}u^\nu(t) = [-\nu A u^\nu(t) - B u^\nu(t)]\mathrm{d}t + \sqrt{\nu}\,\Upsilon_\nu(t, u^\nu(t))\,\mathrm{d}W(t) \\ \quad +\nu\int_{\mathbb{Y}} G(t, u^\nu(t-), y)\tilde{N}^{\nu^{-1}}(\mathrm{d}y, \mathrm{d}t), \\ u^\nu(0) = \varsigma(x), \end{cases} \tag{1.6}$$

with the conditions (1.2) and (1.3) for u^ν. Here Υ_ν and G are measurable mappings and will be specified later. $W(t)$ is a H-cylindrical Brownian motion. $N^{\nu^{-1}}$ is a Poisson random measure on $\mathbb{Y}_T = \mathbb{Y} \times [0, T]$ with intensity measure $\nu^{-1}\vartheta_T = \nu^{-1}\vartheta \otimes \lambda_T$, where $\mathbb{Y}$ is a locally compact Polish space, ϑ is a locally finite measure on $\mathbb{Y}$, λ_T is the Lebesgue measure on $[0, T]$. $\nu > 0$ is the scaling parameter. $\tilde{N}^{\nu^{-1}}$ is the compensated Poisson random measure, i.e., for $O \in \mathcal{B}(\mathbb{Y})$ with $\vartheta(O) < \infty$, $\tilde{N}^{\nu^{-1}}([0, t] \times O) = N^{\nu^{-1}}([0, t] \times O) - \nu^{-1}t\vartheta(O)$.

Our aim in this paper is to establish a large deviation principle (LDP) for the stochastic 2D Navier–Stokes equations (1.6) as $\nu \to 0$ in the Freidlin–Wentzell setting, that is, the exponential concentration of the distribution of the process $u^\nu(t, \cdot)$ for a fixed t when the viscosity coefficient converges to zero. Note that Kuskin [**36**] established asymptotic properties of the invariant measure to the stochastic Navier–Stokes equations with an additive degenerate noise when the viscosity is small.

Due to the pioneering work of Freidlin and Wentzell (cf. [**30**, **53**]), there is a lot of literature on the large deviation principles for small noise diffusion equations. Subsequently, many researchers relaxed the model assumptions and carried out many extensions. For stochastic evolution equations, we refer to [**23**, **40**, **47**, **49**]. Moreover, many authors considered the large deviation estimates for other processes, for example, [**1**, **25**, **51**] for stochastic 2D Navier–Stokes equations, [**28**] for stochastic 2D Bénard convection, [**17**] for stochastic reaction-diffusion systems, [**41**] for the stochastic shell model of turbulence. We also refer to [**20**] for LDP of stochastic evolution equations with non-Lipschitz coefficients and to [**48**] for LDP of the stochastic tamed 3D Navier–Stokes equations and [**50**] for LDP of a reaction-diffusion equation with non-Gaussian perturbations. Noted that the existence of solutions for the stochastic Navier–Stokes equations was studied by many authors, see, for example, [**19**, **29**, **42**, **54**] and the references therein. There are not many studies on large deviation principles for Lévy noise so far, for example, Budhiraja *et al.* [**15**] for stochastic differential equations, Xu and Zhang [**59**] and Zhai and Zhang [**60**] and Dong et al. [**27**] for the stochastic 2D Navier–Stokes equations.

Like the large deviations, many scholars studied moderate deviation principles (MDP). For example, De Acosta [**24**], Chen [**22**] and Ledoux [**39**] for processes with independent increments, Wu [**55**] for Markov processes, Guillin and Liptser [**31**] for stochastic diffusion processes, Wang and Zhang [**57**] for stochastic reaction-diffusion equations, Wang, Zhai and Zhang [**56**] for the stochastic 2D Navier–Stokes equations.

Note that all the papers mentioned above studied LDP and MDP for a fixed positive viscosity coefficient. But Bessaih and Millet [**7**] established a large deviation principle for the inviscid shell model of turbulence when the viscosity coefficient ν decays to 0, where the multiplicative noise intensity is multiplied by $\sqrt{\nu}$. Later, they [**8**] proved LDP for the stochastic 2D Navier–Stokes equations where the noise intensity is multiplied by $\sqrt{\nu}$ when the viscosity ν converges to 0 by using a weak convergence approach introduced by [**17**] for the case of the Poisson random measures (see § 2 for details) and by [**16**] for the case of Gaussian noises and by [**14**]. Currently, this approach becomes a powerful tool which has been applied by many people to prove large deviation principles for various dynamical systems driven by Gaussian noises, see, for example, [**7**, **14**, **21**, **28**, **40**, **41**, **47**, **48**, **51**]. Note that many researchers considered a Donsker–Varadhan type large deviation principle, see [**32**, **34**] and the references therein.

Inspired by the above work, especially [**8**, **13**, **17**, **60**], we will use the weak convergence approach to establish a large deviation principle of (1.6) in a two-dimensional bounded domain. Note that the rate function is described by the solution of the following deterministic controlled Euler equations:

$$\begin{aligned}\partial_t u(t) = &-(u(t)\cdot\nabla)u(t) - \nabla p(t,x) + \Upsilon_0(t,u(t))f(t)\\ &+ \int_{\mathbb{Y}} G(t,u(t),y)(g(t,y)-1)\vartheta(\mathrm{d}y),\end{aligned} \tag{1.7}$$

with (1.2) and (1.3). There is a bulk of literature considering the two-dimensional Euler equations ($G = 0$ in (1.7)), such as [**3**, **4**, **35**, **58**] and the references therein. For getting the uniqueness of the solution to (1.7), we should work in Sobolev space $H^{1,q}$, $q \geqslant 2$ (see § 2.1) as [**8**] and require that the coefficient (Υ, G) satisfies some additional conditions in the later such as Υ be both trace class and Radonifying (e.g., see [**45**]). For more details, see § 2. Among, much studies regarding the stochastic Euler equations, we refer to [**5**, **6**, **12**, **18**] and the references therein. Thanks to the appearance of the jumps when compared with [**8**], it was very difficult for us to obtain the required energy estimates and the convergence of terms related to jumps. This requires more tricks and more detailed estimates. Hence, simply using the weak convergence approach we cannot get the desired results. To overcome the difficulty, we apply a new argument and take a more careful analysis by introducing the norm $\|\tilde{G}(s,y)\|_{i,\cdot}^2, i = 0, 1$ (e.g., see § 2.5) to prove the well-posedness and establish a priori estimates of the solution to (1.7) in $\mathcal{C}([0,T];L^2)\cap L^\infty(0,T;H^{1,q})$ for $q > 2$ with more regular initial data. Therefore, we can establish a LDP for the stochastic 2D Navier–Stokes equations in $L^2(0,T;\mathcal{H})$, where $\mathcal{H}$ is a Hilbert interpolation space between H and V (see § 2.1), via the tightness and the Skorohod–Jakubowski theorem. In particular, our results can be seen as a generalization of [**8**]. Unlike in [**60**], the noise intensity of our study depends on the viscosity coefficient. The estimates of ∇u are missing when $\nu \to 0$. Hence, we cannot directly use the method in [**60**]. We need to use the introduced norm to make a careful analysis in a suitable space to overcome the obstacles.

An outline of this paper is organized as follows. In §2, we introduce function spaces, state some background on the Wiener process, Poisson random measure, and recall the general criteria for a large deviation principle and list some assumptions. In § 3, we establish the existence and uniqueness of solution to the stochastic controlled equation and give a priori estimates in the Hilbert spaces L^2 and $H^{1,2}$ with the free boundary conditions and a small viscosity ν. In § 4, we establish the existence and uniqueness of solution for the inviscid problem in some suitable Sobolev space. In §5, we obtain a priori estimates of the stochastic controlled equations in $H^{1,q}$, which will play an important role in the following. Section 6 is devoted to establishing a large deviation principle for the 2D stochastic Navier–Stokes equations (see theorem 6.2). In the last section, we list some classical Sobolev embedding and some useful results.

2. Preliminaries

In this section, we will recall some Sobolev spaces and basic knowledge in stochastic analysis including Wiener process, Poisson random measure, and a general criteria of large deviation [**17**], etc. We will follow some notations in [**8**, **13**, **17**] for ease of description.

2.1. Basic spaces

To formulate the Navier–Stokes equations in an abstract form (1.5), we introduce the standard spaces as follows. Define the Hilbert space H by

$$H = \left\{u(t,x) \in L^2(D;\mathbb{R}^2) : \nabla \cdot u(t,x) = 0 \text{ in } D,\ u(t,x)\cdot \mathbf{n} = 0 \text{ on } \partial D\right\}.$$

We denote the inner product and the corresponding norm in H by $(\cdot,\cdot)$ and $|\cdot|_H$, respectively. For every integer $k \geqslant 0$ and any $p \in [1,\infty)$, $W^{k,p}$ stands for the completion of $\mathcal{C}_0^\infty(\bar{D})$ with the norm

$$\|u\|_{W^{k,p}} = \left(\sum_{|\alpha|\leqslant k} \int_D |\partial^\alpha u(x)|^p dx\right)^{\frac{1}{p}}.$$

To simplify the notation, $\|\cdot\|_p := \|\cdot\|_{W^{0,p}}$. We denote $W^{-k,p^*} := (W^{k,p})^*$, where $p^* = p/(p-1)$ and for a multi-index $\alpha = (\alpha_1,\alpha_2)$, set $\partial^\alpha u(x) = \frac{\partial^{|\alpha|}u(x)}{\partial x_1^{\alpha_1}\partial x_2^{\alpha_2}}$. For every integer $k \geqslant 0$, let $W^{k+r,p}$ denote the completion of $\mathcal{C}_0^\infty(\bar{D})$ with the following norm:

$$\|u\|^p_{W^{k+r,p}} = \|u\|^p_{W^{k,p}} + \sum_{|\alpha|=k}\int_D\int_D \frac{|\partial^\alpha u(x) - \partial^\alpha u(y)|^p}{|x-y|^{2+2r}}\,\mathrm{d}x\,\mathrm{d}y, r \in (0,1), p \in [1,\infty).$$

Given $0 < \alpha < 1$, let $W^{\alpha,p}(0,T;H)$ denote the Sobolev space of all $u \in L^p(0,T;H)$ such that

$$\int_0^T \int_0^T \frac{|u(t)-u(s)|^p}{|t-s|^{1+\alpha p}}\,\mathrm{d}t\,\mathrm{d}s < \infty.$$

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We define $H^{k,q} = W^{k,q} \cap H$ for any $k \in [0, +\infty)$ and $q \in [2, \infty)$ and denote the norm by $\| \cdot \|_{H^{k,q}}$. Let $V = H^{1,2}$ (the subspace of H) be defined as

$$V = \left\{ u \in W^{1,2}(D; \mathbb{R}^2) : \nabla \cdot u = 0 \text{ in } D, \ u \cdot \mathbf{n} = 0 \text{ on } \partial D \right\}.$$

The corresponding norm and the inner product on V will be defined by

$$\|u\|^2 = ((u,u)) \text{ and } ((u,v)) = \int_D \left[u(x) \cdot v(x) + \nabla u(x) \cdot \nabla v(x) \right] \mathrm{d}x, \quad u, v \in V.$$

Denote the dual space of H by H', and the dual space of V by V'. Then $V \subset H \subset V'$ with continuous dense injections. Let us denote the dual pairing between $u \in V$ and $v \in V'$ by $\langle u, v \rangle$. Note that $(u, v) = \langle u, v \rangle$ when $v \in H$. Let $b(\cdot, \cdot, \cdot) : V \times V \times V \longrightarrow \mathbb{R}$ be the trilinear operator defined as

$$b(u, v, z) = \int_D (u(x) \cdot \nabla v(x)) \cdot z(x) \, \mathrm{d}x.$$

There exists a bilinear operator $B(\cdot, \cdot) : V \times V \to V'$ such that $\langle B(u, v), z \rangle = b(u, v, z)$ for all $z \in V$. By using (1.2), we obtain (see e.g. [**3**, **8**, **38**])

$$\langle B(u, v), z \rangle = -\langle B(u, z), v \rangle \text{ and } \langle B(u, v), v \rangle = 0, \quad u, v, z \in V.$$

We also have

$$\|B(u,u)\|_{V'} \leqslant C|u|_H \|u\|, u \in V.$$

Here C is a constant.

Assuming that $a(\cdot, \cdot) : V \times V \to \mathbb{R}$ is the bilinear continuous operator defined below (see [**3**, **8**])

$$a(u, v) = \int_D \nabla u \cdot \nabla v - \int_{\partial D} k(r) u(r) \cdot v(r) \, \mathrm{d}r,$$

where $k(r)$ is the curvature of the boundary ∂D at the point r, we infer that (see [**8**, **37**])

$$\int_{\partial D} k(r) u(r) \cdot v(r) \, \mathrm{d}r \leqslant C \|u\| \|v\|,$$

and

$$\int_{\partial D} k(r) |u(r)|^2 \mathrm{d}r \leqslant \varepsilon \|u\|^2 + C(\varepsilon) |u|_H^2 \text{ for any } \varepsilon > 0. \tag{2.1}$$

Let $D(A) = \{u \in H^{2,2} : \ \text{curl } u = 0 \text{ on } \partial D\}$. Define the operator $A : D(A) \to H$ by

$$Au = -\Delta u, \text{ i.e., } a(u, v) := (Au, v),$$

and we have (see [**8**])

$$(B(u,u), Au) = 0 \text{ for all } u \in D(A).$$

For $\beta > 0$, denote the β-power of the operator A and its domain by A^β and $D(A^\beta)$, respectively. We also denote the dual of $D(A^\beta)$ by $D(A^{-\beta})$. It follows from

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[12, theorem 3.1] that $H^{k,2} = D(A^{k/2})$ when $k < 3/4$. Set $\mathcal{H} = H^{1/2,2}$ and notice that $\mathcal{H} = D(A^{1/4})$ and $V = D(A^{1/2})$. Then, $V \subset \mathcal{H} \subset H$. Moreover, there exists a constant $C > 0$ such that

$$\|u\|_{\mathcal{H}}^2 \leqslant C|u|_H\|u\|, \text{ for all } u \in V.$$

Here we follow some notations in [3, 8, 38]. We deduce from $\mathcal{H} \subset L^4(D)$ and $\langle B(u,v), w\rangle = -\langle B(u,w), v\rangle$ that

$$|\langle B(u,v), w\rangle| \leqslant C\|u\|_{\mathcal{H}}\|v\|_{\mathcal{H}}\|w\|. \tag{2.2}$$

Hence, B can be extended as a bilinear operator from $\mathcal{H} \times \mathcal{H} \longrightarrow V'$.

2.2. Wiener process

Assume that Q is a linear positive operator in a Hilbert space H, which is trace class, and hence compact. Let $H_0 = Q^{\frac{1}{2}}H$. Then, H_0 is a Hilbert space with the the inner product

$$(\phi, \psi)_0 = (Q^{-\frac{1}{2}}\phi, Q^{-\frac{1}{2}}\psi), \quad \forall \phi,\ \psi \in H_0,$$

and the induced norm $|\cdot|_0 = \sqrt{(\cdot,\cdot)_0}$. Clearly, the embedding of H_0 in H is Hilbert–Schmidt and hence compact, since Q is a trace class operator. Let $L_Q := L_Q(H_0, H)$ be the space of linear operators $S : H_0 \mapsto H$ such that $SQ^{\frac{1}{2}}$ is a Hilbert–Schmidt operator from H to H. Denote the norm on $L_Q(H_0, H)$ by $|S|_{L_Q}^2 = \mathrm{tr}(SQS^*)$, where S^* is the adjoint operator of S. Then, for any orthonormal basis $\{\psi_k\}_{k\geqslant 1} \in H$, we have

$$|S|_{L_Q}^2 = \mathrm{tr}([SQ^{1/2}][SQ^{1/2}]^*) = \sum_{k\geqslant 1} |SQ^{1/2}\psi_k|_H^2 = \sum_{k\geqslant 1} |[SQ^{1/2}]^*\psi_k|_H^2,$$

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ be a filtered probability space. We suppose that $(W(t), t \geqslant 0)$ is a Wiener process taking values in H and with covariance operator Q. Let $\{e_k\}_{k\geqslant 1}$ be an orthonormal basis in H consisting of eigenelements of Q, with $Qe_k = \lambda_k e_k$. Then, we have

$$W(t) = \sum_{k\geqslant 1} \sqrt{\lambda_k}\beta_k(t)e_k,$$

where $\{\beta_k\}_{k\geqslant 1}$ is a sequence of independent standard one-dimensional Brownian motions. For more details, we refer to [23].

2.3. Poisson random measure

Let $\mathbb{Y}$ denote a locally compact Polish space and let $\mathcal{M}_{FC}(\mathbb{Y})$ denote the space of all measures ϑ on $(\mathbb{Y}, \mathcal{B}(\mathbb{Y}))$ such that $\vartheta(K) < \infty$ for every compact $K \subset \mathbb{Y}$. Endow $\mathcal{M}_{FC}(\mathbb{Y})$ with the usual vague topology. This topology can be metrized such that $\mathcal{M}_{FC}(\mathbb{Y})$ is a Polish space (see [17]). Fix $T \in (0, \infty)$ and set $\mathbb{Y}_T = \mathbb{Y} \times [0, T]$. Fix a measure $\vartheta \in \mathcal{M}_{FC}(\mathbb{Y})$, and let $\vartheta_T = \vartheta \otimes \lambda_T$, where λ_T is Lebesgue measure on $[0, T]$.

We know that a Poisson random measure π on $\mathbb{Y}_T$ with intensity measure ϑ_T is a $\mathcal{M}_{FC}(\mathbb{Y}_T)$-valued random variable such that for each $B \in \mathcal{B}(\mathbb{Y}_T)$ with $\vartheta_T(B) < \infty$, $\pi(B)$ is Poisson distributed with mean $\vartheta_T(B)$ and for disjoint $B_1, \ldots, B_k \in \mathcal{B}(\mathbb{Y}_T)$, $\pi(B_1), \ldots, \pi(B_k)$ are mutually independent random variables. Denote the measure induced by π on $(\mathcal{M}_{FC}(\mathbb{Y}_T), \mathcal{B}(\mathcal{M}_{FC}(\mathbb{Y}_T)))$ by $\mathbb{P}$. Then letting $\mathbb{M} = \mathcal{M}_{FC}(\mathbb{Y}_T)$, $\mathbb{P}$ is the unique probability measure on $(\mathbb{M}, \mathcal{B}(\mathbb{M}))$ under which the canonical map, $N : \mathbb{M} \to \mathbb{M}, N(m) \doteq m$, is a Poisson random measure with intensity measure ϑ_T. For $\theta > 0$, $\mathbb{P}_\theta$ will denote a probability measure on $(\mathbb{M}, \mathcal{B}(\mathbb{M}))$ under which N is a Poisson random measure with intensity $\theta\vartheta_T$. $\mathbb{E}$ and $\mathbb{E}_\theta$ denote the corresponding expectation operators, respectively.

Set $\mathbb{X} = \mathbb{Y} \times [0, \infty)$ and $\mathbb{X}_T = \mathbb{X} \times [0, T]$. Let $\bar{\mathbb{M}} = \mathcal{M}_{FC}(\mathbb{X}_T)$ and let $\bar{\mathbb{P}}$ be the unique probability measure on $(\bar{\mathbb{M}}, \mathcal{B}(\bar{\mathbb{M}}))$ under which the canonical map, $\bar{N} : \bar{\mathbb{M}} \to \bar{\mathbb{M}}, \bar{N}(m) \doteq m$, is a Poisson random measure with intensity measure $\bar{\vartheta}_T = \vartheta \otimes \lambda_\infty \otimes \lambda_T$, where λ_∞ is Lebesgue measure on $[0, \infty)$. The corresponding expectation operator will be denoted by $\bar{\mathbb{E}}$. Let $\mathcal{F}_t \doteq \sigma\{\bar{N}((0, s] \times O) : 0 \leqslant s \leqslant t, O \in \mathcal{B}(\mathbb{X})\}$ be the σ -algebra generated by $\bar{N}$, and let $\bar{\mathcal{F}}_t$ denote the completion under $\bar{\mathbb{P}}$. We denote the predictable σ-field on $[0, T] \times \bar{\mathbb{M}}$ with the filtration $\{\bar{\mathcal{F}}_t : 0 \leqslant t \leqslant T\}$ on $(\bar{\mathbb{M}}, \mathcal{B}(\bar{\mathbb{M}}))$ by $\bar{\mathcal{P}}$. Let $\bar{\mathcal{A}}_+$ [resp. $\bar{\mathcal{A}}$] be the class of all $(\mathcal{B}(\mathbb{Y}) \otimes \bar{\mathcal{P}})/\mathcal{B}[0, \infty)$ [resp. $(\mathcal{B}(\mathbb{Y}) \otimes \bar{\mathcal{P}})/\mathcal{B}(\mathbb{R})$]-measurable maps from $\mathbb{Y}_T \times \bar{\mathbb{M}}$ to $[0, \infty)$ [resp. $\mathbb{R}$]. For $\varphi \in \bar{\mathcal{A}}_+$, define a counting process N^φ on $\mathbb{Y}_T$ by (see [**17**])

$$N^\varphi((0, t] \times U) = \int_{U \times [0,\infty) \times [0,t]} 1_{[0,\varphi(x,s)]}(r) \bar{N}(\mathrm{d}x\, \mathrm{d}r\, \mathrm{d}s), \quad t \in [0, T], U \in \mathcal{B}(\mathbb{Y}).$$

Here N^φ can be regarded as a controlled random measure, where φ selects the intensity for the points at position x and time s in a possibly random but non-anticipating way. We denote $N^\varphi = N^\theta$ when $\varphi(x, s, \bar{m}) \equiv \theta \in (0, \infty)$. Noted that the distribution of N^θ with respect to $\bar{\mathbb{P}}$ is the same as that of N to $\mathbb{P}_\theta$.

Set $\mathbb{W} = C([0, T], \mathbb{R}^\infty)$, $\mathbb{V} = \mathbb{W} \times \mathbb{M}$ and $\bar{\mathbb{V}} = \mathbb{W} \times \bar{\mathbb{M}}$. Then let the mapping $N^{\mathbb{V}} : \mathbb{V} \to \mathbb{M}$ be defined by $N^{\mathbb{V}}(\omega, m) = m$ for $(\omega, m) \in \mathbb{V}$, and let $\beta^{\mathbb{V}} = (\beta_i^{\mathbb{V}})_{i=1}^\infty$ by $\beta_i^{\mathbb{V}}(\omega, m) = \omega_i$ for $(\omega, m) \in \mathbb{V}$. The maps $\bar{N}^{\bar{\mathbb{V}}} : \bar{\mathbb{V}} \to \bar{\mathbb{M}}$ and $\beta^{\bar{\mathbb{V}}} = (\beta_i^{\bar{\mathbb{V}}})_{i=1}^\infty$ are defined analogously. Define the σ-filtration $\mathcal{G}_t^{\mathbb{V}} := \sigma\{N^{\mathbb{V}}((0, s] \times O), \beta_i^{\mathbb{V}}(s) : 0 \leqslant s \leqslant t, O \in \mathcal{B}(\mathbb{Y}), i \geqslant 1\}$. For every $\theta > 0$, $\mathbb{P}_\theta^{\mathbb{V}}$ denotes the unique probability measure on $(\mathbb{V}, \mathcal{B}(\mathbb{V}))$ such that:

(a) $(\beta_i^{\mathbb{V}})_{i=1}^\infty$ is an independent and identically distributed family of standard Brownian motions,

(b) $N^{\mathbb{V}}$ is a Poisson random measure with intensity measure $\theta\vartheta_T$.

(c) $(\beta_i^{\mathbb{V}})_{i=1}^\infty$ and $N^{\mathbb{V}}$ are independent.

Analogously, we define $(\bar{\mathbb{P}}_\theta^{\bar{\mathbb{V}}}, \bar{\mathcal{G}}_t^{\bar{\mathbb{V}}})$ and denote $\bar{\mathbb{P}}_{\theta=1}^{\bar{\mathbb{V}}}$ by $\bar{\mathbb{P}}^{\bar{\mathbb{V}}}$. $\bar{\mathcal{F}}_t^{\bar{\mathbb{V}}}$ denotes the $\bar{\mathbb{P}}^{\bar{\mathbb{V}}}$-completion of $\bar{\mathcal{G}}_t^{\bar{\mathbb{V}}}$ and $\bar{\mathcal{P}}^{\bar{\mathbb{V}}}$ denotes the predictable σ-field on $[0, T] \times \bar{\mathbb{V}}$ with the filtration $\bar{\mathcal{F}}_t^{\bar{\mathbb{V}}}$ on $(\bar{\mathbb{V}}, \mathcal{B}(\bar{\mathbb{V}}))$. Let $\bar{\mathbb{A}}$ be the class of all $(\bar{\mathcal{P}}^{\bar{\mathbb{V}}} \otimes \mathcal{B}(\mathbb{Y}))/\mathcal{B}[0, \infty)$-measurable maps $\varphi : \mathbb{Y}_T \times \bar{\mathbb{V}} \to [0, \infty)$. Define $\ell : [0, \infty) \to [0, \infty)$ by

$$\ell(r) = r \log r - r + 1, \quad r \in [0, \infty).$$

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For any $\varphi \in \bar{\mathbb{A}}$ and $t \in [0, T]$, the quantity

$$L_t(\varphi) = \int_{\mathbb{Y}\times[0,t]} \ell(\varphi(x, s, \omega))\vartheta_T(\mathrm{d}x\,\mathrm{d}s)$$

is well defined as a $[0, \infty]$-valued random variable.

Define

$$\mathcal{L}_2 := \left\{\psi : \psi \text{ is } \bar{\mathcal{P}}^{\bar{\mathbb{V}}}/\mathcal{B}(\mathbb{R}^\infty) \text{ measurable and } \int_0^T |\psi(s)|_0^2 \mathrm{d}s < \infty,\ \bar{\mathbb{P}}^{\bar{\mathbb{V}}} - \text{a.s.}\right\}.$$

Set $\mathcal{U} = \mathcal{L}_2 \times \bar{\mathbb{A}}$. Define $\tilde{L}_T(\psi) := \frac{1}{2}\int_0^T |\psi(s)|_0^2 \mathrm{d}s$ for $\psi \in \mathcal{L}_2$ and $\bar{L}_T(\phi) := \tilde{L}_T(\psi) + L_T(\varphi)$ for $\phi = (\psi, \varphi) \in \mathcal{U}$.

We first recall some classical definitions. By convention the infimum over an empty set is $+\infty$. Let $\mathcal{E}$ be a Polish space with the Borel σ-field $\mathcal{B}(\mathcal{E})$.

Definition 2.1 (Rate function). *A function $I : \mathcal{E} \to [0, \infty]$ is called a rate function on $\mathcal{E}$, if for each $M < \infty$, the level set $\{\Psi \in \mathcal{E} : I(\Psi) \leqslant M\}$ is a compact subset of $\mathcal{E}$. For $O \in \mathcal{B}(\mathcal{E})$, we define $I(O) := \inf_{\Psi\in O} I(\Psi)$.*

Definition 2.2. *Let I be a rate function on $\mathcal{E}$. The random family $\{u^\nu\}_{\nu>0}$ is said to satisfy a large deviation principle on $\mathcal{E}$ with the good rate function I if the following conditions hold:*

(1) *Large deviation upper bound. For each closed subset F of $\mathcal{E}$:*

$$\limsup_{\nu\to 0} \nu \log \mathbb{P}(u^\nu \in F) \leqslant -I(F).$$

(2) *Large deviation lower bound. For each open subset G of $\mathcal{E}$:*

$$\liminf_{\nu\to 0} \nu \log \mathbb{P}(u^\nu \in G) \geqslant -I(G).$$

2.4. A general criteria

In this subsection, we recall a general criteria for a large deviation principle established in [**17**]. Let $\{\mathcal{G}^\nu\}_{\nu>0}$ be a family of measurable maps from $\bar{\mathbb{V}}$ to $\mathbb{U}$, where $\bar{\mathbb{V}}$ is introduced in § 2.3 and $\mathbb{U}$ is some Polish space. We present below a sufficient condition for a large deviation principle to hold for the family $Z^\nu = \mathcal{G}^\nu(\sqrt{\nu}W, \nu N^{\nu^{-1}})$ as $\nu \to 0$.

Define

$$S^M = \{g : \mathbb{Y}_T \to [0, \infty) : L_T(g) \leqslant M\},$$
$$\tilde{S}^M = \{f : L^2([0, T], H_0) : \tilde{L}_T(f) \leqslant M\}.$$

A function $g \in S^M$ can be identified with a measure $\vartheta_T^g \in \mathbb{M}$, defined by

$$\vartheta_T^g(O) = \int_O g(s, x)\vartheta_T(\mathrm{d}s, \mathrm{d}x), \quad O \in \mathcal{B}(\mathbb{Y}_T).$$

This identification induces a topology on S^M under which S^M is a compact space, see the appendix of [**13**]. Throughout we use this topology on S^M. Let

$\bar{S}^M = S^M \times \tilde{S}^M$. Define $\mathbb{S} = \cup_{M\geqslant 1}\bar{S}^M$ and let

$$\mathcal{U}^M = \{\phi = (\psi, \varphi) \in \mathcal{U} : \phi(\omega) \in \bar{S}^M, \bar{\mathbb{P}}^{\bar{\mathbb{V}}} \text{ a.e. } \omega\},$$

where $\mathcal{U}$ is introduced in subsection 2.3.

The following condition will be sufficient for establishing a LDP for a family $\{Z^\nu\}_{\nu>0}$ defined by $Z^\nu = \mathcal{G}^\nu(\sqrt{\nu}W, \nu N^{\nu^{-1}})$.

Condition 2.3. *There exists a measurable map $\mathcal{G}^0 : \bar{\mathbb{V}} \to \mathbb{U}$ such that the following hold.*

(1) *For $\forall M \in \mathbb{N}$, let (f_n, g_n), $(f, g) \in \bar{S}^M$ be such that $(f_n, g_n) \to (f, g)$ as $n \to \infty$. Then*

$$\mathcal{G}^0\left(\int_0^\cdot f_n(s)\,\mathrm{d}s, \vartheta_T^{g_n}\right) \to \mathcal{G}^0\left(\int_0^\cdot f(s)\,\mathrm{d}s, \vartheta_T^{g}\right) \text{ in } \mathbb{U}.$$

(2) *For $\forall M \in \mathbb{N}$, let $\phi_\nu = (\psi_\nu, \varphi_\nu)$, $\phi = (\psi, \varphi) \in \mathcal{U}^M$ be such that ϕ_ν converges in distribution to ϕ as $\nu \to 0$. Then*

$$\mathcal{G}^\nu\left(\sqrt{\nu}W + \int_0^\cdot \psi_\nu(s)\,\mathrm{d}s, \nu N^{\nu^{-1}\varphi_\nu}\right) \to \mathcal{G}^0\left(\int_0^\cdot \psi(s)\,\mathrm{d}s, \vartheta_T^{\varphi}\right).$$

For $\phi \in \mathbb{U}$, define $\mathbb{S}_\phi = \{(f, g) \in \mathbb{S} : \phi = \mathcal{G}^0(\int_0^\cdot f(s)\mathrm{d}s, \vartheta_T^g)\}$. Let $I : \mathbb{U} \to [0, \infty)$ be defined by

$$I(\phi) = \inf_{(f,g)\in\mathbb{S}_\phi} \{\bar{L}_T((f, g))\}, \quad \phi \in \mathbb{U}. \tag{2.3}$$

By convention, $I(\phi) = \infty$ if $\mathbb{S}_\phi = \emptyset$. Let $\{K_n \subset \mathbb{Y}, n = 1, 2, \ldots\}$ be an increasing sequence of compact sets such that $\cup_{n=1}^\infty K_n = \mathbb{Y}$. For each n, set

$$\begin{aligned}\bar{\mathbb{A}}_{b,n} = \{\varphi \in \bar{\mathbb{A}} :\ &\text{for all } (t, \omega) \in [0, T] \times \bar{\mathbb{M}}, n \geqslant \varphi(t, x, \omega) \geqslant 1/n \\ &\text{if } x \in K_n \text{ and } \varphi(t, x, \omega) = 1 \text{ if } x \in K_n^c\},\end{aligned}$$

and set $\bar{\mathbb{A}}_b = \cup_{n=1}^\infty \bar{\mathbb{A}}_{b,n}$. we define $\tilde{\mathcal{U}}^M = \mathcal{U}^M \cap \{(\psi, \phi) : \phi \in \bar{\mathbb{A}}_b\}$.

The following criteria was established in [**17**] (see also [**13**, **60**]).

Theorem 2.4. *For $\nu > 0$, let Z^ν be defined by $Z^\nu = \mathcal{G}^\nu(\sqrt{\nu}W, \nu N^{\nu^{-1}})$ and suppose that condition 2.3 or condition 2.3 for replacing $\mathcal{U}^M$ with $\tilde{\mathcal{U}}^M$ holds. Therefore, I defined as in (2.3) is a rate function on $\mathbb{U}$ and the family $\{Z^\nu\}_{\nu>0}$ satisfies a large deviation principle with the rate function I.*

2.5. Assumptions

Given a viscosity coefficient $\nu > 0$, we study the following two-dimensional stochastic Navier–Stokes equations:

$$\begin{aligned}\mathrm{d}u^\nu(t) = &-\left[\nu A u^\nu(t) + B(u^\nu(t), u^\nu(t))\right]\mathrm{d}t + \sqrt{\nu}\,\Upsilon_\nu(t, u^\nu(t))\,\mathrm{d}W(t) \\ &+ \nu\int_{\mathbb{Y}} G(t, u^\nu(t-), y)\tilde{N}^{\nu^{-1}}(\mathrm{d}y, \mathrm{d}t),\end{aligned} \tag{2.4}$$

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where $\Upsilon_\nu : [0,T] \times V \to L_Q(H_0, H)$ and $G : [0,T] \times V \times \mathbb{Y} \to H$ are given measurable maps. We assume that Υ_ν and G satisfy the following growth and Lipschitz conditions:

Condition 2.5. *For $\Upsilon_\nu \in \mathcal{C}([0,T] \times V; L_Q(H_0, H))$, $G \in \mathcal{C}([0,T] \times V \times L^p(\mathbb{Y}); H)$, there exists $K(\cdot) \in L^1([0,T], \mathbb{R}^+)$ such that for every $t \in [0,T]$, $\nu > 0$ and $u, v \in V$, it holds*

(1) $|\Upsilon_\nu(t,u)|^2_{L_Q} \leqslant K(t)(1 + |u|^2_H)$, $|\Upsilon_\nu(t,u) - \Upsilon_\nu(t,v)|^2_{L_Q} \leqslant K(t)|u-v|^2_H$;

(2) $\int_{\mathbb{Y}} |G(t,u,y)|^p_H \vartheta(\mathrm{d}y) \leqslant K(t)(1 + |u|^p_H)$, $\int_{\mathbb{Y}} |G(t,u,y) - G(t,v,y)|^p_H \vartheta(\mathrm{d}y) \leqslant K(t)|u-v|^p_H$ *for* $p \geqslant 2$.

Let $\nu > 0$, $\hbar := (f,g) \in \tilde{\mathcal{U}}^M$. We consider the following stochastic equations:

$$\begin{aligned} du^\nu_\hbar(t) = &-\big[\nu A u^\nu_\hbar(t) + B\big(u^\nu_\hbar(t), u^\nu_\hbar(t)\big)\big]\mathrm{d}t + \sqrt{\nu}\Upsilon_\nu(t, u^\nu_\hbar(t))\,\mathrm{d}W(t) \\ &+ \tilde{\Upsilon}_\nu(t, u^\nu_\hbar(t)) f(t)\,\mathrm{d}t + \nu \int_{\mathbb{Y}} G(t, u^\nu_\hbar(t-), y) \tilde{N}^{g\nu^{-1}}(\mathrm{d}y, \mathrm{d}t) \\ &+ \int_{\mathbb{Y}} \tilde{G}(t, u^\nu_\hbar(t), y)(g(t,y) - 1)\vartheta(\mathrm{d}y)\,\mathrm{d}t \end{aligned} \tag{2.5}$$

with the initial data $u^\nu_\hbar(0) = \varsigma \in H$. Assume that ς is deterministic. In order to define the stochastic controlled equations (2.5), we introduce a family of intensity coefficients $(\tilde{\Upsilon}_\nu, \tilde{G})$ for $\nu \geqslant 0$ which act on a random element $\hbar := (f,g) \in \tilde{\mathcal{U}}^M$. For any $\nu \geqslant 0$, we assume that the coefficient $(\tilde{\Upsilon}_\nu, \tilde{G})$ satisfies the following conditions.

Condition 2.6. *For $\tilde{\Upsilon}_\nu \in \mathcal{C}([0,T] \times V; L(H_0, H))$ and $\tilde{G} \in \mathcal{C}([0,T] \times V \times L^p(\mathbb{Y}); H)$, there exists $K(\cdot) \in L^1([0,T], \mathbb{R}^+)$ such that for every $t \in [0,T]$, $\nu \geqslant 0$ and $u, v \in V$:*

(1) $|\tilde{\Upsilon}_\nu(t,u)|_{L(H_0,H)} \leqslant K(t)(1 + |u|_H)$, $|\tilde{\Upsilon}_\nu(t,u) - \tilde{\Upsilon}_\nu(t,v)|_{L(H_0,H)} \leqslant K(t)|u - v|_H$;

(2) $\int_{\mathbb{Y}} |\tilde{G}(t,u,y)|^p_H \vartheta(\mathrm{d}y) \leqslant K(t)(1 + |u|^p_H)$, $\int_{\mathbb{Y}} |\tilde{G}(t,u,y) - \tilde{G}(t,v,y)|^p_H \vartheta(\mathrm{d}y) \leqslant K(t)|u-v|^p_H$ *for* $p \geqslant 2$.

Note that when $\Upsilon_\nu, \tilde{\Upsilon}_\nu$ have Nemytski form (e.g., see [**12**]) and $G(t,u(t),y), \tilde{G}(t,u(t),y) = \kappa(t)u(t) + \iota(t)\Gamma(y)$, where $\kappa(t), \iota(t), \Gamma(y)$ meet certain assumptions, and $(\Upsilon_\nu, G), (\tilde{\Upsilon}_\nu, \tilde{G})$ satisfy conditions 2.5 and 2.6.

We define

$$\|\tilde{G}(t,y)\|_{0,H} = \sup_{u \in H} \frac{|\tilde{G}(t,u,y)|_H}{1 + |u|_H}, \quad (t,y) \in [0,T] \times \mathbb{Y},$$

$$\|\tilde{G}(t,y)\|_{1,H} = \sup_{u,v \in H, u \neq v} \frac{|\tilde{G}(t,u,y) - \tilde{G}(t,v,y)|_H}{|u - v|_H}, \quad (t,y) \in [0,T] \times \mathbb{Y}.$$

Similarly, we can define $\|\tilde{G}(t,y)\|_{0,L^q}$, $\|\tilde{G}(t,y)\|_{1,L^q}$, $\|\tilde{G}(t,y)\|_{0,V}$ and $\|\tilde{G}(t,y)\|_{1,V}$. In the later proof, we need the following lemmas. Here, we omit the proof. For more details, we refer to [**13**, **60**] and the references therein.

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Condition 2.7. *For $i=0,1$, there exists $\delta_1^i>0$ such that for all $E\in\mathcal{B}([0,T]\times\mathbb{Y})$ satisfying $\vartheta_T(E)<\infty$, it holds that*

$$\int_E e^{\delta_1^i\|\tilde{G}(s,y)\|_{i,\cdot}^2}\vartheta(\mathrm{d}y)\,\mathrm{d}s<\infty.$$

Here $\|\tilde{G}(s,y)\|_{i,\cdot}^2=\|\tilde{G}(s,y)\|_{i,H}^2$ *or* $\|\tilde{G}(s,y)\|_{i,V}^2$ *or* $\|\tilde{G}(s,y)\|_{i,L^q}^2$.

Remark 2.8. *Suppose condition 2.7 holds, for every $\delta_2^i>0$ and for all $E\in\mathcal{B}([0,T]\times\mathbb{Y})$ satisfying $\vartheta_T(E)<\infty$, then*

$$\int_E e^{\delta_2^i\|\tilde{G}(s,y)\|_{i,\cdot}}\vartheta(\mathrm{d}y)\,\mathrm{d}s<\infty.$$

Lemma 2.9. *Suppose that conditions 2.6 and 2.7 hold.*

(1) *For $i=0,1$ and every $M\in\mathbb{N}$, it holds that*

$$\sup_{g\in S^M}\int_{\mathbb{Y}_T}\|\tilde{G}(s,y)\|_{i,\cdot}^2(g(s,y)+1)\vartheta(\mathrm{d}y)\,\mathrm{d}s<\infty,$$

$$\sup_{g\in S^M}\int_{\mathbb{Y}_T}\|\tilde{G}(s,y)\|_{i,\cdot}|g(s,y)-1|\vartheta(\mathrm{d}y)\,\mathrm{d}s<\infty;$$

(2) *For every $\tilde{\eta}>0$, there exists $\delta>0$ such that for any $A\subset[0,T]$ satisfying $\lambda_T(A)<\delta$*

$$\sup_{g\in S^M}\int_A\int_{\mathbb{Y}}\|\tilde{G}(s,y)\|_{i,\cdot}|g(s,y)-1|\vartheta(\mathrm{d}y)\,\mathrm{d}s\leqslant\tilde{\eta}.$$

Lemma 2.10. *Let $h:[0,T]\times\mathbb{Y}\to\mathbb{R}$ be a measurable function such that*

$$\int_{\mathbb{Y}_T}|h(s,y)|^2\vartheta(\mathrm{d}y)\,\mathrm{d}s<\infty,$$

and for all $\delta\in(0,\infty)$ and $E\in\mathcal{B}([0,T]\times\mathbb{Y})$ satisfying $\vartheta_T(E)<\infty$,

$$\int_E\exp(\delta|h(s,y)|)\vartheta(\mathrm{d}y)\,\mathrm{d}s<\infty.$$

(1) *Fix $M\in\mathbb{N}$ and let $g_n,g\in S^M$ be such that $g_n\to g$ as $n\to\infty$. Then we have*

$$\lim_{n\to\infty}\int_{\mathbb{Y}_T}h(s,y)(g_n(s,y)-1)\vartheta(\mathrm{d}y)\,\mathrm{d}s=\int_{\mathbb{Y}_T}h(s,y)(g(s,y)-1)\vartheta(\mathrm{d}y)\,\mathrm{d}s;$$

(2) *Fix $M\in\mathbb{N}$ and given $\varepsilon>0$, there exists a compact set $K\subset\mathbb{Y}$ such that*

$$\sup_{g\in S^M}\int_0^T\int_{K^c}|h(s,y)||g(s,y)-1|\vartheta(\mathrm{d}y)\,\mathrm{d}s\leqslant\varepsilon;$$

(3) *For every compact $K\subset\mathbb{Y}$, then we have*

$$\lim_{\tilde{M}\to\infty}\sup_{g\in S^M}\int_0^T\int_K|h(s,y)|1_{\{h\geqslant\tilde{M}\}}g(s,y)\vartheta(\mathrm{d}y)\,\mathrm{d}s=0.$$

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For fixed $\nu > 0$, well-posedness and a priori estimates of the solution to the equations (2.5) in $\mathcal{D}([0,T];H) \cap L^2(0,T;V)$ are obtained when $u = 0$ on ∂D (see e.g. [**11**, **60**]). For some small ν_0, we will show that the solutions are uniform bounded in $\nu \in (0, \nu_0]$ under free boundary conditions.

Next, we introduce other useful conditions, which we will use later in this paper.

Condition 2.11. *For every $\nu > 0$, $\Upsilon_\nu \in \mathcal{C}([0,T] \times D(A); L_Q(H_0, V))$ and $G \in \mathcal{C}([0,T] \times D(A) \times L^p(\mathbb{Y}); H)$, there exists $K(\cdot) \in L^1([0,T], \mathbb{R}^+)$ such that for every $t \in [0,T]$ and $u, v \in D(A)$, it holds*

(1) $|\text{ curl } \Upsilon_\nu(u,t)|^2_{L_Q} \leqslant K(t)(1 + \|u\|^2_V)$, $|A^{1/2}\Upsilon_\nu(t,u) - A^{1/2}\Upsilon_\nu(t,v)|^2_{L_Q} \leqslant K(t)\|u - v\|^2_V$;

(2) *for* $p \geqslant 2$, $\int_{\mathbb{Y}} |\text{ curl } G(t,u,y)|^p_H \vartheta(\mathrm{d}y) \leqslant K(t)(1 + |\text{ curl } u|^p_H)$, $\int_{\mathbb{Y}} |A^{1/2}G(t,u,y) - A^{1/2}G(t,v,y)|^2_H \vartheta(\mathrm{d}y) \leqslant K(t)|u - v|^2_V$.

Condition 2.12. *For every $\nu \geqslant 0$, $\tilde{\Upsilon}_\nu \in \mathcal{C}([0,T] \times D(A); L(H_0, V))$ and $\tilde{G} \in \mathcal{C}([0,T] \times D(A) \times L^p(\mathbb{Y}); H)$, there exists $K(\cdot) \in L^1([0,T], \mathbb{R}^+)$ such that for every $t \in [0,T]$ and $u, v \in D(A)$, it holds*

(1) $|\text{curl } \tilde{\Upsilon}_\nu(t,u)|_{L(H_0,H)} \leqslant \sqrt{K(t)}(1 + \|u\|_V)$, $|A^{1/2}\tilde{\Upsilon}_\nu(t,u) - A^{1/2}\tilde{\Upsilon}_\nu(t,v)|_{L(H_0,H)} \leqslant \sqrt{K(t)}\|u - v\|_V$.

(2) $|\text{curl } \tilde{G}(t,u,y)|_H \leqslant C|\tilde{G}(t, \text{ curl } u, y)|_H$ *and* $|A^{1/2}\tilde{G}(t,u,y) - A^{1/2}\tilde{G}(t,v,y)|_H \leqslant C|\tilde{G}(t, A^{1/2}u, y) - \tilde{G}(t, A^{1/2}v, y)|_H$.

Again, note that when $\Upsilon_\nu, \tilde{\Upsilon}_\nu$ have Nemytski form (see [**12**]), $G(t,u(t),y)$, $\tilde{G}(t,u(t),y) = \kappa(t)u(t) + \iota(t)\Gamma(y)$ where $\kappa(t), \iota(t), \Gamma(y)$ meet certain assumptions, and $(\Upsilon_\nu, G), (\tilde{\Upsilon}_\nu, \tilde{G})$ satisfy conditions 2.11 and 2.12.

3. Existence and uniqueness of the solution to equations (2.5)

In this section, we want to prove the existence and uniqueness of solutions to the stochastic equations (2.5) with free boundary conditions (1.3) under some additional assumptions.

Let $(\Omega := \bar{\mathbb{V}}, \mathcal{F} := \mathcal{B}(\bar{\mathbb{V}}), \{\mathcal{F}_t\}_{t\geqslant 0} := \{\bar{\mathcal{F}}^{\bar{\mathbb{V}}}_t\}_{t\geqslant 0}, \mathbb{P} := \bar{\mathbb{P}}^{\bar{\mathbb{V}}}, W, N)$ be a fixed stochastic basis. First, we recall the definition of stochastic strong analytically weak solutions (e.g. see [**46**]). If an $\mathcal{F}_t$-progressively measurable stochastic process $u^\nu_\hbar(t,\omega)$ belongs to X $\mathbb{P}$-a.s., and for all $v \in D(A)$ and all $t \in [0,T]$, it holds that $\mathbb{P}$-a.s.

$$\begin{aligned}
&(u^\nu_\hbar(t), v) - (\varsigma, v) + \int_0^t \left[\nu(u^\nu_\hbar(s), Av) + \langle B(u^\nu_\hbar(s), v), u^\nu_\hbar(s)\rangle\right]\mathrm{d}s \\
&= \sqrt{\nu}\int_0^t (\Upsilon_\nu(s, u^\nu_\hbar(s))\,\mathrm{d}W(s), v) + \int_0^t (\tilde{\Upsilon}_\nu(s, u^\nu_\hbar(s))f(s), v)\mathrm{d}s \\
&\quad + \nu\int_0^t\int_{\mathbb{Y}} (G(s, u^\nu_\hbar(s-), y)\tilde{N}^{g\nu^{-1}}(\mathrm{d}y, \mathrm{d}s), v) \\
&\quad + \int_0^t\int_{\mathbb{Y}} (\tilde{G}(s, u^\nu_\hbar(s), y)(g(s,y) - 1), v)\vartheta(\mathrm{d}y)\,\mathrm{d}s,
\end{aligned}$$

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then $u_\hbar^\nu(t,\omega)$ is called a stochastic strong analytically weak solution in $X \subset \mathcal{D}([0,T];H) \cap L^2(0,T;V)$ of (2.5) with deterministic initial data ς satisfying $|\varsigma|_H < \infty$.

PROPOSITION 3.1. *Assume that (Υ_ν, G) and $(\tilde{\Upsilon}_\nu, \tilde{G})$ satisfy conditions 2.5, 2.6 and 2.7 with $K(t) = C$, respectively and $\mathbb{E}|\varsigma|_H^{2p} < \infty$ for some $p \geqslant 2$. Then, for $\nu_0 > 0$ and for any $M > 0$, there exists a positive constant C that depends on M, T and ν_0 such that for any $\nu \in (0, \nu_0]$ and any $\hbar := (f, g) \in \tilde{\mathcal{U}}^M$, (2.5) has a unique stochastic strong analytically weak solution in $\mathcal{D}([0,T];H) \cap L^2(0,T;V)$. Moveover, we have*

$$\sup_{0<\nu\leqslant\nu_0} \sup_{\hbar\in\tilde{\mathcal{U}}^M} \mathbb{E}\left(\sup_{0\leqslant s\leqslant T} |u_\hbar^\nu(s)|_H^{2p}\right) \leqslant C(p,M,T,\nu_0)\left(1 + \mathbb{E}|\varsigma|_H^{2p}\right), \tag{3.1}$$

and

$$\sup_{0<\nu\leqslant\nu_0} \sup_{\hbar\in\tilde{\mathcal{U}}^M} \nu \int_0^T \mathbb{E}\big(\|u_\hbar^\nu(s)\|^2 + \|u_\hbar^\nu(s)\|_{\mathcal{H}}^4\big)\,\mathrm{d}s \leqslant C(M,T,\nu_0)\left(1 + \mathbb{E}|\varsigma|_H^4\right). \tag{3.2}$$

Proof. First, by applying the Fadeo-Galerkin approximation, we can establish the existence of approximation solution $u_{\hbar,n}^\nu$ (for more details, see [**11**, **21**, **28**]). In order to obtain the existence of solution to (2.5), we need to take n tend to infinity. And then we should give a priori estimates of $u_{\hbar,n}^\nu$ uniformly in $n \geqslant 1$ and in $\nu \in (0, \nu_0]$ for some $\nu_0 > 0$ under free boundary conditions as [**21**]. Next, we will give the energy estimates of $u_{\hbar,n}^\nu$. To simplify symbols, we replace the approximation solution $u_{\hbar,n}^\nu$ by $u_\hbar^\nu$.

Let $\nu > 0$, $\hbar = (f,g) \in \tilde{\mathcal{U}}^M$. Define $\tau_N = \inf\{t \geqslant 0, |u_\hbar^\nu(t)|_H \geqslant N\} \wedge T$ for every $N > 0$. By using Itô's formula to the function $|u_\hbar^\nu(t \wedge \tau_N)|_H^{2p}$, we obtain from (2.5)

$$|u_\hbar^\nu(t\wedge\tau_N)|_H^{2p} + 2p\nu \int_0^{t\wedge\tau_N} |u_\hbar^\nu(s)|_H^{2p-2} \|u_\hbar^\nu(s)\|^2 \mathrm{d}s \leqslant |u_\hbar^\nu(0)|_H^{2p} + \sum_{i=1}^{8} I_i(t), \tag{3.3}$$

where

$$I_1(t) = 2p\sqrt{\nu} \int_0^{t\wedge\tau_N} |u_\hbar^\nu(s)|_H^{2p-2} (\Upsilon_\nu(s, u_\hbar^\nu(s)), u_\hbar^\nu(s))\,\mathrm{d}W(s),$$

$$I_2(t) = \int_0^{t\wedge\tau_N} \int_{\mathbb{Y}} \left(|u_\hbar^\nu(s) + \nu G(s, u_\hbar^\nu(s), y)|_H^{2p} - |u_\hbar^\nu(s)|_H^{2p}\right) \tilde{N}^{\nu^{-1}}(\mathrm{d}y, \mathrm{d}s),$$

$$I_3(t) = 2p\nu \int_0^{t\wedge\tau_N} |u_\hbar^\nu(s)|_H^{2p-2} \int_{\partial D} k(r) |u_\hbar^\nu(r)|_H^2 dr ds,$$

$$I_4(t) = 2p \int_0^{t\wedge\tau_N} |u_\hbar^\nu(s)|_H^{2p-2} \langle B(u_\hbar^\nu(s), u_\hbar^\nu(s)), u_\hbar^\nu(s)\rangle \mathrm{d}s,$$

$$I_5(t) = 2p \int_0^{t\wedge\tau_N} |u_\hbar^\nu(s)|_H^{2p-2} \big(\tilde{\Upsilon}_\nu(s, u_\hbar^\nu(s)) f(s), u_\hbar^\nu(s)\big)\,\mathrm{d}s,$$

$$I_6(t) = \nu p(2p-1) \int_0^{t\wedge\tau_N} |u_\hbar^\nu(s)|_H^{2p-2} |\Upsilon_\nu(s, u_\hbar^\nu(s))|_{L_Q}^2 \mathrm{d}s,$$

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$$
\begin{aligned}
I_7(t) = \int_0^{t\wedge\tau_N} \int_{\mathbb{Y}} \Big(& |u_\hbar^\nu(s) + \nu G(s, u_\hbar^\nu(s), y)|_H^{2p} - |u_\hbar^\nu(s)|_H^{2p} \\
& - \nu p |u_\hbar^\nu(s)|^{2p-2} (u_\hbar^\nu(s), G(s, u_\hbar^\nu(s), y))_H \Big) \vartheta(\mathrm{d}y)\,\mathrm{d}s, \\
I_8(t) = 2p \int_0^{t\wedge\tau_N} \int_{\mathbb{Y}} & |u_\hbar^\nu(s)|_H^{2p-2} (\tilde{G}(s, u_\hbar^\nu(s), y)(g(s,y)-1), u_\hbar^\nu(s)) \vartheta(\mathrm{d}y)\,\mathrm{d}s.
\end{aligned}
$$

Next, we estimate the terms $I_1 - I_8$ one by one. For the term I_1, the Burkholder–Davis–Gundy inequality, condition 2.5, Cauchy–Schwarz's and Young's inequalities imply

$$
\begin{aligned}
\mathbb{E}\left(\sup_{s\in[0,t\wedge\tau_N]} I_1(s) \right) &\leqslant C\sqrt{\nu} p \mathbb{E}\left(\int_0^{t\wedge\tau_N} |u_\hbar^\nu(s)|_H^{4p-4} (\Upsilon_\nu(s, u_\hbar^\nu(s)), u_\hbar^\nu(s))^2 \mathrm{d}s \right)^{1/2} \\
&\leqslant C\sqrt{\nu} p \mathbb{E}\left(\int_0^{t\wedge\tau_N} |u_\hbar^\nu(s)|_H^{4p-4} |\Upsilon_\nu(s, u_\hbar^\nu(s))|_H^2 |u_\hbar^\nu(s)|_H^2 \mathrm{d}s \right)^{1/2} \\
&\leqslant C\sqrt{\nu} p \mathbb{E}\left(\int_0^{t\wedge\tau_N} |u_\hbar^\nu(s)|_H^{4p-2} K(s)(1 + |u_\hbar^\nu(s)|_H^2)\,\mathrm{d}s \right)^{1/2} \\
&\leqslant C\sqrt{\nu} p \mathbb{E}\left(\int_0^{t\wedge\tau_N} K(s)(1 + |u_\hbar^\nu(s)|_H^{4p})\,\mathrm{d}s \right)^{1/2} \\
&\leqslant \frac{1}{4} \mathbb{E}\left(\sup_{s\in[0,t\wedge\tau_N]} |u_\hbar^\nu(s)|_H^{2p} \right) \\
&\quad + C\nu p^2 \mathbb{E}\left(\int_0^{t\wedge\tau_N} K(s)(1 + |u_\hbar^\nu(s)|_H^{2p})\,\mathrm{d}s \right) + C.
\end{aligned}
$$

From the Taylor formula, it follows that for every $p \geqslant 2$ there exists a positive constant $C_p > 0$ such that for all $x, h \in H$

$$
\left| |x+h|_H^p - |x|_H^p - p|x|_H^{(p-2)} (x,h)_H \right| \leqslant C_p \left(|x|_H^{(p-2)} + |h|_H^{(p-2)} \right) |h|_H^2. \tag{3.4}
$$

Moreover, we arrive at

$$
||x+h|_H^p - |x|_H^p| \leqslant \frac{p}{2}|x|_H^p + \left(C_p + \frac{p}{2} \right) |x|_H^{(p-2)} |h|_H^2 + C_p |h|_H^p.
$$

By using the Schwarz inequality, we obtain for all $x, h \in H$

$$
\begin{aligned}
(|x+h|_H^p - |x|_H^p)^2 &\leqslant 2\left[p^2 |x|_H^{2p-2} |h|_H^2 + c_p^2 \left(|x|_H^{p-2} + |h|_H^{p-2} \right)^2 |h|_H^4 \right] \\
&\leqslant 2p^2 |x|_H^{2p-2} |h|_H^2 + 4c_p^2 |x|_H^{2p-4} |h|_H^4 + 4C_p^2 |h|_H^{2p}.
\end{aligned} \tag{3.5}
$$

For the term $I_2(t)$, by using the Burkholder–Davis–Gundy inequality, (3.5), condition 2.5, Hölder's and Young's inequalities, we deduce that

$$\begin{aligned}
&\mathbb{E}\left(\sup_{s\in[0,t\wedge\tau_N]} I_2(s)\right)\\
&\leqslant \mathbb{E}\left[\int_0^{t\wedge\tau_N}\int_{\mathbb{Y}}\left(|u_\hbar^\nu(s)+\nu G(s,u_\hbar^\nu(s),y)|_H^{2p}-|u_\hbar^\nu(s)|_H^{2p}\right)^2\vartheta(\mathrm{d}y)\,\mathrm{d}s\right]^{\frac{1}{2}}\\
&\leqslant C(\nu)\mathbb{E}\Bigg(\int_0^{t\wedge\tau_N}\int_{\mathbb{Y}}|u_\hbar^\nu(s)|_H^{4p-2}|G(s,u_\hbar^\nu(s),y)|_H^2\\
&+|u_\hbar^\nu(s)|_H^{4p-4}|G(s,u_\hbar^\nu(s),y)|_H^4+|G(s,u_\hbar^\nu(s),y)|_H^{4p}\Bigg)^{\frac{1}{2}}\\
&\leqslant C(\nu)\mathbb{E}\left(\int_0^{t\wedge\tau_N}K(s)|u_\hbar^\nu(s)|_H^{4p}\mathrm{d}s\right)^{\frac{1}{2}}+C(\nu)\mathbb{E}\left(\int_0^{t\wedge\tau_N}K(s)\,\mathrm{d}s\right)^{\frac{1}{2}}\\
&\leqslant \frac{1}{4}\mathbb{E}\left(\sup_{s\in[0,t\wedge\tau_N]}|u_\hbar^\nu(s)|_H^{2p}\right)+C(\nu)\mathbb{E}\int_0^{t\wedge\tau_N}K(s)|u_\hbar^\nu(s)|_H^{2p}\mathrm{d}s+C(\nu).
\end{aligned}$$

Using (2.1) and Young's inequality, for any $\varepsilon>0$, we bound the term $I_3(t)$ by

$$\mathbb{E}I_3(t)\leqslant 2\nu p\varepsilon\mathbb{E}\int_0^{t\wedge\tau_N}|u_\hbar^\nu(s)|_H^{2p-2}\|u_\hbar^\nu(s)\|^2\mathrm{d}s+2\nu pC(\varepsilon)\mathbb{E}\int_0^{t\wedge\tau_N}|u_\hbar^\nu(s)|_H^{2p}\mathrm{d}s.$$

For the term $I_4(t)$, using (1.2) implies that $\mathbb{E}I_4(t)=0$ for any $t\in[0,T]$. For the term $I_5(t)$, since $(f,g)\in\tilde{\mathcal{U}}^M$, the growth condition 2.6, Cauchy–Schwarz's and Hölder's inequalities yield that

$$\begin{aligned}
\mathbb{E}I_5(t)&\leqslant 2p\mathbb{E}\int_0^{t\wedge\tau_N}|u_\hbar^\nu(s)|_H^{2p-2}|\tilde{\Upsilon}_\nu(s,u_\hbar^\nu(s))|_{L(H_0,H)}|u_\hbar^\nu(s)|_H|f(s)|_0\mathrm{d}s\\
&\leqslant 2p\mathbb{E}\int_0^{t\wedge\tau_N}|u_\hbar^\nu(s)|_H^{2p-2}\sqrt{K(t)}(1+|u_\hbar^\nu(s)|_H^2)|f(s)|_0\mathrm{d}s\\
&\leqslant 2p\mathbb{E}\int_0^{t\wedge\tau_N}(1+|u_\hbar^\nu(s)|_H^{2p})(K(t)+|f(s)|_0^2)\,\mathrm{d}s\\
&\leqslant 2p\mathbb{E}\int_0^{t\wedge\tau_N}(K(s)+|f(s)|_0^2)|u_\hbar^\nu(s)|_H^{2p}\mathrm{d}s+2p\mathbb{E}\int_0^{t\wedge\tau_N}(K(s)+|f(s)|_0^2)\,\mathrm{d}s.
\end{aligned}$$

For the term I_6, using the growth condition 2.5, we get

$$\begin{aligned}
\mathbb{E}I_6(t)\leqslant{}&\nu p(2p-1)\mathbb{E}\int_0^{t\wedge\tau_N}K(s)\,\mathrm{d}s\\
&+\nu p(2p-1)\mathbb{E}\int_0^{t\wedge\tau_N}K(s)|u_\hbar^\nu(s)|_H^{2p}\mathrm{d}s,\text{ for }\nu\in(0,\nu_0].
\end{aligned}$$

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For the term $I_7(t)$, by using (3.4) and condition 2.5, we have

$$\begin{aligned}\mathbb{E}I_7(t) &\leqslant C\mathbb{E}\Big[\int_0^{t\wedge\tau_N}\int_{\mathbb{Y}}\Big(|u_\hbar^\nu(s)|_H^{2p-2}+|\nu G(s,u_\hbar^\nu(s),y)|_H^{2p-2}\Big)\\&\quad\times|\nu G(s,u_\hbar^\nu(s),y)|_H^2\vartheta(\mathrm{d}y)\,\mathrm{d}s\Big]\\&\leqslant C(\nu)\mathbb{E}\Big(\int_0^{t\wedge\tau_N}|u_\hbar^\nu(s)|_H^{2p-2}\int_{\mathbb{Y}}|G(s,u_\hbar^\nu(s),y)|_H^2\vartheta(\mathrm{d}y)\,\mathrm{d}s\\&\quad+\int_0^{t\wedge\tau_N}\int_{\mathbb{Y}}|G(s,u_\hbar^\nu(s),y)|_H^{2p}\vartheta(\mathrm{d}y)\,\mathrm{d}s\Big)\\&\leqslant C(\nu)\mathbb{E}\int_0^{t\wedge\tau_N}K(s)|u_\hbar^\nu(s)|_H^{2p}\mathrm{d}s+C(\nu)\mathbb{E}\int_0^{t\wedge\tau_N}K(s)\,\mathrm{d}s.\end{aligned}$$

For the term $I_8(t)$, by using Hölder's and Young's inequalities, one has

$$\begin{aligned}&\mathbb{E}\int_0^{t\wedge\tau_N}\int_{\mathbb{Y}}|u_\hbar^\nu(s)|_H^{2p-2}(\tilde G(s,u_\hbar^\nu(s),y)(g(s,y)-1),u_\hbar^\nu(s))\vartheta(\mathrm{d}y)\,\mathrm{d}s\\&\leqslant\mathbb{E}\int_0^{t\wedge\tau_N}\int_{\mathbb{Y}}|u_\hbar^\nu(s)|_H^{2p-2}|\tilde G(s,u_\hbar^\nu(s),y)|_H|g(s,y)-1||u_\hbar^\nu(s)|_H\vartheta(\mathrm{d}y)\,\mathrm{d}s\\&\leqslant\mathbb{E}\int_0^{t\wedge\tau_N}\int_{\mathbb{Y}}|u_\hbar^\nu(s)|_H^{2p-2}\frac{|\tilde G(s,u_\hbar^\nu(s),y)|_H}{1+|u_\hbar^\nu(s)|_H}|g(s,y)-1|(1+|u_\hbar^\nu(s)|_H^2)\vartheta(\mathrm{d}y)\,\mathrm{d}s\\&\leqslant\mathbb{E}\int_0^{t\wedge\tau_N}\int_{\mathbb{Y}}\frac{|\tilde G(s,u_\hbar^\nu(s),y)|_H}{1+|u_\hbar^\nu(s)|_H}|g(s,y)-1|(|u_\hbar^\nu(s)|_H^{2p-2}+|u_\hbar^\nu(s)|_H^{2p})\vartheta(\mathrm{d}y)\,\mathrm{d}s\\&\leqslant C\mathbb{E}\int_0^{t\wedge\tau_N}\int_{\mathbb{Y}}\|\tilde G(s,y)\|_{0,H}|g(s,y)-1|\vartheta(\mathrm{d}y)\,\mathrm{d}s\\&\quad+C\mathbb{E}\int_0^{t\wedge\tau_N}|u_\hbar^\nu(s)|_H^{2p}\int_{\mathbb{Y}}\|\tilde G(s,y)\|_{0,H}|g(s,y)-1|\vartheta(\mathrm{d}y)\,\mathrm{d}s.\end{aligned}$$

Thus, for any $t\in[0,T]$, $\varepsilon\in(0,1)$, (3.3) and the estimates of I_1-I_8 imply that

$$\begin{aligned}&\mathbb{E}|u_\hbar^\nu(t\wedge\tau_N)|_H^{2p}+2\nu p\big(1-\varepsilon\big)\mathbb{E}\int_0^{t\wedge\tau_N}|u_\hbar^\nu(s)|_H^{2p-2}\|u_\hbar^\nu(s)\|^2\mathrm{d}s\\&\leqslant C(\nu)\mathbb{E}\int_0^{t\wedge\tau_N}(K(s)+|f(s)|_0^2)|u_\hbar^\nu(s)|_H^{2p}\mathrm{d}s+C(\nu)\mathbb{E}\int_0^{t\wedge\tau_N}(K(s)+|f(s)|_0^2)\,\mathrm{d}s\\&\quad+C\mathbb{E}\int_0^{t\wedge\tau_N}\int_{\mathbb{Y}}\|\tilde G(s,y)\|_{0,H}|g(s,y)-1|\vartheta(\mathrm{d}y)\,\mathrm{d}s+C+\mathbb{E}|\varsigma|_H^{2p}\\&\quad+C\mathbb{E}\int_0^{t\wedge\tau_N}|u_\hbar^\nu(s)|_H^{2p}\int_{\mathbb{Y}}\|\tilde G(s,y)\|_{0,H}|g(s,y)-1|\vartheta(\mathrm{d}y)\,\mathrm{d}s.\end{aligned}$$

Applying Gronwall's inequality, condition 2.7 and lemma 2.9, we obtain

$$\mathbb{E}|u_\hbar^\nu(t\wedge\tau_N)|_H^{2p}+2\nu p\big(1-\varepsilon\big)\mathbb{E}\int_0^{t\wedge\tau_N}|u_\hbar^\nu(s)|_H^{2p-2}\|u_\hbar^\nu(s)\|^2\mathrm{d}s\leqslant C\left(1+\mathbb{E}|\varsigma|_H^{2p}\right).$$

Due to the estimates of the right-hand side of (3.3) does not depend on N, taking $N \to \infty$, we can infer that $\tau_N \to T$ $\mathbb{P}$-a.s. Hence, there exists a constant $C := C(\mathbb{E}|\varsigma|_H^{2p}, \nu_0, M, T, p)$ such that

$$\sup_n \mathbb{E}\left(\sup_{0\leqslant t\leqslant T} |u^\nu_{\hbar,n}(t)|_H^{2p} + \nu \int_0^T \left[\|u^\nu_{\hbar,n}(t)\|^4_{\mathcal{H}} + \|u^\nu_{\hbar,n}(s)\|^2\right]\mathrm{d}s\right) \leqslant C,$$

for any n, $\nu \in (0, \nu_0]$ and $(f, g) \in \tilde{\mathcal{U}}^M$. Letting $n \to \infty$ and using the classical argument as [**11**], we obtain the existence of solution to (2.5). Taking $n \to \infty$ and combining the above estimates and the weakly lower semi-continuity of the norm, we can get (3.1) and (3.2).

Next, we begin to prove the uniqueness. When condition 2.5 with $K(\cdot) = C$ holds, we know that equations (2.4) has a unique strong solution $u^\nu \in \mathcal{D}([0,T]; H) \cap L^2(0,T;V)$ by [**11**] for any $T > 0$. Set $\phi_\nu = (\psi_\nu, \varphi_\nu) \in \tilde{\mathcal{U}}^M$ and $\vartheta_\nu = \frac{1}{\varphi_\nu}$. We conclude from lemma 2.3 in [**17**] that $\mathcal{E}^\nu_t(\vartheta_\nu) := \exp\{\int_{[0,t]\times\mathbb{Y}\times[0,\nu^{-1}]} \log(\vartheta_\nu(s,x))\bar{N}(dsdxdr) + \int_{[0,t]\times\mathbb{Y}\times[0,\nu^{-1}]}(-\vartheta_\nu(s,x)+1)\bar{\vartheta}_T (dsdxdr)\}$, $\tilde{\mathcal{E}}^\nu_t(\psi_\nu) := \exp\{\frac{1}{\nu}\int_0^t \psi_\nu(s)d\beta(s) - \frac{1}{2\nu}\int_0^t \|\psi_\nu(s)\|^2\mathrm{d}s\}$ are $\{\bar{\mathcal{F}}^{\bar{\mathbb{V}}}_t\}$-martingales and $\mathbb{Q}^\nu_t(G) = \int_G \bar{\mathcal{E}}^\nu_t(\psi_\nu, \vartheta_\nu)d\bar{\mathbb{P}}^{\bar{\mathbb{V}}}$ defines a probability measure on $\bar{\mathbb{V}}$, where $G \in \mathcal{B}(\bar{\mathbb{V}})$ and $\mathbb{Q}^\nu_t(G) = \tilde{\mathcal{E}}^\nu_t(\psi_\nu)\mathcal{E}^\nu_t(\vartheta_\nu)$. Then, we get the uniqueness from the fact that (2.4) has a unique stochastic analytically strong solution. Therefore, we complete the proof of proposition 3.1. □

PROPOSITION 3.2. *Suppose the assumptions of proposition 3.1 are satisfied for $p = 1$ or some $p \in [2, \infty)$. Furthermore, we assume that $\mathbb{E}\|\varsigma\|^{2p} < \infty$, and (Υ_ν, G) and $(\tilde{\Upsilon}_\nu, \tilde{G})$ satisfy conditions 2.7, 2.11 and 2.12, respectively. Hence, for any fixed $M > 0$, $\hbar := (f, g) \in \tilde{\mathcal{U}}^M$, and for $\nu \in (0, \nu_0]$, there is a positive constant $C(p, M, T)$ such that the solution $u^\nu_\hbar$ of (2.5) satisfies*:

$$\mathbb{E}\left(\sup_{0\leqslant t\leqslant T} \|u^\nu_\hbar(t)\|^{2p} + \nu\int_0^T |Au^\nu_\hbar(s)|_H^2 \mathrm{d}s\right) \leqslant C(p,M,T)\big(1 + \mathbb{E}\|\varsigma\|^{2p}\big). \tag{3.6}$$

Proof. Denote $\xi^\nu_\hbar =$ curl $u^\nu_\hbar$. We know that $u^\nu_\hbar$ is a solution of the following elliptic problem (for details, see [**6**] and the references therein):

$$\begin{cases} -\Delta u^\nu_\hbar = \nabla^\perp \xi^\nu_\hbar & \text{in } D, \\ u^\nu_\hbar \cdot \mathbf{n} = \xi^\nu_\hbar = 0 & \text{on } \partial D, \end{cases} \tag{3.7}$$

where $\nabla^\perp = (\partial_2, -\partial_1)$. By (3.7), we have

$$-(\Delta u^\nu_\hbar, \Delta u^\nu_\hbar) = (\nabla^\perp \xi^\nu_\hbar, \quad \Delta u^\nu_\hbar) = -(\nabla^\perp \xi^\nu_\hbar,\ \nabla^\perp \xi^\nu_\hbar).$$

Then, by direct calculations, we can get

$$|\Delta u^\nu_\hbar|_H^2 = |\nabla^\perp \xi^\nu_\hbar|_H^2 = \|\partial_2 \xi^\nu_\hbar\|^2_{L^2(D)} + \|\partial_1 \xi^\nu_\hbar\|^2_{L^2(D)} = |\nabla \xi^\nu_\hbar|_H^2.$$

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In order to prove (3.6), by using (A.3), we only need to prove that there exists a constant $C(M,T) := C$ such that for any $\nu \in (0,\nu_0]$ and $\hbar = (f,g) \in \tilde{\mathcal{U}}^M$,

$$\mathbb{E}\left(\sup_{0\leqslant t\leqslant T} |\xi_\hbar^\nu(t)|_H^{2p} + \nu \int_0^T |\nabla \xi_\hbar^\nu(s)|_H^2 \mathrm{d}s\right) \leqslant C(1+\mathbb{E}|\ \mathrm{curl}\ \varsigma|_H^{2p}). \tag{3.8}$$

Let $\bar\tau_N = \inf\{t \geqslant 0 : |\xi_\hbar^\nu(t)|_H \geqslant N\} \wedge T$ for fixed $N > 0$. Applying the curl operator to (2.5), we obtain $\xi_\hbar^\nu(0) =\ \mathrm{curl}\ \varsigma$ and

$$\begin{aligned}
d\xi_\hbar^\nu(t) &= \nu A\xi_\hbar^\nu(t)\,\mathrm{d}t + \ \mathrm{curl} B(u_\hbar^\nu(t), u_\hbar^\nu(t))\,\mathrm{d}t \\
&\quad + \sqrt{\nu}\ \mathrm{curl}\ \Upsilon_\nu(s, u_\hbar^\nu(t))\mathrm{d}W(t) + \ \mathrm{curl}\ \tilde\Upsilon_\nu(s, u_\hbar^\nu(t)) f(t)\,\mathrm{d}t \\
&\quad + \nu \int_{\mathbb{Y}} \mathrm{curl} G(t, u_\hbar^\nu(t-), y)\tilde N^{\nu^{-1}}(\mathrm{d}y, \mathrm{d}t) \\
&\quad + \int_{\mathbb{Y}} \mathrm{curl}\ \tilde G(t, u_\hbar^\nu(t), y)(g(t,y)-1)\vartheta(\mathrm{d}y)\,\mathrm{d}t.
\end{aligned} \tag{3.9}$$

We note that for $u \in D(A)$, $(\ \mathrm{curl}\ B(u_\hbar^\nu, u_\hbar^\nu), \xi_\hbar^\nu) = 0$ by using (A.7) when $q = 2$. Employing Itô's formula to the function $|\xi_\hbar^\nu(s\wedge\bar\tau_N)|_H^{2p}, p \in [2,\infty)$, for $t \in [0,T]$, we obtain from (3.9)

$$|\xi_\hbar^\nu(s\wedge\bar\tau_N)|_H^{2p} + 2p\nu \int_0^{t\wedge\bar\tau_N} |\nabla\xi_\hbar^\nu(s)|_H^2 |\xi_\hbar^\nu(s)|_H^{2p-2}\mathrm{d}s = |\ \mathrm{curl}\varsigma|_H^{2p} + \sum_{i=1}^7 J_i(t),$$

where

$$\begin{aligned}
J_1(t) &= 2p\sqrt{\nu}\int_0^{t\wedge\bar\tau_N} |\xi_\hbar^\nu(s)|_H^{2p-2}\big(\ \mathrm{curl}\ \Upsilon_\nu(s, u_\hbar^\nu(s)), \xi_\hbar^\nu(s)\big)\,\mathrm{d}W(s), \\
J_2(t) &= \int_0^{t\wedge\bar\tau_N}\int_{\mathbb{Y}} \Big(|\xi_\hbar^\nu(s) + \nu\mathrm{curl} G(s, u_\hbar^\nu(s), y)|_H^{2p} - |\xi_\hbar^\nu(s)|_H^{2p}\Big)\,\tilde N^{\nu^{-1}}(\mathrm{d}y, \mathrm{d}s), \\
J_3(t) &= 2p\int_0^{t\wedge\bar\tau_N} |\xi_\hbar^\nu(s)|_H^{2p-2}\big(\ \mathrm{curl}\ \tilde\Upsilon_\nu(s, u_\hbar^\nu(s)) f(s), \xi_\hbar^\nu(s)\big)\,\mathrm{d}s, \\
J_4(t) &= \nu p(2p-1)\int_0^{t\wedge\bar\tau_N} |\xi_\hbar^\nu(s)|_H^{2p-2}|\ \mathrm{curl}\ \Upsilon_\nu(s, u_\hbar^\nu(s))|_{L_Q}^2\mathrm{d}s, \\
J_5(t) &= \int_0^{t\wedge\bar\tau_N}\int_{\mathbb{Y}} \Big(|\xi_\hbar^\nu(s) + \nu\mathrm{curl}\ G(s, u_\hbar^\nu(s), y)|_H^{2p} - |\xi_\hbar^\nu(s)|_H^{2p} \\
&\qquad - \nu p|\xi_\hbar^\nu(s)|_H^{2p-2}(\xi_\hbar^\nu(s), \mathrm{curl}\ G(s, u_\hbar^\nu(s), y))_H\Big)\vartheta(\mathrm{d}y)\,\mathrm{d}s, \\
J_6(t) &= 2p\int_0^{t\wedge\bar\tau_N}\int_{\mathbb{Y}} |\xi_\hbar^\nu(s)|_H^{2p-2}(\mathrm{curl}\ \tilde G(s, u_\hbar^\nu(s), y)(g(s,y)-1), \xi_\hbar^\nu(s))\vartheta(\mathrm{d}y)\,\mathrm{d}s.
\end{aligned}$$

For the term $J_1(t)$, it follows from the Burkholder–Davis–Gundy inequality, condition 2.11, (A.3) ($q = 2$), Cauchy–Schwarz's, Hölder's and Young's inequalities

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that

$$
\begin{aligned}
&\mathbb{E}\left(\sup_{s\in[0,t\wedge\bar{\tau}_N]} J_1(s)\right)\\
&\leqslant 2p\sqrt{\nu}\mathbb{E}\left(\int_0^{t\wedge\bar{\tau}_N} |\xi_\hbar^\nu(s)|_H^{4p-2}|\ \mathrm{curl}\ \Upsilon_\nu(u_\hbar^\nu(s)|_{L_Q}^2 \mathrm{d}s\right)^{\frac{1}{2}}\\
&\leqslant 2p\sqrt{\nu}\mathbb{E}\left(\int_0^{t\wedge\bar{\tau}_N} |\xi_\hbar^\nu(s)|_H^{4p-2}K(s)(1+\|u_\hbar^\nu(s)\|_V^2)\,\mathrm{d}s\right)^{\frac{1}{2}}\\
&\leqslant 2p\sqrt{\nu}\mathbb{E}\left(\int_0^{t\wedge\bar{\tau}_N} |\xi_\hbar^\nu(s)|_H^{4p-2}K(s)(1+|u_\hbar^\nu(s)|_H^2+C|\xi_\hbar^\nu(s)|_H^2 ds\right)^{\frac{1}{2}}\\
&\leqslant \frac{1}{4}\mathbb{E}\left(\sup_{s\in[0,t\wedge\tau_N]}|\xi_\hbar^\nu(s)|_H^{2p}\right)+2p\sqrt{\nu}C\mathbb{E}\left(\int_0^{t\wedge\bar{\tau}_N}|\xi_\hbar^\nu(s)|_H^{2p}K(s)\,\mathrm{d}s\right)\\
&\quad+2p\sqrt{\nu}C\mathbb{E}\left(1+\sup_{s\in[0,T]}|u_\hbar^\nu(s)|_H^{2p}\right)\left(\int_0^{t\wedge\bar{\tau}_N}K(s)\,\mathrm{d}s\right)^{\frac{1}{2}}.
\end{aligned}
$$

For the term $J_2(t)$, by applying the Burkholder–Davis–Gundy inequality, condition 2.11, Hölder's and Young's inequalities, we obtain

$$
\begin{aligned}
&\mathbb{E}\left(\sup_{s\in[0,t\wedge\bar{\tau}_N]} J_2(s)\right)\\
&\leqslant \mathbb{E}\left[\int_0^{t\wedge\bar{\tau}_N}\int_{\mathbb{Y}}\left(|\xi_\hbar^\nu(s)+\nu\mathrm{curl}\ G(s,u_\hbar^\nu(s),y)|_H^{2p}-|\xi_\hbar^\nu(s)|_H^{2p}\right)^2\vartheta(\mathrm{d}y)\,\mathrm{d}s\right]^{\frac{1}{2}}\\
&\leqslant C(\nu)\mathbb{E}\Bigg(\int_0^{t\wedge\bar{\tau}_N}\int_{\mathbb{Y}}|\xi_\hbar^\nu(s)|_H^{4p-2}|\mathrm{curl}\ G(s,u_\hbar^\nu(s),y)|_H^2\\
&\quad+|\xi_\hbar^\nu(s)|_H^{4p-4}|\mathrm{curl}\ G(s,u_\hbar^\nu(s),y)|_H^4+|\mathrm{curl}\ G(s,u_\hbar^\nu(s),y)|_H^{4p}\vartheta(\mathrm{d}y)\,\mathrm{d}s\Bigg)^{\frac{1}{2}}\\
&\leqslant C(\nu)\mathbb{E}\left(\int_0^{t\wedge\bar{\tau}_N}K(s)|\xi_\hbar^\nu(s)|_H^{4p}\mathrm{d}s\right)^{\frac{1}{2}}+C(\nu)\mathbb{E}\left(\int_0^{t\wedge\bar{\tau}_N}K(s)\,\mathrm{d}s\right)^{\frac{1}{2}}\\
&\leqslant \frac{1}{4}\mathbb{E}\Big(\sup_{s\in[0,t\wedge\bar{\tau}_N]}|\xi_\hbar^\nu(s)|_H^{2p}\Big)+C(\nu)\mathbb{E}\int_0^{t\wedge\tau_N}K(s)|\xi_\hbar^\nu(s)|_H^{2p}\mathrm{d}s\\
&\quad+C(\nu)\mathbb{E}\left(\int_0^{t\wedge\bar{\tau}_N}K(s)\,\mathrm{d}s\right)^{\frac{1}{2}}.
\end{aligned}
$$

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For the term $J_3(t)$, we use Cauchy-Schwarz's inequality, condition 2.12, (A.3) with $q=2$, Hölder's and Young's inequalities to get

$$\begin{aligned}\mathbb{E}J_3(t) &\leqslant 2p\mathbb{E}\int_0^{t\wedge\bar{\tau}_N} |\xi_\hbar^\nu(s)|_H^{2p-1}|\operatorname{curl}\tilde{\Upsilon}_\nu(s,u_\hbar^\nu(s))|_{L(H_0,H)}|f(s)|_0\mathrm{d}s\\ &\leqslant 2p\mathbb{E}\int_0^{t\wedge\bar{\tau}_N} \left(\sqrt{K(s)}+C\sqrt{K(s)}|\xi^\nu(s)|_H^{2p}+\sqrt{K(s)}|u_\hbar^\nu(s)|_H|\xi_\hbar^\nu(s)|_H^{2p-1}\right)\\ &\quad\times|f(s)|_0\mathrm{d}s\\ &\leqslant 2p\mathbb{E}\int_0^{t\wedge\bar{\tau}_N} \left(1+|\xi_\hbar^\nu(s)|_H^{2p}+|u_\hbar^\nu(s)|_H|\xi_\hbar^\nu(s)|_H^{2p-1}\right)\left(K(s)+|f(s)|_0^2\right)\mathrm{d}s\\ &\leqslant 2pC\mathbb{E}\int_0^{t\wedge\bar{\tau}_N} \left(K(s)+|f(s)|_0^2\right)\left(1+\sup_{s\in[0,T]}|u_\hbar^\nu(s)|_H^{2p}+|\xi_\hbar^\nu(s)|_H^{2p}\right)\mathrm{d}s.\end{aligned}$$

For the term $J_4(t)$, condition 2.11, (A.3) with $q=2$, Hölder's and Young's inequalities yield that for $\nu\in(0,\nu_0]$

$$\begin{aligned}\mathbb{E}J_4(t) &\leqslant \nu p(2p-1)\mathbb{E}\int_0^{t\wedge\bar{\tau}_N} |\xi_\hbar^\nu(s)|_H^{2p-2}K(s)\left(1+|u_\hbar^\nu(s)|_H^2+C|\xi_\hbar^\nu(s)|_H^2\right)\mathrm{d}s\\ &\leqslant \nu(2p-1)\mathbb{E}\int_0^{t\wedge\bar{\tau}_N} K(s)\left(1+\sup_{s\in[0,T]}|u_\hbar^\nu(s)|_H^{2p}+|\xi_\hbar^\nu(s)|_H^{2p}\right)\mathrm{d}s.\end{aligned}$$

For the term $J_5(t)$, it follows from condition 2.11 and Hölder's inequality that

$$\begin{aligned}\mathbb{E}J_5(t) &\leqslant C\mathbb{E}\Bigg[\int_0^{t\wedge\bar{\tau}_N}\int_{\mathbb{Y}}\left(|\xi_\hbar^\nu(s)|_H^{2p-2}+|\nu\operatorname{curl} G(s,u_\hbar^\nu(s),y)|_H^{2p-2}\right)\\ &\quad\times|\operatorname{curl} G(s,u_\hbar^\nu(s),y)|_H^2\vartheta(\mathrm{d}y)\,\mathrm{d}s\Bigg]\\ &\leqslant C(\nu)\mathbb{E}\Bigg(\int_0^{t\wedge\bar{\tau}_N}\int_{\mathbb{Y}}|\xi_\hbar^\nu(s)|_H^{2p-2}|\operatorname{curl}G(s,u_\hbar^\nu(s),y)|_H^2\\ &\quad+|\operatorname{curl} G(s,u_\hbar^\nu(s),y)|_H^{2p}\vartheta(\mathrm{d}y)\,\mathrm{d}s\Bigg)\\ &\leqslant C(\nu)\mathbb{E}\int_0^{t\wedge\bar{\tau}_N}K(s)|\xi_\hbar^\nu(s)|_H^{2p}\mathrm{d}s+C(\nu)\mathbb{E}\int_0^{t\wedge\bar{\tau}_N}K(s)\,\mathrm{d}s.\end{aligned}$$

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For the term $J_6(t)$, condition 2.12, Hölder's and Young's inequalities imply that

$$\begin{aligned}
&\mathbb{E}\int_0^{t\wedge\bar{\tau}_N}\int_{\mathbb{Y}}|\xi_\hbar^\nu(s)|_H^{2p-2}(\mathrm{curl}\tilde{G}(s,u_\hbar^\nu(s),y)(g(s,y)-1),\xi_\hbar^\nu(s))\vartheta(\mathrm{d}y)\,\mathrm{d}s\\
&\quad\leqslant C\mathbb{E}\int_0^{t\wedge\bar{\tau}_N}\int_{\mathbb{Y}}|\xi_\hbar^\nu(s)|_H^{2p-2}|\mathrm{curl}\tilde{G}(s,u_\hbar^\nu(s),y)|_H|g(s,y)-1||\xi_\hbar^\nu(s)|_H\vartheta(\mathrm{d}y)\,\mathrm{d}s\\
&\quad\leqslant C\mathbb{E}\int_0^{t\wedge\bar{\tau}_N}\int_{\mathbb{Y}}|\xi_\hbar^\nu(s)|_H^{2p-2}\frac{|\mathrm{curl}\tilde{G}(s,u_\hbar^\nu(s),y)|_H}{1+|\xi_\hbar^\nu(s)|_H}\\
&\qquad\times|g(s,y)-1|(1+|\xi_\hbar^\nu(s)|_H^2)\vartheta(\mathrm{d}y)\,\mathrm{d}s\\
&\quad\leqslant C\mathbb{E}\int_0^{t\wedge\bar{\tau}_N}\int_{\mathbb{Y}}|\xi_\hbar^\nu(s)|_H^{2p-2}\frac{|\tilde{G}(s,\xi_\hbar^\nu(s),y)|_H}{1+|\xi_\hbar^\nu(s)|_H}|g(s,y)-1|(1+|\xi_\hbar^\nu(s)|_H^2)\vartheta(\mathrm{d}y)\,\mathrm{d}s\\
&\quad\leqslant C\mathbb{E}\int_0^{t\wedge\bar{\tau}_N}\int_{\mathbb{Y}}|\xi_\hbar^\nu(s)|_H^{2p-2}|\tilde{G}(s,y)|_{0,H}|g(s,y)-1|(1+|\xi_\hbar^\nu(s)|_H^2)\vartheta(\mathrm{d}y)\,\mathrm{d}s\\
&\quad\leqslant C\mathbb{E}\int_0^{t\wedge\bar{\tau}_N}\int_{\mathbb{Y}}\|\tilde{G}(s,y)\|_{0,H}|g(s,y)-1|\vartheta(\mathrm{d}y)\,\mathrm{d}s\\
&\qquad+C\mathbb{E}\int_0^{t\wedge\bar{\tau}_N}|\xi_\hbar^\nu(s)|_H^{2p}\int_{\mathbb{Y}}\|\tilde{G}(s,y)\|_{0,H}|g(s,y)-1|\vartheta(\mathrm{d}y)\,\mathrm{d}s.
\end{aligned}$$

Then, combining the estimates of J_1-J_6, we obtain

$$\begin{aligned}
&\mathbb{E}\Big(\sup_{s\in[0,t]}|\xi_\hbar^\nu(s\wedge\bar{\tau}_N)|_H^{2p}\Big)+2p\nu\mathbb{E}\int_0^{t\wedge\bar{\tau}_N}|\nabla\xi_\hbar^\nu(s)|_H^2|\xi_\hbar^\nu(s)|_H^{2p-2}\mathrm{d}s\\
&\quad\leqslant\mathbb{E}\Big(|\ \mathrm{curl}\varsigma|_H^{2p}\Big)+C(\nu)\mathbb{E}\int_0^{t\wedge\bar{\tau}_N}\Big(K(s)+|f(s)|_0^2\Big)|\xi_\hbar^\nu(s)|^{2p}\\
&\qquad+C(\nu)\mathbb{E}\left(\int_0^{t\wedge\bar{\tau}_N}K(s)\,\mathrm{d}s\right)^{\frac{1}{2}}\\
&\qquad+C(\nu)\mathbb{E}\int_0^{t\wedge\bar{\tau}_N}\Big(K(s)+|f(s)|_0^2\Big)\left(1+\sup_{s\in[0,T]}|u_\hbar^\nu(s)|_H^{2p}\right)\mathrm{d}s\\
&\qquad+C\mathbb{E}\int_0^{t\wedge\bar{\tau}_N}|\xi_\hbar^\nu(s)|_H^{2p}\int_{\mathbb{Y}}\|\tilde{G}(s,y)\|_{0,H}|g(s,y)-1|\vartheta(\mathrm{d}y)\,\mathrm{d}s\\
&\qquad+C\mathbb{E}\int_0^{t\wedge\bar{\tau}_N}\int_{\mathbb{Y}}\|\tilde{G}(s,y)\|_{0,H}|g(s,y)-1|\vartheta(\mathrm{d}y)\,\mathrm{d}s\\
&\qquad+C(\nu)\mathbb{E}\Big(1+\sup_{s\in[0,T]}|u_\hbar^\nu(s)|_H^{2p}\Big)\left(\int_0^{t\wedge\bar{\tau}_N}K(s)\,\mathrm{d}s\right)^{\frac{1}{2}}.
\end{aligned}$$

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By using Gronwall's inequality, lemma 2.9 and proposition 3.1, we can infer that

$$\mathbb{E}\left(\sup_{s\in[0,t\wedge\bar{\tau}_N)]}|\xi^\nu_\hbar(s|^{2p}_H\right)+2p\nu\mathbb{E}\int_0^{t\wedge\bar{\tau}_N}|\nabla\xi^\nu_\hbar(s)|^2_H|\xi^\nu_\hbar(s)|^{2p-2}_H\mathrm{d}s \tag{3.1}$$

$$\leqslant C(1+\mathbb{E}|\operatorname{curl}\varsigma|^{2p}_H). \tag{3.10}$$

From the fact that estimate (3.10) is uniform in N, we can deduce that $\bar{\tau}_N \to T$ as $N \to \infty$. This together with the monotone convergence theorem yields that (3.8). Then, the proof of proposition 3.2 is completed. □

Combining propositions 3.1 and 3.2, we obtain the following well-posedeness result for the stochastic equations (2.5).

THEOREM 3.3. *Let $p \in [2,\infty)$ be such that $\mathbb{E}(\|\varsigma\|^{2p}) < \infty$. Assume that (Υ_ν, G) satisfies conditions 2.5, 2.11 and $(\tilde{\Upsilon}_\nu, \tilde{G})$ satisfies conditions 2.6, 2.7 and 2.12. For every $M > 0$, $\hbar := (f,g) \in \tilde{\mathcal{U}}^M$ and $\nu \in (0,\nu_0]$, there exists a unique stochastic strong analytically weak solution $u^\nu_\hbar \in \mathcal{D}([0,T];H) \cap L^2(0,T;V)$ to (2.5) with deterministic initial data $u^\nu_\hbar(0) = \varsigma \in V$. Moreover, $u^\nu_\hbar \in \mathcal{D}([0,T];V)$ $\mathbb{P}$-a.s. and satisfies inequalities (3.1), (3.2) and (3.6).*

4. Existence and uniqueness of solutions for the inviscid problem

In this section, we study the inviscid problem when $\nu = 0$. More precisely, we consider the following Euler equations in $[0,T] \times D$:

$$\begin{cases} \mathrm{d}u^0_\hbar(t) + B(u^0_\hbar(t), u^0_\hbar(t))\,\mathrm{d}t = \tilde{\Upsilon}_0(t, u^0_\hbar(t))f(t)\,\mathrm{d}t \\ \quad + \displaystyle\int_{\mathbb{Y}} \tilde{G}(t, u^0_\hbar(t), y)(g(t,y)-1)\vartheta(\mathrm{d}y)\,\mathrm{d}t, \\ u^0_\hbar(0) = \varsigma. \end{cases} \tag{4.1}$$

We have the following theorem.

THEOREM 4.1. *Suppose that $\varsigma \in V$ and that $(\tilde{\Upsilon}_0, \tilde{G})$ satisfies conditions 2.6, 2.7 and 2.12. For all $M > 0$, $\hbar = (f,g) \in \bar{S}^M$ and $T > 0$, there exists a weak solution $u^0_\hbar \in C([0,T];H) \cap L^\infty(0,T;V)$ for (4.1) such that for $t \in [0,T]$ and all $\varphi \in V$, it holds that*

$$\left(u^0_\hbar(t),\varphi\right) = \int_0^t \langle B(u^0_\hbar(s),u^0_\hbar(s)),\varphi\rangle\mathrm{d}s + \int_0^t \left(\tilde{\Upsilon}_0(s,u^0_\hbar(s))f(s),\varphi\right)\mathrm{d}s$$
$$+\int_0^t\int_{\mathbb{Y}}(\tilde{G}(t,u^0_\hbar(s),y)(g(s,y)-1),\varphi)\vartheta(\mathrm{d}y)\,\mathrm{d}s.$$

Furthermore, there is a positive constant $C(M,T)$ such that for every $\hbar = (f,g) \in \bar{S}^M$, it holds that

$$\sup_{0\leqslant t\leqslant T}\|u^0_\hbar(t)\| \leqslant C(M,T)(1+\|\varsigma\|). \tag{4.2}$$

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Proof. We consider the following Navier–Stokes equations:

$$\begin{aligned} \mathrm{d}u_\hbar^{0\nu}(t) = -\big[\nu A u_\hbar^{0\nu}(t) + B(u_\hbar^{0\nu}(t), u_\hbar^{0\nu}(t))\big]\,\mathrm{d}t + \tilde{\Upsilon}_0(t, u_\hbar^{0\nu}(t)) f(t)\,\mathrm{d}t \\ + \int_{\mathbb{Y}} \tilde{G}(t, u_\hbar^{0\nu}(t), y)(g(t,y)-1)\vartheta(\mathrm{d}y)\,\mathrm{d}t \end{aligned} \tag{4.3}$$

with $\nu > 0$, the initial data $u_\hbar^{0\nu}(0) = \varsigma$, $\nabla \cdot u_\hbar^{0\nu} = 0$ and (1.3). When $\varsigma \in H$, $(\tilde{\Upsilon}_0, \tilde{G})$ satisfies conditions 2.6, 2.7 and $\hbar = (f,g) \in \bar{S}^M$ with $M > 0$, similar to the proof of proposition 3.1, we can conclude that (4.3) has a unique weak solution $u_\hbar^{0\nu} \in \mathcal{C}([0,T]; H) \cap L^2(0,T;V)$. Furthermore, when $\varsigma \in V$ and $(\tilde{\Upsilon}_0, \tilde{G})$ satisfies conditions 2.7 and 2.12, similar to the proof of proposition 3.2, we can infer that $u_\hbar^{0\nu} \in \mathcal{C}([0,T]; V)$.

By taking $\nu \to 0$, we need to get some uniform estimates of $u_\hbar^{0\nu}$ about $\nu > 0$ for proving the existence of solutions to (4.1). Similar to the proof of proposition 3.1, multiplying (4.3) by $2u_\hbar^{0\nu}$ and integrating over $[0,t] \times D$ and applying (A.5), Hölder's and Young's inequalities, condition 2.6, we obtain for every $\nu > 0$

$$\begin{aligned} &|u_\hbar^{0\nu}(t)|_H^2 + 2\nu \int_0^t \|u_\hbar^{0\nu}(s)\|^2 \mathrm{d}s \\ &\leqslant |\varsigma|_H^2 + 2\int_0^t \big(\tilde{\Upsilon}_0(s, u_\hbar^{0\nu}(s)) f(s), u_\hbar^{0\nu}(s)\big)\,\mathrm{d}s \\ &\quad + 2\int_0^t \int_{\mathbb{Y}} (\tilde{G}(t, u_\hbar^{0\nu}(s), y)(g(s,y)-1), u_\hbar^{0\nu}(s))\vartheta(\mathrm{d}y)\,\mathrm{d}s \\ &\leqslant |\varsigma|_H^2 + 2\int_0^t K(s)\,\mathrm{d}s + 2\int_0^t \Big(K(s) + |f(s)|_0^2\Big)|u_\hbar^{0\nu}(s)|_H^2 \mathrm{d}s \\ &\quad + C\int_0^t \int_{\mathbb{Y}} \|\tilde{G}(s,y)\|_{0,H}|g(s,y)-1|\vartheta(\mathrm{d}y)\,\mathrm{d}s \\ &\quad + C\int_0^t |u_\hbar^{0\nu}(s)|_H^2 \int_{\mathbb{Y}} \|\tilde{G}(s,y)\|_{0,H}|g(s,y)-1|\vartheta(\mathrm{d}y)\,\mathrm{d}s. \end{aligned}$$

Then, using condition 2.7, remark 2.8, lemma 2.9 and Gronwall's inequality, we deduce that there exists a constant C such that

$$\sup_{\nu>0} \sup_{0 \leqslant t \leqslant T} |u_\hbar^{0\nu}(t)|_H^2 \leqslant C(M,T)(1 + |\varsigma|_H^2). \tag{4.4}$$

Denote $\xi_\hbar^{0\nu}(t) :=$ curl $u_\hbar^{0\nu}(t)$. Similar to the proof of proposition 3.2, employing the curl operator to (4.3) and applying (A.6), we get

$$\begin{aligned} d\xi_\hbar^{0\nu}(t) = -\nu A \xi_\hbar^{0\nu}(t) - B(u_\hbar^{0\nu}(t), \xi_\hbar^{0\nu}(t))\,\mathrm{d}t + \operatorname{curl} \tilde{\Upsilon}_0(t, u_\hbar^{0\nu}(t)) f(t)\,\mathrm{d}t \\ + \int_{\mathbb{Y}} \operatorname{curl} \tilde{G}(t, u_\hbar^{0\nu}(t), y)(g(t,y)-1)\vartheta(\mathrm{d}y)\,\mathrm{d}t \end{aligned} \tag{4.5}$$

with the initial data $\xi_\hbar^{0\nu}(0) =$ curl ς.

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Multiplying (4.5) by $2\xi_\hbar^{0\nu}$ and integrating over $[0,T]\times D$ and using condition 2.12, (A.7) ($q=2$), (A.3), Hölder's and Young's inequalities, one deduces that

$$
\begin{aligned}
&|\xi_\hbar^{0\nu}(t)|_H^2 + 2\nu\int_0^t \|\xi_\hbar^{0\nu}(s)\|^2\mathrm{d}s \\
&\quad\leqslant |\operatorname{curl}\varsigma|_H^2 + 2\int_0^t |\operatorname{curl}\tilde{\Upsilon}_0(s,u_\hbar^{0\nu}(s))|_{L(H_0,H)}|f(s)|_0|\xi_\hbar^{0\nu}(s)|_H\mathrm{d}s \\
&\qquad + 2\int_0^t\int_{\mathbb{Y}} |\operatorname{curl}\tilde{G}(s,u_\hbar^{0\nu}(s),y)|_H|g(s,y)-1||\xi_\hbar^{0\nu}(s)|_H\vartheta(\mathrm{d}y)\,\mathrm{d}s \\
&\quad\leqslant \|\varsigma\|^2 + \Big(1+\sup_{s\in[0,T]}|u_\hbar^{0\nu}(s)|_H^2\Big)\int_0^t\Big(K(s)+|f(s)|_0^2\Big)\,\mathrm{d}s \\
&\qquad + 2\int_0^t\big(K(s)+|f(s)|_0^2\big)\,|\xi_\hbar^{0\nu}(s)|_H^2\mathrm{d}s \\
&\qquad + C\int_0^t\int_{\mathbb{Y}}\|\tilde{G}(s,y)\|_{0,H}|g(s,y)-1|\vartheta(\mathrm{d}y)\,\mathrm{d}s \\
&\qquad + C\int_0^t|\xi_\hbar^{0\nu}(s)|_H^2\int_{\mathbb{Y}}\|\tilde{G}(s,y)\|_{0,H}|g(s,y)-1|\vartheta(\mathrm{d}y)\,\mathrm{d}s.
\end{aligned}
$$

It follows from condition 2.7, remark 2.8, lemma 2.9, (4.4) and Gronwall's inequality that there exists a constant C such that for every $\hbar=(f,g)\in\bar{S}^M$, it holds that

$$
\sup_{\nu>0}\sup_{0\leqslant t\leqslant T}|\xi_\hbar^{0\nu}(t)|_H^2 \leqslant C(M,T)(1+\|\varsigma\|^2). \tag{4.6}
$$

This together with (4.4) and (A.3) yields that

$$
\sup_{\nu>0}\sup_{0\leqslant t\leqslant T}\|u_\hbar^{0\nu}\| \leqslant C(M,T)(1+\|\varsigma\|). \tag{4.7}
$$

Moreover, for every $\nu>0$, we can infer that $u_\hbar^{0\nu}\in\mathcal{C}([0,T];H)\cap L^\infty(0,T;V)$ and it holds that

$$
\begin{aligned}
u_\hbar^{0\nu}(t) = \varsigma - \nu\int_0^t Au_\hbar^{0\nu}(s) - \int_0^t B(u_\hbar^{0\nu}(s),u_\hbar^{0\nu}(s))\,\mathrm{d}s + \int_0^t \tilde{\Upsilon}_0(s,u_\hbar^{0\nu}(s))f(s)\,\mathrm{d}s \\
+ \int_0^t\int_{\mathbb{Y}}\tilde{G}(s,u_\hbar^{0\nu}(s),y)(g(s,y)-1)\vartheta(\mathrm{d}y)\,\mathrm{d}s.
\end{aligned}
$$

By applying (4.4), (4.7), conditions 2.6, 2.7 and 2.12 on $(\tilde{\Upsilon}_0,\tilde{G})$ and (A.4) when $q=2$ and $r=1$, remark 2.8, lemma 2.9, one deduces that for any $\nu\in(0,1]$ and $\hbar=(f,g)\in\bar{S}^M$:

$$
\|u_\hbar^{0\nu}\|_{W^{1,2}(0,T;V')} \leqslant C(1+\|\varsigma\|).
$$

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Then, there exists a function $v \in W^{1,2}(0,T;V') \cap L^\infty(0,T;V)$ such that

$$\begin{aligned} &u_\hbar^{0\nu} \to v \text{ weakly in } L^2\big(0,T;V\big) \cap W^{1,2}\big(0,T;V'\big),\\ &u_\hbar^{0\nu} \to v \text{ strongly in } L^2\big(0,T;H\big),\\ &u_\hbar^{0\nu} \to v \text{ weak star in } L^\infty\big(0,T;V\big),\end{aligned}$$

as $\nu \to 0$. Taking $\nu \to 0$ in (4.3), we conclude that $u_\hbar^0 := v$ is a solution of (4.1) as [**52**, theorem 3.1] (see also [**60**]). Moreover, since (4.7) is uniformly bounded in $\nu > 0$, taking $\nu \to 0$ in (4.7), we can get (4.2). □

As far as we know, it is difficult to prove the uniqueness of solution to the Euler equations in Hilbert–Sobolev spaces. But scholars get the uniqueness of solution in the non-Hilbert–Sobolev spaces $H^{1,q}, q \in [2,+\infty)$ for the deterministic case. Inspired by them such as [**8**], for proving uniqueness, then we require the coefficient $(\tilde{\Upsilon}_\nu, \tilde{G})$ to meet some conditions (see conditions 4.2 and 4.3).

For $k \geqslant 0$, $q \in [2,\infty)$, we denote $R(H_0, W^{k,q})$ by the space of all γ-radonifying mappings from H_0 into $W^{k,q}$. We note that $R(H_0, W^{k,q})$ are analogues of Hilbert–Schmidt operators when $W^{k,2}$ is replaced by the general Banach spaces $W^{k,q}$. We refer to [**10**, **12**, **26**, **43**, **44**] for the definitions and some basic properties of stochastic calculus in the Banach spaces.

CONDITION 4.2. *Let $q \in [2,\infty)$; $\Upsilon_\nu \in \mathcal{C}([0,T] \times H^{2,q}; R(H_0, H^{1,q}))$ for $\nu > 0$ and $G \in \mathcal{C}([0,T] \times H^{2,q} \times L^q(\mathbb{Y}); H^{1,q})$, there exists $K(t) \in L^1([0,T],\mathbb{R}^+)$ such that for every $u \in H \cap H^{2,q}$ and $\nu > 0$, if $\xi =$ curl u, it holds*

$$\begin{aligned} \|\textit{curl } \Upsilon_\nu(t,u)\|^2_{R(H_0,L^q)} &\leqslant K(t)(1 + \|u\|_q^2 + \|\xi\|_q^2),\\ \int_{\mathbb{Y}} \|\textit{curl } G(t,u(t),y)\|^q_{L^q} \vartheta(\mathrm{d}y) &\leqslant \int_{\mathbb{Y}} \|G(t,\textit{curl } u(t),y)\|^q_{L^q}\vartheta(\mathrm{d}y)\\ &\leqslant K(t)(1 + \|\textit{curl } u\|_q^q).\end{aligned}$$

CONDITION 4.3. *Let $q \in [2,\infty)$; $\tilde{\Upsilon}_\nu \in \mathcal{C}([0,T] \times H^{1,q}; L(H_0, H^{1,q}))$ for $\nu \geqslant 0$, $\tilde{G} \in \mathcal{C}([0,T] \times H^{1,q} \times L^1(\mathbb{Y}); H^{1,q})$, and there exists $K(t) \in L^1([0,T],\mathbb{R}^+)$ such that for every $u \in H^{1,q}$ and $\nu > 0$ (resp. $u \in H^{2,q}$ for $\nu = 0$) if $\xi =$ curl u, it holds*

$$\begin{aligned} \| \textit{ curl } \tilde{\Upsilon}_\nu(t,u)\|_{L(H_0,L^q)} &\leqslant \sqrt{K(t)}(1 + \|u\|_q + \|\xi\|_q),\\ \int_{\mathbb{Y}} \|\textit{curl } \tilde{G}(t,u(t),y)\|_q \vartheta(\mathrm{d}y) &\leqslant \int_{\mathbb{Y}} \|\tilde{G}(t,\textit{curl } u(t),y)\|_q \vartheta(\mathrm{d}y).\end{aligned}$$

Note that examples of Nemytski operators in [**12**] for $\Upsilon_\nu, \tilde{\Upsilon}_\nu$ and $G(t,u(t),y), \tilde{G}(t,u(t),y) = \kappa(t)u(t) + \iota(t)\Gamma(y)$, where $\kappa(t), \iota(t) \in L^1([0,T],\mathbb{R}^+)$, $\Gamma(y) \in L^q(\mathbb{Y})$, $\int_{\mathbb{Y}} \vartheta(\mathrm{d}y) < C$, satisfy the above conditions. If curl ς is bounded, we have the following theorem.

THEOREM 4.4. *Suppose the hypotheses of theorem 4.1 are satisfied. Moreover, we assume that curl $\varsigma \in (L^\infty(D))^2$ and that condition 4.3 holds. For every $M > 0$*

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and $\hbar = (f,g) \in \bar{S}^M$, *then the weak solution of* (4.1) *is unique in* $\mathcal{C}([0,T];H) \cap L^\infty(0,T;H^{1,q}), q \in [2,\infty)$. *Moreover, we have*

$$\sup_{0\leqslant t\leqslant T} \| \textit{ curl } u_\hbar^0(t\|_q \leqslant C(1 + \|\varsigma\| + \| \textit{ curl } \varsigma\|_q), \tag{4.8}$$

$$\sup_{0\leqslant t\leqslant T} \|\nabla u_\hbar^0(t)\|_q \leqslant Cq(1 + \|\varsigma\| + \| \textit{ curl } \varsigma\|_q). \tag{4.9}$$

Here $C > 0$ *is a constant, which depends on* M, T, *and* $\|\varsigma\|_{L^\infty(D;\mathbb{R}^2)}$.

Proof. First, we verify the estimates (4.8) and (4.9). Thanks to (A.3), we only need to check the $L^q(D)$ upper bounds of curl $u_\hbar^0(t)$. Now, we establish the a priori estimates of the Galerkin approximation solution $u_{\hbar,n}^0$, and assume that $u_{\hbar,n}^0 \in H^{2,q}$. To simplify the notation, we drop the index n.

Denote $\xi_\hbar^0(t) :=$ curl $u_\hbar^0(t)$. Applying the curl operator to (4.1) and using (A.6), we have

$$\begin{aligned} d\xi_\hbar^0(t) &= -B\big(u_\hbar^0(t), \xi_\hbar^0(t)\big)\,\mathrm{d}t + \text{ curl } \tilde{\Upsilon}_0(t, u_\hbar^0(t))f(t)\,\mathrm{d}t \\ &\quad + \int_{\mathbb{Y}} \text{curl } \tilde{G}(t, u_\hbar^0(t), y)(g(t,y) - 1)\vartheta(\mathrm{d}y)\,\mathrm{d}t,\ \xi_\hbar^0(0) = \text{ curl } \varsigma. \end{aligned} \tag{4.10}$$

Multiplying (4.10) by $q|\xi_\hbar^0(t)|^{q-2}\xi_\hbar^0(t)$ and integrating over $[0,t]\times D$, we obtain

$$\begin{aligned} &\|\xi_\hbar^0(t)\|_q^q + q\int_0^t \int_D (u_\hbar^0(s)\cdot\nabla)\xi_\hbar^0(s)|\xi_\hbar^0(s)|^{q-2}\xi_\hbar^0(s)dxds \\ &\quad = \| \text{ curl } \varsigma\|_q^q + q\int_0^t\int_D \text{ curl } \tilde{\Upsilon}_0(s, u_\hbar^0(s))f(s)|\xi_\hbar^0(s)|^{q-2}\xi_\hbar^0(s)\mathrm{d}x\mathrm{d}s \\ &\qquad + q\int_0^t\int_D\int_{\mathbb{Y}} \text{ curl } \tilde{G}(t, u_\hbar^0(t), y)(g(t,y) - 1)|\xi_\hbar^0(t)|^{q-2}\xi_\hbar^0(t)\vartheta(\mathrm{d}y)dxds. \end{aligned}$$

We note that for every s, $\int_D (u_\hbar^0(s)\cdot\nabla)\xi_\hbar^0(s)|\xi_\hbar^0(s)|^{q-2}\xi_\hbar^0(s)\mathrm{d}x = 0$ by using the fact that $\xi_\hbar^0(t) =$ curl $u_\hbar^0(t)$, and (A.7). Then, by using Hölder's and Young's inequalities and condition 4.3, one has

$$\begin{aligned} \|\xi_\hbar^0(t)\|_q^q &\leqslant \| \text{ curl } \varsigma\|_q^q + q\int_0^t \| \text{ curl } \tilde{\Upsilon}_\nu(s, u_\hbar^0(s))\|_{L(H_0,L^q)}|f(s)|_0\|\xi_\hbar^0(s)\|_q^{q-1}\mathrm{d}s \\ &\quad + q\int_0^t \int_D\int_{\mathbb{Y}} \text{ curl } \tilde{G}(t, u_\hbar^0(t), y)(g(t,y) - 1)|\xi_\hbar^0(t)|^{q-2}\xi_\hbar^0(t)\vartheta(\mathrm{d}y)dxds \\ &\leqslant \| \text{ curl } \varsigma\|_q^q + q\Big(1 + \sup_{0\leqslant s\leqslant T}\|u_\hbar^0(s)\|_q^q\Big)\int_0^t \left(K(s) + |f(s)|_0^2\right)\,\mathrm{d}s \\ &\quad + q\int_0^t \left(K(s) + |f(s)|_0^2\right)\|\xi_\hbar^0(s)\|_q^q ds \\ &\quad + C\int_0^t\int_{\mathbb{Y}} \|\tilde{G}(s,y)\|_{0,q}|g(s,y) - 1|\vartheta(\mathrm{d}y)\,\mathrm{d}s \\ &\quad + C\int_0^t \|\xi_\hbar^0(s)\|_q^q\int_{\mathbb{Y}} \|\tilde{G}(s,y)\|_{0,q}|g(s,y) - 1|\vartheta(\mathrm{d}y)\,\mathrm{d}s. \end{aligned}$$

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From the fact that $H^{1,2} \hookrightarrow L^q(D)$, (4.2), lemma 2.9 and Gronwall's inequality, we conclude that for any $n \geqslant 1$ and $\hbar = (f, g) \in \bar{S}^M$

$$\sup_{0 \leqslant t \leqslant T} \|\xi^0_{\hbar,n}(t)\|^q_q \leqslant C\Big(\| \operatorname{curl} \varsigma\|^q_q + 1 + \|\varsigma\|^q\Big). \tag{4.11}$$

Note that the upper bounds of inequality 4.11 do not depend on n. We can take $n \to \infty$ in (4.11) and use classical arguments to get (4.9). For more details, see e.g. [8].

Next, we prove the uniqueness of the solution $u^0_\hbar$. By using (4.9) for some $q > 2$ and (A.2), we can infer that any solution $u^0_\hbar$ to (4.1) belongs to $L^\infty((0, T) \times D)$.

Let $u^0_\hbar$ and $v^0_\hbar$ be two solutions of (4.1) with the same initial data. Denote $\varpi := u^0_\hbar - v^0_\hbar$. Then ϖ is a weak solution of the following equations:

$$\begin{aligned} d\varpi(s) = &- \big[B(u^0_\hbar(s), u^0_\hbar(s)) - B(v^0_\hbar(s), v^0_\hbar(s))\big] \,\mathrm{d}s \\ &+ \Big[\tilde{\Upsilon}_0(s, u^0_\hbar(s)) - \tilde{\Upsilon}_0(s, v^0_\hbar(s))\Big] f(s) \,\mathrm{d}s \\ &+ \int_{\mathbb{Y}} \Big[\tilde{G}(s, u^0_\hbar(s), y) - \tilde{G}(s, v^0_\hbar(s), y)\Big] (g(s, y) - 1)\vartheta(\mathrm{d}y) \,\mathrm{d}s, \quad \varpi(0) = 0. \end{aligned}$$

Multiplying the above equations by $\varpi(t)$ and integrating on D, and using condition 2.6 on $(\tilde{\Upsilon}_0, \tilde{G})$, Schwarz's and Hölder's inequalities and (4.9), we obtain for any $q \in (1, \infty)$

$$\begin{aligned} &\frac{1}{2}\frac{d}{\mathrm{d}t}|\varpi(t)|^2_H \\ &\quad = -(B(\varpi(t), u^0_\hbar(t)), \varpi(t)) + \Big(\Big[\tilde{\Upsilon}_0(t, u^0_\hbar(t)) - \tilde{\Upsilon}_0(t, v^0_\hbar(t))\Big] f(t), \varpi(t)\Big) \\ &\qquad + \int_{\mathbb{Y}} (\Big[\tilde{G}(t, u^0_\hbar(t), y) - \tilde{G}(t, v^0_\hbar(t), y)\Big] (g(t, y) - 1), \varpi(t))\vartheta(\mathrm{d}y) \\ &\quad \leqslant \int_D |\varpi(t)|^2(x)|\nabla u^0_\hbar(t)|(x) \,\mathrm{d}x \\ &\qquad + \Big|\Big(\tilde{\Upsilon}_0(t, u^0_\hbar(t)) - \tilde{\Upsilon}_0(t, v^0_\hbar(t))\Big)\Big|_{L(H_0,H)}|f(t)|_0|\varpi(t)|_H \\ &\qquad + C|\varpi(t)|^2_H \int_{\mathbb{Y}} \|\tilde{G}(t, y)\|_{1,H}|g(t, y) - 1|\vartheta(\mathrm{d}y) \\ &\quad \leqslant \|\nabla u^0_\hbar(t)\|_q \|\varpi(t)\|^{\frac{2}{q}}_{L^\infty(D)}|\varpi(t)|^{\frac{2(q-1)}{q}}_H + \sqrt{K(t)}|u^0_\hbar(t) - v^0_\hbar(t)|_H|f(t)|_0|\varpi(t)|_H \\ &\qquad + C|\varpi(t)|^2_H \int_{\mathbb{Y}} \|\tilde{G}(t, y)\|_{1,H}|g(t, y) - 1|\vartheta(\mathrm{d}y). \end{aligned}$$

Applying the fact that $\| \operatorname{curl} \varsigma\|_q \leqslant C\| \operatorname{curl} \varsigma\|_\infty, q \in [2, \infty), C \geqslant 1,\ \varpi(0) = 0$ and (4.9), we obtain for $t \in [0, T]$

$$\begin{aligned} \mathcal{W}'(t) \leqslant{} & 2qC(M)(1 + \|\varsigma\| + \| \operatorname{curl} \varsigma\|_{L^\infty(D)})\bar{\mathcal{W}}^{\frac{2}{q}}\mathcal{W}(t)^{1-\frac{1}{q}} + 2(K(t) + |f(t)|^2_0)\mathcal{W}(t) \\ &+ \mathcal{W}(t) \int_{\mathbb{Y}} \|\tilde{G}(t, y)\|_{1,H}|g(t, y) - 1|\vartheta(\mathrm{d}y), \end{aligned}$$

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where $\bar{\mathcal{W}} := \sup_{0\leqslant t\leqslant T} \|\varpi(t)\|_{L^\infty(D)}$ and $\mathcal{W}(t) := |\varpi(t)|_H^2$. Then

$$\mathcal{W}(t)^{1/q} \leqslant 2Cq\,\bar{C}_4(M)[1 + \|\varsigma\| + \|\operatorname{curl}\varsigma\|_{L^\infty(D)}]\bar{\mathcal{W}}^{\frac{2}{q}} t + \int_0^t 2(K(s) + |f(s)|_0^2)\mathcal{W}(s)^{1/q}\mathrm{d}s + C\int_0^t \mathcal{W}(s)^{1/q}\int_{\mathbb{Y}} \|G(s,y)\|_{1,H}|g(s,y)-1|\vartheta(\mathrm{d}y)\,\mathrm{d}s.$$

Hence, by using Gronwall's inequality, for $q \in [2,\infty)$ and any $T^* \in [0,T]$, we deduce that

$$\sup_{0\leqslant t\leqslant T^*} |\varpi(t)|_H^2 \leqslant C\Bigg(2C(M)[1 + \|\varsigma\| + \|\operatorname{curl}\varsigma\|_{L^\infty(D)}]\bar{\mathcal{W}}^2 T^* \times \exp\left\{\int_0^{T^*}\left[(K(t) + |f(t)|_0^2) + \int_{\mathbb{Y}} \|\tilde{G}(t,y)\|_{1,H}|g(t,y)-1|\vartheta(\mathrm{d}y)\right]\mathrm{d}t\right\}\Bigg)^q.$$

Choosing $T_1^* > 0$ small enough and taking $q \to \infty$, for every $t \in [0, T_1^*]$, we can infer that $|\varpi(t)|_H^2 = 0$. Repeating this process by using the value at time T_1^* as the initial data and employing (A.2), (4.9) and (4.2), one concludes that there is a $T^* > 0$ such that for every integer $k = 0, 1, \cdots$ and any $t \in [T_1^* + kT^*, T_1^* + (k+1)T^*] \cap [0,T]$, it holds that $|\varpi(t)|_H^2 = 0$. This yields the uniqueness. We complete the proof of theorem 4.4. □

5. A priori estimates

In this section, we should give an a priori estimate and more regularity of the solution $u_\hbar^\nu$ in $H^{1,q}, q \in [2,+\infty)$ to the stochastic controlled equations (2.5) for establishing a large deviation principle of the solution u^ν to equations (1.6).

PROPOSITION 5.1. *We assume that* $\mathbb{E}|\varsigma|_H^{2p} < \infty$ *for some* $p \in [2,\infty)$, *and* $E\|\varsigma\|_{H^{1,q}}^q < \infty$ *for* $q \in [2,\infty)$. *Furthermore, suppose that* (Υ_ν, G) *and* $(\tilde{\Upsilon}_\nu, \tilde{G})$ *satisfy conditions 2.5–4.3, respectively. For* $M > 0$, $\hbar = (f,g) \in \tilde{\mathcal{U}}^M$ *and* $\nu \in (0,\nu_0]$, *then the stochastic strong analytically weak solution* $u_\hbar^\nu$ *of (2.5) belongs to* $L^\infty(0,T;H^{1,q})$ $\mathbb{P}$-a.s.. *Moreover, we have*

$$\sup_{0<\nu\leqslant\nu_0}\sup_{\hbar\in\tilde{\mathcal{U}}^M}\mathbb{E}\left(\sup_{0\leqslant t\leqslant T}\|u_\hbar^\nu(t)\|_{H^{1,q}}^q\right) \leqslant C(M,q)\big(1 + \mathbb{E}\|\varsigma\|_{H^{1,q}}^q\big). \tag{5.1}$$

Proof. Now, we study the energy estimation of the approximation solution $u_{\hbar,n}^\nu$ in $H^{1,q}$. To simplify the notation, we drop the index n. It follows from (A.1) and proposition 3.2 that for $0 < \nu \leqslant \nu_0$ and $\hbar = (f,g) \in \tilde{\mathcal{U}}^M$, $\mathbb{E}(\sup_{0\leqslant t\leqslant T}\|u_\hbar^\nu(t)\|_q^q) \leqslant$

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$C(q,M)(1+\mathbb{E}\|\varsigma\|^q)$. Denote $\xi^\nu_\hbar :=$ curl $u^\nu_\hbar$. By using (A.3), we have $\|\nabla u^\nu_\hbar\|_q \leqslant \|\mathrm{curl}\ u^\nu_\hbar\|_q$. Then, for getting (5.1), we only need to prove

$$\sup_{0<\nu\leqslant\nu_0}\sup_{\hbar\in\tilde{\mathcal{U}}^M}\mathbb{E}\Bigg(\sup_{0\leqslant t\leqslant T}\|\xi^\nu_\hbar(t)\|^q_q\Bigg)\leqslant C(M,q)\big(1+\mathbb{E}\|\ \mathrm{curl}\ \varsigma\|^q_q\big). \tag{5.2}$$

Denote $\langle\cdot,\cdot\rangle$ by the duality between $L^q(D)$ and $L^{q^*}(D)$, where $\frac{1}{q}+\frac{1}{q^*}=1$. For fixed $N>0$, set $\tau_N=\inf\{t\geqslant 0:\|\xi^\nu_\hbar(t)\|_q\geqslant N\}\wedge T$. By using Itô's formula, we obtain

$$\|\xi^\nu_\hbar(t\wedge\tau_N)\|^q_q\leqslant\|\mathrm{curl}\varsigma\|^q_q+\sum_{i=1}^8 K_i(t),$$

where

$$\begin{aligned}
K_1(t)&=q\sqrt{\nu}\int_0^{t\wedge\tau_N}\langle|\xi^\nu_\hbar(s)|^{q-2}\xi^\nu_\hbar(s),\ \mathrm{curl}\ \Upsilon_\nu(s,u^\nu_\hbar(s))\rangle\mathrm{d}W(s),\\
K_2(t)&=\int_0^{t\wedge\tau_N}\int_{\mathbb{Y}}\big(\|\xi^\nu_\hbar(s)+\nu\mathrm{curl}\ G(s,u^\nu_\hbar(s),y)\|^q_q-\|\xi^\nu_\hbar(s)\|^q_q\big)\,\tilde{N}^{\nu^{-1}}(\mathrm{d}y,\mathrm{d}s),\\
K_3(t)&=-q\nu\int_0^{t\wedge\tau_N}\langle|\xi^\nu_\hbar(s)|^{q-2}\xi^\nu_\hbar(s),A\xi^\nu_\hbar(s)\rangle\mathrm{d}s,\\
K_4(t)&=-q\int_0^{t\wedge\tau_N}\langle|\xi^\nu_\hbar(s)|^{q-2}\xi^\nu_\hbar(s),\ \mathrm{curl}\ B(u^\nu_\hbar(s),u^\nu_\hbar(s))\rangle\mathrm{d}s,\\
K_5(t)&=q\int_0^{t\wedge\tau_N}\langle|\xi^\nu_\hbar(s)|^{q-2}\xi^\nu_\hbar(s),\mathrm{curl}\ \tilde{\Upsilon}_\nu(s,u^\nu_\hbar(s))f(s)\rangle\mathrm{d}s,\\
K_6(t)&=\frac{q}{2}(q-1)\nu\int_0^{t\wedge\tau_N}\|\mathrm{curl}\ \Upsilon_\nu(s,u^\nu_\hbar(s))\|^2_{R(H_0,L^q)}\|\xi^\nu_\hbar(s)\|^{q-2}_q ds,\\
K_7(t)&=\nu\int_0^{t\wedge\tau_N}\int_{\mathbb{Y}}\Bigg(\|\xi^\nu_\hbar(s)+\nu\mathrm{curl}\ G(s,u^\nu_\hbar(s),y)\|^q_q-\|\xi^\nu_\hbar(s)\|^q_q\\
&\qquad-\nu q|\xi^\nu_\hbar(s)|^{q-2}\langle\xi^\nu_\hbar(s),\mathrm{curl}\ G(s,u^\nu_\hbar(s),y)\rangle\Bigg)\vartheta(\mathrm{d}y)\,\mathrm{d}s,\\
K_8(t)&=q\int_0^{t\wedge\tau_N}\int_{\mathbb{Y}}|\xi^\nu_\hbar(s)|^{q-2}(\mathrm{curl}\ \tilde{G}(s,u^\nu_\hbar(s),y)(g(s,y)-1),\xi^\nu_\hbar(s))\vartheta(\mathrm{d}y)\,\mathrm{d}s.
\end{aligned}$$

For the term $K_1(t)$, by applying the Burkholder–Davies–Gundy inequality, condition 4.2, Hölder's and Young's inequalities, we deduce that

$$\begin{aligned}
&\mathbb{E}\Bigg(\sup_{s\in[0,t\wedge\tau_N]}K_1(s)\Bigg)\\
&\quad\leqslant\sqrt{\nu}Cq\mathbb{E}\Bigg(\int_0^{t\wedge\tau_N}\|\ \mathrm{curl}\ \Upsilon_\nu(s,u^\nu_\hbar(s))\|^2_{R(H_0,L_q)}\|\xi^\nu_\hbar(s)\|^{2(q-1)}_q\mathrm{d}s\Bigg)^{\frac{1}{2}}
\end{aligned}$$

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$$
\begin{aligned}
&\leqslant \sqrt{\nu} C q \mathbb{E}\Bigg(\sup_{0\leqslant s\leqslant t} \|\xi^\nu_\hbar(s\wedge\tau_N)\|_q^{\frac{q}{2}} \Bigg[\int_0^{t\wedge\tau_N} \|\xi^\nu(s)\|_q^{q-2} K(s) \\
&\times \Big(1+\|u^\nu_\hbar(s)\|_q^2+\|\xi^\nu_\hbar(s)\|_q^2\Big)\,\mathrm{d}s\Bigg]^{\frac{1}{2}}\Bigg) \\
&\leqslant \frac{1}{2}\mathbb{E}\Bigg(\sup_{0\leqslant s\leqslant t} \|\xi^\nu_\hbar(s\wedge\tau_N)\|_q^q\Bigg) + \nu C q^2 \mathbb{E}\int_0^{t\wedge\tau_N} K(s)\|\xi^\nu_\hbar(s)\|_q^q ds \\
&\quad + \nu C q^2 \Bigg(1+\mathbb{E}\big(\sup_{0\leqslant t\leqslant T}\|u^\nu_\hbar(t)\|_q^q\big)\Bigg)\int_0^{t\wedge\tau_N} K(s)\,\mathrm{d}s.
\end{aligned}
$$

For the term $K_2(t)$, we use the Burkholder–Davis–Gundy inequality, condition 4.2, Hölder's and Young's inequalities again to obtain

$$
\begin{aligned}
&\mathbb{E}\left(\sup_{s\in[0,t\wedge\tau_N]} K_2(s)\right) \\
&\leqslant \mathbb{E}\left[\int_0^{t\wedge\tau_N}\int_{\mathbb{Y}} \left(\|\xi^\nu_\hbar(s)+\nu \mathrm{curl}\, G(s,u^\nu_\hbar(s),y)\|_q^q - \|\xi^\nu_\hbar(s)\|_q^q\right)^2 \vartheta(\mathrm{d}y)\,\mathrm{d}s\right]^{\frac{1}{2}} \\
&\leqslant C(\nu)\mathbb{E}\Bigg(\int_0^{t\wedge\tau_N}\int_{\mathbb{Y}} \|\xi^\nu_\hbar(s)\|_q^{2q-2}\|\mathrm{curl} G(s,u^\nu_\hbar(s),y)\|_q^2 \\
&\quad + \|\xi^\nu_\hbar(s)\|_q^{2q-4}\|\mathrm{curl}\, G(s,u^\nu_\hbar(s),y)\|_q^4 + \|\mathrm{curl}\, G(s,u^\nu_\hbar(s),y)\|_q^{2q}\vartheta(\mathrm{d}y)\,\mathrm{d}s\Bigg)^{\frac{1}{2}} \\
&\leqslant C(\nu)\mathbb{E}\left(\int_0^{t\wedge\tau_N} K(s)\|\xi^\nu_\hbar(s)\|_q^{2q}\mathrm{d}s\right)^{\frac{1}{2}} + C(\nu)\mathbb{E}\left(\int_0^{t\wedge\tau_N} K(s)\,\mathrm{d}s\right)^{\frac{1}{2}} \\
&\leqslant \frac{1}{2}\mathbb{E}\left(\sup_{s\in[0,t\wedge\tau_N]}\|\xi^\nu_\hbar(s)\|_q^q\right) + C(\nu)\mathbb{E}\int_0^{t\wedge\tau_N} K(s)\|\xi^\nu_\hbar(s)\|_q^q\mathrm{d}s \\
&\quad + C(\nu)\mathbb{E}\left(\int_0^{t\wedge\tau_N} K(s)\,\mathrm{d}s\right)^{\frac{1}{2}}.
\end{aligned}
$$

By using the facts that $\xi^\nu_\hbar = 0$ on ∂D and $A=-\Delta$, one deduces that

$$
\begin{aligned}
\mathbb{E}K_3(t) &= -q\nu\mathbb{E}\int_0^{t\wedge\tau_N}\int_D \langle\nabla\big(|\xi^\nu_\hbar(s)|^{q-2}\xi^\nu_\hbar(s)\big),\nabla\xi^\nu_\hbar(s)\rangle dxds \\
&= -q(q-1)\nu\mathbb{E}\int_0^{t\wedge\tau_N}\int_D |\xi^\nu_\hbar(s)|^{q-2}|\nabla\xi^\nu_\hbar(s)|^2 dxds.
\end{aligned}
$$

It follows from $\xi_\hbar^\nu = \text{ curl } u_\hbar^\nu$ and (A.7) that $\mathbb{E}K_4(t) = 0$. For the term $K_5(t)$, we employ Hölder's and Young's inequalities and condition 4.3 to bound

$$
\begin{aligned}
\mathbb{E}K_5(t) &\leqslant q\mathbb{E}\int_0^{t\wedge\tau_N} \||\xi_\hbar^\nu(s)|^{q-1}\|_{q^*}\| \text{ curl } \tilde{\Upsilon}_\nu(s, u_\hbar^\nu(s))f(s)\|_q ds \\
&\leqslant \mathbb{E}\left[\Big(1 + \sup_{s\in[0,T]} \|u_\hbar^\nu(s)\|_q^q\Big) \int_0^{t\wedge\tau_N} \Big(K(s) + |f(s)|_0^2\Big)\,\mathrm{d}s \right.\\
&\qquad \left. + q\int_0^{t\wedge\tau_N} \|\xi_\hbar^\nu(s)\|_q^q\Big(K(s) + |f(s)|_0^2\Big)\,\mathrm{d}s\right].
\end{aligned}
$$

Condition 4.2, Hölder's and Young's inequalities imply that for any $\nu \in (0, \nu_0]$

$$
\begin{aligned}
\mathbb{E}K_6(t) &\leqslant \frac{q(q-1)}{2}\nu\mathbb{E}\int_0^{t\wedge\tau_N} \|\xi_\hbar^\nu(s)\|_q^{q-2}K(s)\Big(1 + \|u_\hbar^\nu(s)\|_q^2 + \|\xi_\hbar^\nu(s)\|_q^2\Big)\,\mathrm{d}s \\
&\leqslant \frac{q(q-1)}{2}\nu\mathbb{E}\Big(1 + \frac{2}{q}\sup_{s\in[0,T]} \|u_\hbar^\nu(s)\|_q^q\Big)\int_0^{t\wedge\tau_N} K(s)\,\mathrm{d}s \\
&\quad + \nu(q-1)^2\mathbb{E}\int_0^{t\wedge\tau_N} K(s)\|\xi_\hbar^\nu(s)\|_q^q\mathrm{d}s.
\end{aligned}
$$

For the term $K_7(t)$, by using condition 4.2, Hölder's and Young's inequalities, we have

$$
\begin{aligned}
\mathbb{E}K_7(t) &\leqslant C\mathbb{E}\left[\int_0^{t\wedge\tau_N} \int_{\mathbb{Y}} \big(\|\xi_\hbar^\nu(s)\|_q^{q-2} + \|\nu\text{curl } G(s, u_\hbar^\nu(s), y)\|_q^{q-2}\big)\, \| \right.\\
&\qquad \left. \times\nu\text{curl } G(s, u_\hbar^\nu(s), y)\|_q^2\vartheta(\mathrm{d}y)\,\mathrm{d}s\right] \\
&\leqslant C(\nu)\mathbb{E}\left(\int_0^{t\wedge\tau_N} \int_{\mathbb{Y}} (\|\xi_\hbar^\nu(s)\|_q^{q-2}\|\text{curl } G(s, u_\hbar^\nu(s), y)\|_q^2 \right.\\
&\qquad \left. + \|\text{curl } G(s, u_\hbar^\nu(s), y)\|_q^q)\vartheta(\mathrm{d}y)\,\mathrm{d}s\right) \\
&\leqslant C(\nu)\mathbb{E}\int_0^{t\wedge\tau_N} K(s)\|\xi_\hbar^\nu(s)\|_q^q\mathrm{d}s + C(\nu)\mathbb{E}\int_0^{t\wedge\tau_N} K(s)\,\mathrm{d}s.
\end{aligned}
$$

For the term $K_8(t)$, applying condition 4.3, Hölder's and Young's inequalities, one has

$$
\begin{aligned}
&\mathbb{E}\int_0^{t\wedge\tau_N} \int_{\mathbb{Y}} \|\xi_\hbar^\nu(s)\|_q^{q-2}(\text{curl } \tilde{G}(s, u_\hbar^\nu(s), y)(g(s,y) - 1), \xi_\hbar^\nu(s))\vartheta(\mathrm{d}y)\,\mathrm{d}s \\
&\leqslant C\mathbb{E}\int_0^{t\wedge\tau_N} \int_{\mathbb{Y}} \|\xi_\hbar^\nu(s)\|_q^{q-2}\|\text{curl } \tilde{G}(s, u_\hbar^\nu(s), y)\|_q|g(s,y) - 1|\|\xi_\hbar^\nu(s)\|_q\vartheta(\mathrm{d}y)\,\mathrm{d}s \\
&\leqslant C\mathbb{E}\int_0^{t\wedge\tau_N} \int_{\mathbb{Y}} \|\xi_\hbar^\nu(s)\|_q^{q-2}\frac{\|\text{curl } \tilde{G}(s, u_\hbar^\nu(s), y)\|_q}{1 + \|\xi_\hbar^\nu(s)\|_q} \\
&\quad \times |g(s,y) - 1|(1 + \|\xi_\hbar^\nu(s)\|_q^2)\vartheta(\mathrm{d}y)\,\mathrm{d}s
\end{aligned}
$$

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$$\leqslant C\mathbb{E}\int_0^{t\wedge\tau_N}\int_{\mathbb{Y}}\|\xi_\hbar^\nu(s)\|_q^{q-2}\frac{\|\tilde{G}(s,\xi_\hbar^\nu(s),y)\|_q}{1+\|\xi_\hbar^\nu(s)\|_q}|g(s,y)-1|(1+\|\xi_\hbar^\nu(s)\|_q^2)\vartheta(\mathrm{d}y)\,\mathrm{d}s$$

$$\leqslant C\mathbb{E}\int_0^{t\wedge\tau_N}\int_{\mathbb{Y}}\|\xi_\hbar^\nu(s)\|_q^{q-2}|\tilde{G}(s,y)|_{0,q}|g(s,y)-1|(1+\|\xi_\hbar^\nu(s)\|_q^2)\vartheta(\mathrm{d}y)\,\mathrm{d}s$$

$$\leqslant C\mathbb{E}\int_0^{t\wedge\tau_N}\int_{\mathbb{Y}}|\tilde{G}(s,y)|_{0,q}|g(s,y)-1|\vartheta(\mathrm{d}y)\,\mathrm{d}s$$

$$+C\mathbb{E}\int_0^{t\wedge\tau_N}\|\xi_\hbar^\nu(s)\|_q^q\int_{\mathbb{Y}}|\tilde{G}(s,y)|_{0,q}|g(s,y)-1|\vartheta(\mathrm{d}y)\,\mathrm{d}s.$$

Then, we obtain

$$\begin{aligned}
&\mathbb{E}\|\xi_\hbar^\nu(t\wedge\tau_N)\|_q^q\\
&\leqslant \mathbb{E}\|\mathrm{curl}\ \varsigma\|_q^q + C(\nu,q)\mathbb{E}\left[\left(1+\sup_{s\in[0,T]}\|u_\hbar^\nu(s)\|_q^q\right)\int_0^{t\wedge\tau_N}\left(K(s)+|f(s)|_0^2\right)\mathrm{d}s\right]\\
&\quad + C(\nu,q)\mathbb{E}\int_0^{t\wedge\tau_N}\|\xi_\hbar^\nu(s)\|_q^q\left(K(s)+|f(s)|_0^2\right)\mathrm{d}s + C(\nu)\mathbb{E}\int_0^{t\wedge\tau_N}K(s)\,\mathrm{d}s\\
&\quad + C\mathbb{E}\int_0^{t\wedge\tau_N}\int_{\mathbb{Y}}|\tilde{G}(s,y)|_{0,q}|g(s,y)-1|\vartheta(\mathrm{d}y)\,\mathrm{d}s + C(\nu)\mathbb{E}\left(\int_0^{t\wedge\tau_N}K(s)\,\mathrm{d}s\right)^{\frac{1}{2}}\\
&\quad + C\mathbb{E}\int_0^{t\wedge\tau_N}\|\xi^\nu(s)\|_q^q\int_{\mathbb{Y}}|\tilde{G}(s,y)|_{0,q}|g(s,y)-1|\vartheta(\mathrm{d}y)\,\mathrm{d}s.
\end{aligned}$$

Using Grownall's inequality and lemma 2.9, we have

$$\sup_{0<\nu\leqslant\nu_0}\sup_{\hbar\in\tilde{\mathcal{U}}^M}\mathbb{E}\Big(\sup_{0\leqslant t\leqslant T}\|\xi_\hbar^\nu(t)\|_q^q\Big)\leqslant C(M,q)\big(1+\mathbb{E}\|\ \mathrm{curl}\ \varsigma\|_q^q\big).$$

Then, one deduces that (5.2) holds for the approximation sequence $\xi_{\hbar,n}^\nu$. We notice that the constant $C(M,q)$ does not depend on n. Hence, letting $n\to\infty$, we can obtain (5.1) by using the weak convergence discussion. Therefore, we complete the proof. □

6. Large deviations principle

In this section, we establish a large deviation principle of (2.4) by using a weak convergence approach in [**13**, **15**]. First, for every $\nu>0$ and for every $q\in[2,+\infty)$, we assume that $\Upsilon_\nu=\tilde{\Upsilon}_\nu, G=\tilde{G}$ satisfy conditions 2.6, 2.7, 2.12 and 4.3. Moreover, we assume that (Υ_0,G) satisfies the following condition:

CONDITION 6.1. *For every $q\in[2,+\infty)$, we assume that (Υ_0,G) satisfies conditions 2.5, 2.7, 2.11 and 4.2 and Υ_0 satisfies*

$$\sup_{0\leqslant t\leqslant T}\big|\Upsilon_\nu(t,u)-\Upsilon_0(t,u)\big|_{L(H_0,H)}\leqslant C_\nu\big(1+|u|_H\big),\ \textit{for}\ u\in H\ \textit{and}\ \nu>0,$$

where $C_\nu\in[0,+\infty)$ and it converges to 0 as $\nu\to 0$.

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Here, we notice that $\|\Psi\|_{L(H_0,L^q)} \leqslant C\|\Psi\|_{R(H_0,L^q)}$. Then, for $\nu \geqslant 0$, (Υ_ν, G) also satisfies conditions 2.6, 2.7, 2.12 and 4.3 with appropriate coefficients.

Let $\mathcal{B}$ be the Borel $\sigma-$field of the following Polish space:

$$\mathcal{X} = D\big([0,T];H\big) \cap L^\infty\big(0,T;H^{1,q} \cap V\big) \cap L^2\big(0,T;\mathcal{H}\big)$$

with the norm $\|u\|_{\mathcal{X}}^2 := \int_0^T \|u(t)\|_{\mathcal{H}}^2 \mathrm{d}t$ and define

$$\mathcal{Y} = \{\varsigma \in V, \text{ such that curl } \varsigma \in L^\infty(D)\}$$

with the norm $\|\varsigma\|_{\mathcal{Y}}^2 := \|\varsigma\|^2 + \|\text{ curl } \varsigma\|_{L^\infty}^2$.

By using (A.3) and (A.1), for any $q \in [2,\infty)$, we obtain $\mathcal{Y} \subset H^{1,q}$. Next, we establish a large deviation principle of equations (2.4) with deterministic initial data $u^\nu(0) = \varsigma \in \mathcal{Y}$.

Now, we state our main result in this section.

Theorem 6.2. *Assume that $\varsigma \in \mathcal{Y}$, and for $\nu > 0$, (Υ_ν, G) satisfies conditions 2.5, 2.7, 2.11 and 4.2 for any $q \in [2,+\infty)$, $K(t) = C$ and condition 6.1 is satisfied. Then, the stochastic strong analytically weak solution u^ν of equations (2.4) with deterministic initial data ς satisfies a large deviation principle in $\mathcal{X}$ with the good rate function I defined in definition 2.1.*

Notice that inspired by [**20**], we will study the construction of the rate function I in the further work. For fixed $q, p \in [4,\infty)$, $M > 0$ and $\nu_0 > 0$, let $\hbar_\nu = (f_\nu, g_\nu), 0 < \nu \leqslant \nu_0$ be a family of random elements taking values in $\tilde{\mathcal{U}}^M$. Suppose that $u_{\hbar_\nu}^\nu$ is the stochastic strong analytically weak solution of the following stochastic controlled equations:

$$\begin{aligned}
\mathrm{d}u_{\hbar_\nu}^\nu(t) = &-\big[\nu A u_{\hbar_\nu}^\nu(t) + B(u_{\hbar_\nu}^\nu(t), u_{\hbar_\nu}^\nu(t))\big]\mathrm{d}t + \sqrt{\nu}\Upsilon(t, u_{\hbar_\nu}^\nu(t))\,\mathrm{d}W(t)\\
&+ \Upsilon(t, u_{\hbar_\nu}^\nu(t)) f_\nu(t)\,\mathrm{d}t,\\
&+ \nu\int_{\mathbb{Y}} G(t, u_{\hbar_\nu}^\nu(t-), y)\tilde{N}^{\nu^{-1}g_\nu}(\mathrm{d}y, \mathrm{d}t)\\
&+ \int_{\mathbb{Y}} G(t, u_{\hbar_\nu}^\nu(t), y)(g_\nu(t,y) - 1)\vartheta(\mathrm{d}y)\,\mathrm{d}t
\end{aligned} \tag{6.1}$$

with deterministic initial data $u_{\hbar_\nu}^\nu(0) = \varsigma \in \mathcal{Y}$. Due to the uniqueness of the solution, we can conclude that for $(f_\nu, g_\nu) \in \tilde{\mathcal{U}}^M$, $u_{\hbar_\nu}^\nu(s) = \mathcal{G}_\varsigma^\nu(\sqrt{\nu}(W(s) + \frac{1}{\sqrt{\nu}}\int_0^\cdot f_\nu(s)\mathrm{d}s), \nu N^{\nu^{-1}g_\nu})$. Recall that $u_h^0 = \mathcal{G}_\varsigma^0(\int_0^\cdot f(s)\mathrm{d}s, \vartheta^g)$.

Now, we proceed to verify the second part of condition 2.3.

Proposition 6.3. *Assume that for $\nu > 0$ the coefficients Υ_ν, G satisfy conditions 2.5, 2.7, 2.11 and 4.2 for all $q \in [2,+\infty)$ and that condition 6.1 holds. Furthermore, we suppose that for every $p \in [2,\infty)$, $\mathbb{E}(|\varsigma|_H^p + \|\varsigma\|_{\mathcal{Y}}^p) < +\infty$, and (f_ν, g_ν) converges in distribution to (f, g) in $\tilde{\mathcal{U}}^M$. Therefore, the process $\mathcal{G}_\varsigma^\nu(\sqrt{\nu}(W(\cdot) + \frac{1}{\sqrt{\nu}}\int_0^\cdot f_\nu(s)\mathrm{d}s), \nu N^{\nu^{-1}g_\nu})$ converges in distribution to $\mathcal{G}_\varsigma^0(\int_0^\cdot f(s)\mathrm{d}s, \vartheta^g)$ in $\mathcal{X}$ as $\nu \to 0$.*

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Proof. We divide the following four steps to prove the proposition.

Step 1: We note that $u^\nu_{\hbar_\nu} = \varsigma + \sum_{i=1}^6 H_i$, where

$$H_1 = -\nu \int_0^t Au^\nu_{\hbar_\nu}(s)\,\mathrm{d}s, \quad H_2 = -\int_0^t B(u^\nu_{\hbar_\nu}(s), u^\nu_{\hbar_\nu}(s))\,\mathrm{d}s,$$

$$H_3 = \sqrt{\nu}\int_0^t \Upsilon_\nu(s, u^\nu_{\hbar_\nu}(s))\,\mathrm{d}W(s), \quad H_4 = \int_0^t \Upsilon_\nu(s, u^\nu_{\hbar_\nu}(s)) f_\nu(s)\,\mathrm{d}s,$$

$$H_5 = \nu \int_0^t \int_{\mathbb{Y}} G(t, u^\nu_{\hbar_\nu}(t-), y)\tilde{N}^{\nu^{-1}g_\nu}(\mathrm{d}y, \mathrm{d}t),$$

$$H_6 = \int_0^t \int_{\mathbb{Y}} G(t, u^\nu_{\hbar_\nu}(t), y)(g_\nu(t,y) - 1)\vartheta(\mathrm{d}y)\,\mathrm{d}t.$$

For $\nu \in (0, \nu_0]$, by using Minkowski's and Cauchy-Schwarz's inequalities, we obtain

$$\begin{aligned}\|H_1\|^2_{W^{1,2}(0,T;H)} &= \nu \int_0^T \Big| \int_0^t Au^\nu_{\hbar_\nu}(s)\,\mathrm{d}s\Big|^2_H \mathrm{d}t + \nu \int_0^T |Au^\nu_{\hbar_\nu}(t)|^2_H \mathrm{d}t \\ &\leqslant C(T,p)\nu \int_0^T |Au^\nu_{\hbar_\nu}(s)|^2_H \mathrm{d}s.\end{aligned}$$

Hence, for $\nu \in (0, \nu_0]$, by virtue of (3.6), one deduces

$$\mathbb{E}\big(\|H_1\|^2_{W^{1,2}(0,T;H)}\big) \leqslant C(M,T,\nu_0)\left(1 + \mathbb{E}\|\varsigma\|^4\right).$$

Similarly, it follows from (3.6) that for all $p \in [2,\infty)$ and $\nu \in (0, \nu_0]$

$$\begin{aligned}\mathbb{E}\|H_1\|^p_{W^{1,p}(0,T;V')} &\leqslant \nu C(T)\mathbb{E}\int_0^T \|Au^\nu_{\hbar_\nu}(s)\|^p_{V'}\mathrm{d}s \leqslant \nu C(T)\mathbb{E}\int_0^T \|u^\nu_{\hbar_\nu}(s)\|^p ds \\ &\leqslant C(T,p,\nu_0)(1 + \mathbb{E}\|\varsigma\|^p).\end{aligned}$$

Minkowski's and Hölder's inequalities and (A.8) imply that for $p, q \in [4, \infty)$ and $\nu \in (0, \nu_0]$

$$\|H_2\|^p_{W^{1,p}(0,T;H)} \leqslant C(T,p,\nu_0)\int_0^T \|u^\nu_{\hbar_\nu}(t)\|^p_{H^{1,q}}\|u^\nu_{\hbar_\nu}(t)\|^p \mathrm{d}t.$$

Then, for $\nu \in (0, \nu_0]$, by using Hölder's inequality, (3.6) and (5.1), we have

$$\mathbb{E}\big(\|H_2\|^p_{W^{1,p}(0,T;H)}\big) \leqslant C(T,M,p,q)\left(1 + \mathbb{E}\|\varsigma\|^{pq/(q-p)}\right)^{1-p/q}\left(1 + \mathbb{E}\|\varsigma\|^q_{H^{1,q}}\right)^{p/q}.$$

By the Burkholder–Davis–Gundy inequality and Hölder's inequality, it holds that

$$\begin{aligned}\mathbb{E}\int_0^T |H_3(t)|^p_H \mathrm{d}t &\leqslant C_p\nu^{p/2}\int_0^T \mathbb{E}\left(\int_0^T |\Upsilon_\nu(s, u^\nu_{\hbar_\nu}(s))|^2_{L_Q}\mathrm{d}s\right)^{p/2}\mathrm{d}t \\ &\leqslant C_p\nu^{p/2}\Big(1 + \mathbb{E}\sup_{s\in[0,T]}|u^\nu_{\hbar_\nu}(s)|_H\Big)\int_0^T\Big(\int_0^T K(t)\,\mathrm{d}t\Big)^{\frac{p}{2}}\mathrm{d}t \\ &\leqslant C(T,p,\nu_0)(1 + \mathbb{E}|\varsigma|^p_H).\end{aligned}$$

Let $p \in [4,\infty)$, $\alpha \in (0,\frac{1}{2})$. By virtue of the Burkholder–Davis–Gundy inequality and Hölder's inequality, one deduces that

$$\begin{aligned}
&\mathbb{E}\int_0^T\int_0^T \frac{|H_3(t)-H_3(s)|_H^p}{|t-s|^{1+\alpha p}}\,\mathrm{d}t\,\mathrm{d}s\\
&\quad= \nu^{p/2}\int_0^T\int_0^T \frac{\mathbb{E}|\int_{s\wedge t}^{s\vee t}\Upsilon_\nu(r,u^\nu_{\hbar_\nu}(r))\,\mathrm{d}W(r)|_H^p}{|t-s|^{1+\alpha p}}\,\mathrm{d}s\,\mathrm{d}t\\
&\quad\leqslant C(T,p,\nu_0)(1+\mathbb{E}|\varsigma|_H^p)\int_0^T\int_0^T |t-s|^{-(1+\alpha p)}\,\mathrm{d}s\,\mathrm{d}t\\
&\quad\leqslant C(T,p,\nu_0)(1+\mathbb{E}|\varsigma|_H^p).
\end{aligned}$$

Then, we obtain

$$\mathbb{E}\big(\|H_3\|^p_{W^{\alpha,p}(0,T;H)}\big) \leqslant C(p,T)\nu_0^{p/2}\left(1+\mathbb{E}|\varsigma|_H^p\right).$$

Furthermore, we use Hölder's inequality and condition 2.5 to infer that for $\nu \in (0,\nu_0]$ and $p \in [4,\infty)$

$$\int_0^T |H_4(t)|_H^p \mathrm{d}t \leqslant C\Big(1+\sup_{s\in[0,T]}|u^\nu_{\hbar_\nu}(s)|_H^p\Big)\int_0^T\Big(\int_0^T K(t)\,\mathrm{d}t\Big)^p dt.$$

Let $\alpha \in (0,\frac{1}{2})$. By using Minkowski's and Hölder's inequalities, condition 2.5 and Fubini's theorem, we deduce that for $\nu \in (0,\nu_0]$

$$\begin{aligned}
&\int_0^T\int_0^T \frac{|H_4(t)-H_4(s)|_H^p}{(t-s)^{1+\alpha p}}\mathrm{d}s\mathrm{d}t\\
&\quad\leqslant 2\int_0^T\int_0^T (t-s)^{-1-\alpha p}\Big|\int_s^t |\Upsilon_\nu(r,u^\nu_{\hbar_\nu}(r))|_{L_Q}|f_\nu(r)|_0\mathrm{d}r\Big|^p\,\mathrm{d}s\,\mathrm{d}t\\
&\quad\leqslant C\Big(1+\sup_{r\in[0,T]}|u^\nu_{\hbar_\nu}(r)|_H^p\Big)\int_0^T\int_0^T (t-s)^{-1-\alpha p}\,\mathrm{d}s\,\mathrm{d}t.
\end{aligned}$$

Using the two above estimates and (3.1), for $\alpha \in (0,\frac{1}{2})$, $p \in [4,\infty)$ and $\nu \in (0,\nu_0]$, we get

$$\mathbb{E}\big(\|H_4\|^p_{W^{\alpha,p}(0,T;H)}\big) \leqslant C(p,\alpha,T,M)\left(1+\mathbb{E}|\varsigma|_H^p\right).$$

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For the term H_5, by the Burkholder–Davis–Gundy inequality and Hölder's inequality, and proposition 3.1, we obtain

$$\begin{aligned}\mathbb{E}\int_0^T |H_5(t)|_H^p \mathrm{d}t &\leqslant C_p\nu^{p/2}\int_0^T \mathbb{E}\left(\int_0^T \int_{\mathbb{Y}} |G(t, u_{\hbar_\nu}^\nu(t), y)|_H^2 \vartheta(\mathrm{d}y)\,\mathrm{d}t\right)^{p/2} \mathrm{d}t \\ &\leqslant C_p\nu^{p/2}\int_0^T \mathbb{E}\left(\int_0^T |u_{\hbar_\nu}^\nu|_H^2 \mathrm{d}t\right)^{p/2} \mathrm{d}t \\ &\quad + C(p,T)\nu^{p/2}\left(\int_0^T K(t)\,\mathrm{d}t\right)^{\frac{p}{2}} \\ &\leqslant C(p,T)\nu^{p/2}\left(1+\sup_{0\leqslant t\leqslant T}\mathbb{E}|u_{\hbar_\nu}^\nu(t)|_H^p\right) + C(p,T)\nu^{p/2} \\ &\leqslant C(p,T)\nu_0^{p/2}\left(1+\mathbb{E}|\varsigma|_H^p\right).\end{aligned}$$

Let $p \in [4, \infty)$, $\alpha \in (0, \frac{1}{2})$. The Burkholder–Davis–Gundy and Hölder's inequalities, conditions 2.5 and proposition 3.1 imply that

$$\begin{aligned}&\mathbb{E}\int_0^T \int_0^T \frac{|H_5(t)-H_5(s)|_H^p}{|t-s|^{1+\alpha p}}\,\mathrm{d}s\,\mathrm{d}t \\ &= \nu^{p/2}\int_0^T \int_0^T \frac{\mathbb{E}|\int_{s\wedge t}^{s\vee t}\int_{\mathbb{Y}} G(r, u_{\hbar_\nu}^\nu(r), y)\tilde{N}^{\nu^{-1}}(\mathrm{d}y, \mathrm{d}r)|_H^p}{|t-s|^{1+\alpha p}}\,\mathrm{d}s\,\mathrm{d}t \\ &\leqslant C_p\nu^{p/2}\int_0^T \int_0^T \mathbb{E}\left|\int_{s\wedge t}^{s\vee t} |G(r, u_{\hbar_\nu}^\nu(r-), y)|_H^2 dr\right|^{p/2} |t-s|^{-(1+\alpha p)}\,\mathrm{d}s\,\mathrm{d}t \\ &\leqslant C_p\nu^{p/2}\left(1+\sup_{0\leqslant t\leqslant T}\mathbb{E}|u_{\hbar_\nu}^\nu(t)|_H^p\right)\int_0^T \int_0^T |t-s|^{-(1+\alpha p)}\,\mathrm{d}s\,\mathrm{d}t \\ &\leqslant C(p,T)\nu_0^{p/2}\left(1+\mathbb{E}|\varsigma|_H^p\right).\end{aligned}$$

For the term H_6, by using conditions 2.5, 2.7, lemma 2.9 and proposition 3.1, we obtain

$$\begin{aligned}&\mathbb{E}\int_0^T \int_0^T \frac{|H_6(t)-H_6(s)|_H^p}{|t-s|^{1+\alpha p}}\,\mathrm{d}s\,\mathrm{d}t \\ &= \int_0^T \int_0^T \frac{\mathbb{E}|\int_{s\wedge t}^{s\vee t}\int_{\mathbb{Y}} G(r, u_{\hbar_\nu}^\nu(r), y)(g_\nu(r,y)-1)\vartheta(\mathrm{d}y)\,\mathrm{d}r|_H^p}{|t-s|^{1+\alpha p}}\,\mathrm{d}s\,\mathrm{d}t\end{aligned}$$

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$$
\begin{aligned}
&\leqslant C_p \int_0^T \int_0^T \mathbb{E}\left|\int_{s\wedge t}^{s\vee t}\int_{\mathbb{Y}} |G(r,u^{\nu}_{\hbar_\nu}(r-),y)|_H |g_\nu(r,y)-1|\vartheta(\mathrm{d}y)\,\mathrm{d}r\right|^p \\
&\quad \times |t-s|^{-(1+\alpha p)}\,\mathrm{d}s\,\mathrm{d}t \\
&\leqslant C_p \int_0^T \int_0^T \mathbb{E}\left|\int_{s\wedge t}^{s\vee t}\int_{\mathbb{Y}} |G(r,y)|_{0,H} |g_\nu(r,y)-1|(1+|u^{\nu}_{\hbar_\nu}|_H)\vartheta(\mathrm{d}y)\,\mathrm{d}r\right|^p \\
&\quad \times |t-s|^{-(1+\alpha p)}\,\mathrm{d}s\,\mathrm{d}t \\
&\leqslant C_p \left(1+\sup_{0\leqslant t\leqslant T}\mathbb{E}|u^{\nu}_{\hbar_\nu}(t)|^p_H\right)\int_0^T\int_0^T |t-s|^{-(1+\alpha p)}\,\mathrm{d}s\,\mathrm{d}t \\
&\leqslant C_p\left(1+\mathbb{E}|\varsigma|^p_H\right).
\end{aligned}
$$

Combining the estimates of $H_1 - H_6$, we can infer that for $p\in[4,\infty)$, $\alpha\in(0,1/2)$, there is a constant $C(p,M,T)>0$ such that for any $\nu\in(0,\nu_0]$, it holds that

$$
\mathbb{E}\big(\|u^{\nu}_{\hbar_\nu}\|^2_{W^{\alpha,2}(0,T;H)}\big)+\mathbb{E}\big(\|u^{\nu}_{\hbar_\nu}\|^p_{W^{\alpha,p}(0,T;V')}\big)\leqslant C(p,M,T). \tag{6.2}
$$

Step 2: Using lemmas 4.3 and 4.4 in [**60**], we deduce $u^{\nu}_{\hbar_\nu}$ is tight in $\mathcal{D}([0,T];D(A^{-\beta}),\beta>1/2$. It follows from (3.6) and (6.2) that the process $\{u^{\nu}_{\hbar_\nu}\}_{\nu\in(0,\nu_0]}$ is bounded in probability in

$$
W^{\alpha,2}(0,T;H)\cap L^2(0,T;V)\cap W^{\alpha,p}(0,T;V').
$$

Using theorem 2.1 in [**29**] (see also [**12**] and the references therein), we can infer that the space $W^{\alpha,2}(0,T;H)\cap L^2(0,T;V)$ $(W^{\alpha,2}(0,T;V')\cap L^2(0,T;V))$ is compactly embedded in $L^2([0,T];\mathcal{H})$ $(L^2([0,T];H))$.

Therefore, for $\nu\in(0,\nu_0]$, using the Prokhorov theorem, we know that the distribution $\mathcal{L}((f_\nu,g_\nu),u^{\nu}_{\hbar_\nu})$ of the process $((f_\nu,g_\nu),u^{\nu}_{\hbar_\nu})$ is tight in

$$
\mathcal{Z}:=\tilde{\mathcal{U}}^M\times L^2([0,T];\mathcal{H})(\text{or } L^2([0,T];H))\cap\mathcal{D}([0,T];D(A^{-\beta})).
$$

Let $\{\nu_n\}_{n\geqslant 0}$ be a sequence in $(0,\nu_0]$ such that $\nu_n\to 0$. Thus, we can choose a subsequence, still denoted by $((f_{\nu_n},g_{\nu_n}),u^{\nu_n}_{\hbar_{\nu_n}})$, that converges in distribution to $((f,g),u)$ in $\mathcal{Z}$ as $n\to\infty$.

Step 3: Applying the Jakubowski–Skorohod Theorem in [**33**], we infer that there exists a stochastic basis $(\tilde{\Omega},\tilde{\mathcal{F}},\{\tilde{\mathcal{F}}_t\}_{t\geqslant 0},\tilde{\mathbb{P}})$ and on this basis, $\mathcal{Z}$-valued random variables $((\tilde{f},\tilde{g}),\tilde{u})$ and $\{((\tilde{f}_{\nu_n},\tilde{g}_{\nu_n}),\tilde{u}^{\nu_n}_{\hbar_{\nu_n}})\}_{n\geqslant 0}$ such that $((\tilde{f}_{\nu_n},\tilde{g}_{\nu_n}),\tilde{u}^{\nu_n}_{\hbar_{\nu_n}})$ and $((f_{\nu_n},g_{\nu_n}),u^{\nu_n}_{\hbar_{\nu_n}})$ have the same distribution on $\mathcal{Z}$, and as $n\to\infty$, $((\tilde{f}_{\nu_n},\tilde{g}_{\nu_n}),\tilde{u}^{\nu_n}_{\hbar_{\nu_n}})\to((\tilde{f},\tilde{g}),\tilde{u})$ in $\mathcal{Z}$ $\tilde{\mathbb{P}}$-a.s. For simplicity, we drop the tilde and the index n. Denote $\varsigma:=u^{\nu}_{\hbar_\nu}(0)$. Notice that here the corresponding solution is the stochastic weak solution.

Applying (3.1), (3.6) and (5.1), for $\nu\in(0,\nu_0]$, $\alpha\in(0,1/2)$ and $q\in[2,\infty)$, we get

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$$\tilde{\mathbb{E}}\bigg(\sup_{0\leqslant t\leqslant T}|u^{\nu}_{\hbar_\nu}(t)|^2_H\bigg)\leqslant C,\quad \tilde{\mathbb{E}}\left(\int_0^T\|u^{\nu}_{\hbar_\nu}(t)\|^2\mathrm{d}t\right)\leqslant C,$$
$$\tilde{\mathbb{E}}\Big(\sup_{0\leqslant t\leqslant T}\|u^{\nu}_{\hbar_\nu}(t)\|^q_{H^{1,q}(D)}\Big)\leqslant C.$$

Then, there is a subsequence which converges weakly to u in $L^2(\tilde{\Omega}\times(0,T);V)\cap L^q(\tilde{\Omega}\times(0,T);H^{1,q})$ as $n\to\infty$. Hence,

$$u\in L^2(0,T;V)\cap L^\infty\big(0,T;H\cap H^{1,q}\big)\ \tilde{\mathbb{P}}-\text{a.s.}$$

Step 4: We want to check that u is a weak solution of the following equations:

$$\begin{aligned}\mathrm{d}u(t)+B(u(t),u(t))\,\mathrm{d}t&=\Upsilon_0(t,u(t))f(t)\,\mathrm{d}t\\&\quad+\int_{\mathbb{Y}}G(t,u(t),y)(g(t,y)-1)\vartheta(\mathrm{d}y)\,\mathrm{d}t,\quad u(0)=\varsigma.\end{aligned}$$

Taking $\varphi\in D(A^\beta)$ with $\beta>1/2$, then we have

$$(u^{\nu}_{\hbar_\nu}(t)-\varsigma,\varphi)+\int_0^t\ \big(B(u^{\nu}_{\hbar_\nu}(s),u^{\nu}_{\hbar_\nu}(s))-\Upsilon_0(s,u^{\nu}_{\hbar_\nu}(s))f(s),\varphi\big)\mathrm{d}s=\sum_{i=1}^8 L_i,$$

where

$$\begin{aligned}
L_1&=-\nu\int_0^t\big(Au^{\nu}_{\hbar_\nu}(s),\varphi\big)\ \mathrm{d}s,\quad L_2=\sqrt{\nu}\int_0^t\big(\Upsilon(s,u^{\nu}_{\hbar_\nu}(s)),\varphi\big)\ \mathrm{d}W(s),\\
L_3&=\nu\int_0^t\int_{\mathbb{Y}}(G(s,u^{\nu}_{\hbar_\nu}(s-),y)\tilde{N}^{\nu^{-1}}(\mathrm{d}y,\mathrm{d}s),\varphi),\\
L_4&=-\int_0^t\big[\big\langle B(u^{\nu}_{\hbar_\nu}(s)-u(s),u^{\nu}_{\hbar_\nu}(s)),\varphi\big\rangle+\big\langle B(u(s),u^{\nu}_{\hbar_\nu}(s)-u(s)),\varphi\big\rangle\big]\mathrm{d}s,\\
L_5&=\int_0^t\big(\big[\Upsilon_\nu(s,u^{\nu}_{\hbar_\nu}(s)-\Upsilon_0(s,u^{\nu}_{\hbar_\nu}(s))\big]f_\nu(s),\varphi\big)\ \mathrm{d}s,\\
L_6&=\int_0^t\big(\big[\Upsilon_0(s,u^{\nu}_{\hbar_\nu}(s))-\Upsilon_0(s,u(s))\big]f_\nu(s),\varphi\big)\ \mathrm{d}s,\\
L_7&=\int_0^t(\Upsilon_0(s,u(s))\left[f_\nu(s)-f(s)\right],\varphi)\ \mathrm{d}s,\\
L_8&=\int_0^t\int_{\mathbb{Y}}\big[(G(s,u^{\nu}_{\hbar_\nu}(s),y)(g_\nu(s,y)-1),\varphi)\\
&\quad-(G(s,u(s),y)(g(s,y)-1),\varphi)\big]\,\vartheta(\mathrm{d}y)\,\mathrm{d}s.
\end{aligned}$$

For the term L_1, the fact that $D(A^\beta) \subset V, \beta > 1/2$, Cauchy–Schwarz's inequality and (3.6) imply that for $t \in [0,T]$ and $\nu \in (0,\nu_0]$

$$\begin{aligned}\tilde{\mathbb{E}}|L_1| &\leqslant \nu\tilde{\mathbb{E}}\int_0^t \|u^\nu_{\hbar_\nu}(s)\|\|\varphi\|\mathrm{d}s \leqslant \nu\sqrt{t}\|\varphi\|\left(\tilde{\mathbb{E}}\int_0^t \|u^\nu_{\hbar_\nu}(s)\|^2\mathrm{d}s\right)^{1/2}\\ &\leqslant \nu C(T,M)\|\varphi\|\left(1+\tilde{\mathbb{E}}\|\varsigma\|^4\right)^{1/2} \to 0 \text{ as } \nu \to 0.\end{aligned}$$

For the term L_2, by using the Burkholder–Davis–Gundy inequality, Cauchy–Schwarz's inequality, condition 2.5 and (3.1), we have

$$\begin{aligned}\tilde{\mathbb{E}}|L_2| &\leqslant \sqrt{\nu}\tilde{\mathbb{E}}\left(\int_0^t |\Upsilon(s,u^\nu_{\hbar_\nu}(s))|^2_{L_Q}\|\varphi\|^2\right)^{1/2}\\ &\leqslant \sqrt{\nu}\|\varphi\|C(T,M)\left(1+\tilde{\mathbb{E}}|\varsigma|^4_H\right)^{1/2} \to 0 \text{ as } \nu \to 0.\end{aligned}$$

For the term L_3, we use the Burkholder–Davis–Gundy inequality, Hölder's inequality and condition 2.5 to deduce

$$\begin{aligned}\tilde{\mathbb{E}}|L_3| &\leqslant \nu\tilde{\mathbb{E}}\left(\sup_{t\in[0,T]}\int_0^t\int_{\mathbb{Y}}(G(s,u^\nu_{\hbar_\nu}(s-),y)\tilde{N}^{\nu^{-1}}(\mathrm{d}y,\mathrm{d}s),\varphi)\right)\\ &\leqslant \nu\tilde{\mathbb{E}}\left(\int_0^T\int_{\mathbb{Y}}(G(s,u^\nu_{\hbar_\nu}(s-),y),\varphi)^2\vartheta(\mathrm{d}y)\,\mathrm{d}t\right)^{\frac{1}{2}}\\ &\leqslant \nu\|\varphi\|C(T,M)\left(1+\tilde{\mathbb{E}}|\varsigma|^4_H\right)^{1/2} \to 0 \text{ as } \nu \to 0.\end{aligned}$$

For the term L_4, using (2.2), Hölder's inequality and (3.6), the fact that $u^\nu_{\hbar_\nu} - u \to 0$ in $L^2(0,T;\mathcal{H})$ $\tilde{\mathbb{P}}$-a.s. as $\nu \to 0$, (3.6), (4.2), the Vitali convergence theorem, we obtain

$$\begin{aligned}\tilde{\mathbb{E}}|L_4| &\leqslant C\tilde{\mathbb{E}}\int_0^t \|u^\nu_{\hbar_\nu}(s)-u(s)\|_{\mathcal{H}}\left(\|u^\nu_{\hbar_\nu}(s)\|_{\mathcal{H}}+\|u(s)\|_{\mathcal{H}}\right)\|\varphi\|\mathrm{d}s\\ &\leqslant C\|\varphi\|\left(\tilde{\mathbb{E}}\int_0^t \|u^\nu_{\hbar_\nu}(s)-u(s)\|^2_{\mathcal{H}}\mathrm{d}s\right)^{1/2}\left(\tilde{\mathbb{E}}\int_0^t\left[\|u^\nu_{\hbar_\nu}(s)\|^2_V+\|u(s)\|^2_V\right]\mathrm{d}s\right)^{1/2}\\ &\leqslant C(T,M)\|\varphi\|\left(1+\tilde{\mathbb{E}}\|\varsigma\|^4\right)^{1/2}\left(\tilde{\mathbb{E}}\int_0^t \|u^\nu_{\hbar_\nu}(s)-u(s)\|^2_{\mathcal{H}}\mathrm{d}s\right)^{1/2} \to 0.\end{aligned}$$

For the term L_5, condition 6.1, Cauchy–Schwarz's inequality and (3.1) yield that

$$\begin{aligned}\tilde{\mathbb{E}}|L_5| &\leqslant \tilde{\mathbb{E}}\int_0^t \left|\Upsilon_\nu(s,u^\nu_{\hbar_\nu}(s))-\Upsilon_0(s,u^\nu_{\hbar_\nu}(s))\right|_{L(H_0,H)}|f_\nu(s)|_0|\varphi|_H\mathrm{d}s\\ &\leqslant |\varphi|_H\sqrt{MT}\tilde{\mathbb{E}}\Big(\int_0^t \left|\Upsilon_\nu(s,u^\nu_{\hbar_\nu}(s))-\Upsilon_0(,u^\nu_{\hbar_\nu}(s))\right|^2_{L(H_0,H)}\mathrm{d}s\Big)^{1/2}\end{aligned}$$

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$$\leqslant C_\nu \, |\varphi|_H \sqrt{MT} \tilde{\mathbb{E}} \left(\int_0^t \left[1 + |u^\nu_{\hbar_\nu}(s)|^2_H \right] \mathrm{d}s \right)^{1/2}$$

$$\leqslant C_\nu |\varphi|_H C(T,M) \left(1 + \tilde{\mathbb{E}} |\varsigma|^2_H \right)^{1/2} \to 0 \text{ as } \nu \to 0.$$

For $\nu \in (0, \nu_0]$, by condition 2.5 and Hölder's inequality, we obtain

$$\tilde{\mathbb{E}}|L_6| \leqslant \tilde{\mathbb{E}} \int_0^t \left| \Upsilon_0(s, u^\nu_{\hbar_\nu}) - \Upsilon_0(s, u(s)) \right|_{L_Q(H_0,H)} |f_\nu(s)|_0 |\varphi|_H \mathrm{d}s$$

$$\leqslant C |\varphi|_H \tilde{\mathbb{E}} \left(\int_0^t \sqrt{K(s)} |u^\nu_{\hbar_\nu}(s) - u(s)|_H |f_\nu(s)|_0 \mathrm{d}s \right).$$

For any $\varepsilon > 0$, let $A_{\nu,\varepsilon} = \{t \in [0,T] : |u^\nu_{\hbar_\nu}(s) - u(s)|_H > \varepsilon\}$, then we have

$$\lim_{\nu \to 0} \lambda_T(A_{\nu,\varepsilon}) \leqslant \lim_{\nu \to 0} \frac{\tilde{\mathbb{E}} \int_0^T |u^\nu_{\hbar_\nu}(t) - u(t)|^2_H \mathrm{d}t}{\varepsilon^2} = 0.$$

Set $\tilde{M} = \sup_{\nu \in (0,\nu_0]} \tilde{\mathbb{E}}(\sup_{t\in[0,T]} |u^\nu_{\hbar_\nu}(t)|_H) \vee \tilde{\mathbb{E}}(\sup_{t\in[0,T]} |u(t)|_H) < \infty$. Then, one gets

$$\tilde{\mathbb{E}}|L_6| \leqslant 2\tilde{M}|\varphi|_H \sqrt{\int_{A_{\nu,\varepsilon}} K(t)\,\mathrm{d}t} \sqrt{\int_0^T |f(t)|^2_0 \mathrm{d}t} + \varepsilon |\varphi|_H \int_{A^c_{\nu,\varepsilon}} \sqrt{K(t)} |f(t)|_0 \mathrm{d}t$$

$$\leqslant 2\tilde{M}|\varphi|_H \sqrt{\int_{A_{\nu,\varepsilon}} K(t)\,\mathrm{d}t} \sqrt{\int_0^T |f(t)|^2_0 \mathrm{d}t}$$

$$+ \varepsilon |\varphi|_H \left(\int_0^T K(t)\,\mathrm{d}t + \int_0^T |f(t)|^2_0 \mathrm{d}t \right)$$

$$\to 0 \text{ as } \nu \to 0.$$

For the term L_7, we have

$$\tilde{\mathbb{E}}|L_7| = \tilde{\mathbb{E}} \left| \int_0^t \left(f_\nu(s) - f(s), \Upsilon^*_0(s, u(s))\varphi \right) \mathrm{d}s \right|.$$

It follows from condition 2.5 that

$$\int_0^T |\Upsilon^*_0(s, u(s))\varphi|^2_0 \mathrm{d}s \leqslant |\varphi|^2_H \int_0^T K(s)\big(1 + |u(s)|^2_H\big)\,\mathrm{d}s \leqslant C.$$

We note that $f_\nu - f \to 0$ in $L^2(0,T;H_0)$ for the weak topology $\tilde{\mathbb{P}}$-a.s. as $\nu \to 0$. Hence, $\int_0^t (f_\nu(s) - f(s), \Upsilon^*_0(s, u(s))\varphi)\mathrm{d}s \to 0$ $\tilde{\mathbb{P}}$-a.s. as $\nu \to 0$. On the other hand, (4.2) implies that $\tilde{\mathbb{E}}(\int_0^t (f_\nu(s) - f(s), \Upsilon^*_0(s, u(s))\varphi)\mathrm{d}s)^2 \leqslant C$. Applying the Vitali convergence theorem and (6), we can infer that $\tilde{\mathbb{E}}|L_7| \to 0$ as $\nu \to 0$.

Finally, by condition 2.6, lemma 2.9 and the fact that $u^{\nu}_{\hbar_\nu} \to u$ in $L^2(0,T;H)$ $\tilde{\mathbb{P}}$-a.s. as $\nu \to 0$, we get for fixed $\tilde{\omega} \in \tilde{\Omega}$

$$\int_0^t \int_{\mathbb{Y}} [(G(s, u^{\nu}_{\hbar_\nu}(s), y)(g(s,y)-1), \varphi) \\ - (G(s, u(s), y)(g(s,y)-1), \varphi)]\vartheta(\mathrm{d}y)\,\mathrm{d}s \to 0.$$

In fact, we have

$$\begin{aligned}
&\int_0^t \int_{\mathbb{Y}} |(G(s, u^{\nu}_{\hbar_\nu}(s), y)(g(s,y)-1), \varphi) - (G(s, u(s), y)(g(s,y)-1), \varphi)|\vartheta(\mathrm{d}y)\,\mathrm{d}s \\
&\quad \leqslant C \int_0^t \int_{\mathbb{Y}} \|G(s,y)\|_{1,H} |u^{\nu}_{\hbar_\nu}(s) - u(s)|_H |g(s,y)-1|\vartheta(\mathrm{d}y)\,\mathrm{d}s \\
&\quad \leqslant C\tilde{M} \int_{A_{\nu,\varepsilon}} \int_{\mathbb{Y}} \|G(s,y)\|_{1,H} |g(s,y)-1|\vartheta(\mathrm{d}y)\,\mathrm{d}s \\
&\qquad + \varepsilon \int_{A^c_{\nu,\varepsilon}} \int_{\mathbb{Y}} \|G(s,y)\|_{1,H} |g(s,y)-1|\vartheta(\mathrm{d}y)\,\mathrm{d}s \\
&\quad \leqslant C\tilde{M}\tilde{\eta} + C\varepsilon \to 0,
\end{aligned}$$

since $\tilde{\eta}$ and ε are arbitrary. Moreover, condition 2.5, lemmas 2.9 and 2.10 yields that as $\nu \to 0$, it holds

$$\int_0^t \int_{\mathbb{Y}} (G(s, u^{\nu}_{\hbar_\nu}(s), y)(g_\nu(s,y)-1), \varphi)\vartheta(\mathrm{d}y)\,\mathrm{d}s \\ \to \int_0^t \int_{\mathbb{Y}} (G(s, u^{\nu}_{\hbar_\nu}(s), y)(g(s,y)-1), \varphi)\vartheta(\mathrm{d}y)\,\mathrm{d}s.$$

For the completeness, we give the details of the proof. First, from lemma 2.10, we know that for every $\varepsilon > 0$, there exists a compact set $\mathbb{K} \subset \mathbb{Y}$ so that

$$\int_0^t \int_{\mathbb{K}^c} (G(s, u^{\nu}_{\hbar_\nu}(s), y)(g_{\cdot}(s,y)-1), \varphi)\vartheta(\mathrm{d}y)\,\mathrm{d}s \\ \leqslant C \sup \int_{[0,T]\times\mathbb{K}^c} \|G(s, u^{\nu}_{\hbar_\nu}(s), y)\|_{0,H} |g_{\cdot}(s,y)-1)|\vartheta(\mathrm{d}y)\,\mathrm{d}s \leqslant \varepsilon.$$

Here $g_{\cdot}(s,y) = g_\nu(s,y)$ or $g(s,y)$. Then, to prove the above convergence, we need to prove that for every compact $\mathbb{K} \subset \mathbb{Y}$, it holds that

$$\int_{\mathbb{K}_T} (G(s, u^{\nu}_{\hbar_\nu}(s), y)(g_\nu(s,y)-1), \varphi)\vartheta(\mathrm{d}y)\,\mathrm{d}s \\ \to \int_{\mathbb{K}_T} (G(s, u^{\nu}_{\hbar_\nu}(s), y)(g(s,y)-1), \varphi)\vartheta(\mathrm{d}y)\,\mathrm{d}s.$$

Here $\mathbb{K}_T := [0,T] \times \mathbb{K}$. If $G(s,\cdot,y)$ is bounded, then $\int_{\mathbb{Y}_T} (G(s,\cdot,y), \varphi)\vartheta(\mathrm{d}y)\mathrm{d}s < \infty$, and we just need to prove that

$$\int_{\mathbb{K}_T} (G(s, u^{\nu}_{\hbar_\nu}(s), y)g_\nu(s,y), \varphi)\vartheta(\mathrm{d}y)\,\mathrm{d}s \to \int_{\mathbb{K}_T} (G(s, u^{\nu}_{\hbar_\nu}(s), y)g(s,y), \varphi)\vartheta(\mathrm{d}y)\,\mathrm{d}s.$$

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For this, we introduce $\theta(\cdot) = \frac{\vartheta(\cdot\cap[0,T]\times\mathbb{K})}{\vartheta([0,T]\times\mathbb{K})}$ as appendix in [**13**]. Hence, we know that θ is a probability measure on $[0,T]\times\mathbb{K}$. By Hölder's inequality and lemma 2.9, we have

$$\begin{aligned}\int_{\mathbb{K}_T} &|(G(s,u^\nu_{\hbar_\nu}(s),y)-G(s,u(s),y),\varphi)|\theta(\mathrm{d}y)\,\mathrm{d}s\\ &\leqslant C\frac{\vartheta(\mathbb{K})}{\vartheta([0,T]\times\mathbb{K})}\left(\int_{\mathbb{K}_T}\|G(s,u(s),y),\varphi)\|^2_{1,H}\vartheta(\mathrm{d}y)\,\mathrm{d}s\right)^{\frac{1}{2}}\\ &\quad\times\left(\int_0^T\|u^\nu_{\hbar_\nu}(s)-u(s)\|^2_H\mathrm{d}s\right)^{\frac{1}{2}}\to 0.\end{aligned}$$

Then, we deduce that $(G(s,u^\nu_{\hbar_\nu}(s),y),\varphi)\to G(s,u(s),y),\varphi)$ θ-a.s. Setting $\tilde{\vartheta}^\nu = \frac{\vartheta^{g^\nu}(\cdot\cap\mathbb{K})}{\int_{\mathbb{K}} g_\nu\vartheta(\mathrm{d}y)\mathrm{d}s}$ and $\tilde{\vartheta} = \frac{\vartheta^{g}(\cdot\cap\mathbb{K})}{\int_{\mathbb{K}} g\vartheta(\mathrm{d}y)\mathrm{d}s}$, we know that $\tilde{\vartheta}^\nu$ converges weakly to $\tilde{\vartheta}$. By lemma 2.8 in [**9**] and the fact that $\int_{\mathbb{K}} g_\nu\vartheta(\mathrm{d}y)\mathrm{d}s\to\int_{\mathbb{K}} g\vartheta(\mathrm{d}y)\mathrm{d}s$, we get

$$\int_{\mathbb{K}_T}(G(s,u^\nu_{\hbar_\nu}(s),y)g_\nu(s,y),\varphi)\vartheta(\mathrm{d}y)\,\mathrm{d}s\to\int_{\mathbb{K}_T}(G(s,u(s),y)g(s,y),\varphi)\vartheta(\mathrm{d}y)\,\mathrm{d}s,$$

and

$$\int_{\mathbb{K}_T}(G(s,u^\nu_{\hbar_\nu}(s),y)g(s,y),\varphi)\vartheta(\mathrm{d}y)\,\mathrm{d}s\to\int_{\mathbb{K}_T}(G(s,u(s),y)g(s,y),\varphi)\vartheta(\mathrm{d}y)\,\mathrm{d}s.$$

For general $G(s,\cdot,y)$, it is enough to get

$$\tilde{\Gamma} = \sup\int_{\mathbb{K}_T}\|G(s,\cdot,y)\|_{0,H}1_{\{\|G(s,\cdot,y)\|_{0,H}>\mathcal{M}\}}g_\cdot(s,y)\vartheta(\mathrm{d}y)\,\mathrm{d}s\to 0.$$

Since $ab\leqslant e^{La}+\frac{1}{L}l(b)$ for $a,b\in(0,\infty), L\in[1,\infty)$, where $l(b)=b\ln b-b+1$ (see remark 3.3 in [**13**]) and using the definition of S^M, we have

$$\begin{aligned}|\Gamma| &\leqslant \sup\int_{\mathbb{K}_T\cap\{\|G\|_{0,H}>\mathcal{M}\}} e^{L\|G(s,\cdot,y)\|_{0,H}}\vartheta(\mathrm{d}y)\,\mathrm{d}s+\frac{1}{L}\sup_\nu\int_{\mathbb{Y}_T} l(g_\cdot(s,y))\vartheta(\mathrm{d}y)\,\mathrm{d}s\\ &\leqslant \sup\int_{\mathbb{K}_T\cap\{\|G\|_{0,H}>\mathcal{M}\}} e^{L\|G(s,\cdot,y)\|_{0,H}}\vartheta(\mathrm{d}y)\,\mathrm{d}s+\frac{M}{L}.\end{aligned}$$

For any $\varepsilon>0$, we can choose a sufficiently large L such that $\frac{M}{L}<\varepsilon$ and take $\mathcal{M}_0$ large enough so that $\int_{([0,T]\times\mathbb{K})\cap\{\|G\|_{0,H}>\mathcal{M}\}} e^{L\|G(s,\cdot,y)\|_{0,H}}\vartheta(\mathrm{d}y)\mathrm{d}s\leqslant\varepsilon$ for $\mathcal{M}>\mathcal{M}_0$ (see remark 2.8). Since ε is arbitrary, we have $\tilde{\Gamma}\to 0$ as $\mathcal{M}\to\infty$. Hence, for fixed

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$\tilde{\omega} \in \tilde{\Omega}$, combining the above arguments, we arrive at

$$\int_0^t \int_{\mathbb{Y}} (G(s, u^{\nu}_{\hbar_\nu}(s), y)(g_\nu(s, y) - 1), \varphi)\vartheta(\mathrm{d}y)\,\mathrm{d}s \\ \to \int_0^t \int_{\mathbb{Y}} (G(s, u(s), y)(g(s, y) - 1), \varphi)\vartheta(\mathrm{d}y)\,\mathrm{d}s.$$

Therefore, as $\nu \to 0$, for any $t \in [0, T]$ and $\varphi \in D(A^\beta)$ with $\beta > 1/2$

$$\tilde{\mathbb{E}}\Bigg((u^{\nu}_{\hbar_\nu}(t), \varphi) - \int_0^t \big\langle - B(u(s), u(s)) + \sigma_0(s, u(s))f(s), \varphi\big\rangle \mathrm{d}s \\ + \int_0^t \int_{\mathbb{Y}} (G(s, u(s), y)(g(s, y) - 1), \varphi)\vartheta(\mathrm{d}y)\,\mathrm{d}s\Bigg) \to 0.$$

On the other hand, taking $\varphi \in D(A^\beta)$, we deduce that

$$\sup_{t\in[0,T]} |(u^{\nu}_{\hbar_\nu}(t) - u(t), \varphi)| \to 0, \quad \tilde{\mathbb{P}}\text{-a.s. as } \nu \to 0.$$

By (3.6), (4.2) and the Vitali convergence theorem, we can infer that

$$\tilde{\mathbb{E}}\left(\sup_{t\in[0,T]} |(u^{\nu}_{\hbar_\nu}(t) - u(t), \varphi)|\right) \to 0 \text{ as } \nu \to 0.$$

Then, u is a solution of (4.1). By applying theorem 4.4, we obtain $u = u^0_\hbar$. It follows from theorems 4.1 and 4.4 that $u^0_\hbar$ belongs to $\mathcal{C}([0, T]; H) \cap L^\infty(0, T; V \cap H^{1,q})$. Note that $\tilde{u}^{\nu_n}_{\hbar_{\nu_n}}$ and $u^{\nu_n}_{\hbar_{\nu_n}}$ has the same law. Hence, when $\nu_n \to 0$, we can chose a subsequence $\{\nu_{n_k}\}_{k\geqslant 0}$ such that $u^{\nu_{n_k}}_{\hbar_{\nu_{n_k}}}$ converges to $u^0_\hbar$ in distribution in $\mathcal{X}$. Then, $u^{\nu}_{\hbar_\nu}$ converges in distribution to $u^0_\hbar$. By applying the facts that $u^{\nu}_{\hbar_\nu}(s) = \mathcal{G}^{\nu}_{\xi}(\sqrt{\nu}(W(s) + \frac{1}{\sqrt{\nu}} \int_0^{\cdot} f_\nu(s)\mathrm{d}s), \nu N^{\nu^{-1} g_\nu})$ and $u^0_\hbar = \mathcal{G}^0_\varsigma(\int_0^{\cdot} f(s)\mathrm{d}s, \vartheta^g)$, we get the desired result. Therefore, we complete the proof. □

Next, we verify the first part of condition 2.3.

PROPOSITION 6.4. *For all $q \in [2, +\infty)$, we assume that $(\tilde{\Upsilon}_0, \tilde{G})$ satisfies condition 2.6, 2.7, 2.12, 4.3. For fixed $M > 0$, deterministic initial data $\varsigma \in \mathcal{Y}$ and let $(f, g) \in \bar{S}^M$ be such that $(f_n, g_n) \to (f, g)$ as $n \to 0$. Then*

$$\mathcal{G}_0\left(\int_0^{\cdot} f_n(s)\,\mathrm{d}s, \vartheta^{g_n}_T\right) \to \mathcal{G}_0\left(\int_0^{\cdot} f(s)\,\mathrm{d}s, \vartheta^{g}_T\right).$$

Proof. Let $\{u_n\}_{n\geqslant 1}$ be a sequence corresponding to the solutions of (4.1) with control $\{(f_n, g_n)\}_{n\geqslant 1}$ in $\bar{S}^M$:

$$\begin{aligned} &\mathrm{d}u_n(t) + B(u_n(t), u_n(t))\,\mathrm{d}t \\ &\quad = \tilde{\Upsilon}_0(t, u_n(t))f_n(t)\,\mathrm{d}t + \int_{\mathbb{Y}} \tilde{G}(t, u_n(t), y)(g_n(t,y)-1)\vartheta(\mathrm{d}y)\,\mathrm{d}t \end{aligned}$$

with the initial data $u_n(0) = \varsigma$. First, for any $p, q > 2$ and $\alpha < \frac{1}{2}$, we will prove that u_n is bounded in $W^{1,2}(0,T;L^q)\cap W^{\alpha,p}(0,T;L^q)\cap L^2(0,T;H^{1,q})$. In fact, we can rewrite the above equality as $u_n(t) = \varsigma + W_1(t) + W_2(t) + W_3(t)$, where

$$\begin{aligned} &W_1(t) = -\int_0^t B(u_n(s), u_n(s))\,\mathrm{d}s, W_2(t) = \int_0^t \tilde{\Upsilon}_0(s, u_n(s))f_n(s)\,\mathrm{d}s, \\ &W_3(t) = \int_0^t \int_{\mathbb{Y}} \tilde{G}(s, u_n(s), y)(g_n(s,y)-1)\vartheta(\mathrm{d}y)\,\mathrm{d}s. \end{aligned}$$

For the term W_1, by using Hölder's inequality, (A.8), and (4.9), we have

$$\|W_1\|^q_{W^{1,q}(0,T;L^q)} \leqslant C(T)\sup_{t\in[0,T]}\|u_n(t)\|^{2q}_{H^{1,q}} \leqslant C(q,T,M)\left(1+\|\varsigma\|+\|\operatorname{curl}\varsigma\|_q\right)^{2q}.$$

For the term W_2, it follows from Minkowski's inequality, (A.1), (A.3), condition 2.6 and (4.2) that

$$\begin{aligned} \|W_2\|^2_{L^q} &\leqslant C(T,q)\left(\int_0^T \|\tilde{\Upsilon}_0(t,u_n(t))f_n(t)\|_q dt\right)^2 \\ &\leqslant C(T,q)\left(\int_0^T \|\tilde{\Upsilon}_0(t,u_n(t))f_n(t)\|\mathrm{d}t\right)^2 \\ &\leqslant C(T,q)\left(\int_0^T |\operatorname{curl}\tilde{\Upsilon}_0(t,u_n(t))|_{L(H_0,H)}|f_n(t)|_0\mathrm{d}t\right)^2 \\ &\leqslant C(T,q)\left(1+\sup_{t\in[0,T]}\|u_n(t)\|^2\right)\int_0^T K(t)\,\mathrm{d}t\int_0^T |f(t)|_0^2\mathrm{d}t \\ &\leqslant C(T,q,M)\left(1+\|\varsigma\|^2\right). \end{aligned}$$

Applying Minkowski's and Hölder's inequalities, the Sobolev embedding theorem, (A.3), conditions 2.6 and 2.12, (4.2) and (A.1), we obtain

$$\begin{aligned} &\int_0^T \|W_2(t)\|_q^p\mathrm{d}t \\ &\quad\leqslant \int_0^T \left|\int_0^t \|\tilde{\Upsilon}_0(s,u_n(s))f_n(s)\|_q ds\right|^p dt \\ &\quad\leqslant C(q)\int_0^T \left|\int_0^t \|\tilde{\Upsilon}_0(s,u_n(s))f_n(s)\|\mathrm{d}s\right|^p dt \end{aligned}$$

$$
\begin{aligned}
&\leqslant C(q) \sup_{t\in[0,T]} |\tilde{\Upsilon}_0(t,u_n(t))|^p_{L(H_0,V)} \int_0^T \Big| \int_0^t \Big(K(s)+|f_n(s)|_0^2\Big)\,\mathrm{d}s\Big|^p dt \\
&\leqslant C(p,q,T,M)\left(1+\sup_{t\in[0,T]}\|u(t)\|^p\right) \\
&\leqslant C(p,q,T,M)\left(1+\|\varsigma\|^p\right).
\end{aligned}
$$

Moreover, for $\alpha\in(0,\frac{1}{2})$, we use conditions 2.6 and 2.12 to get

$$
\begin{aligned}
&\int_0^T\int_0^T \frac{\|W_2(t)-W_2(s)\|_q^p}{(t-s)^{1+\alpha p}}\,\mathrm{d}s\,\mathrm{d}t \\
&\quad\leqslant 2C(q)\int_0^T\int_0^T (t-s)^{-1-\alpha p}\Big|\int_s^t \|\tilde{\Upsilon}_0(r,u_n(r))f_n(r)\|\mathrm{d}r\Big|^p\,\mathrm{d}s\,\mathrm{d}t \\
&\quad\leqslant 2C(q)C\left(1+\sup_{r\in[0,T]}\|u_n(r)\|^p\right)\int_0^T\int_0^T (t-s)^{-1-\alpha p}\,\mathrm{d}s\,\mathrm{d}t \\
&\quad\leqslant C(q,T,M)\left(1+\|\varsigma\|^p\right).
\end{aligned}
$$

For the term W_3, by using condition 2.6, 2.7 and lemma 2.9, one deduces that

$$
\begin{aligned}
&\|W_3(t)\|^2_{L^2(0,T;L^q)} \\
&\leqslant C(T,q)\int_0^T\left\|\int_0^T\int_{\mathbb{Y}}\tilde{G}(t,u_n(t),y)(g_n(t,y)-1)\vartheta(\mathrm{d}y)\,\mathrm{d}t\right\|_q^2\mathrm{d}t \\
&\leqslant C\Big(1+\sup_{t\in[0,T]}\|u_n(t)\|^2\Big)\int_0^T\left(\int_0^T\int_{\mathbb{Y}}\|\tilde{G}(t,y)\|_{0,q}(g_n(t,y)-1)\vartheta(\mathrm{d}y)\,\mathrm{d}t\right)^2\mathrm{d}t \\
&\leqslant C(T,q,M)\left(1+\|\varsigma\|^2\right).
\end{aligned}
$$

Then, for $\alpha\in(0,\frac{1}{2})$, we have

$$
\begin{aligned}
&\int_0^T\int_0^T\frac{\|W_3(t)-W_3(s)\|_q^p}{(t-s)^{1+\alpha p}}\,\mathrm{d}s\,\mathrm{d}t \\
&\quad\leqslant 2C(q)\int_0^T\int_0^t (t-s)^{-1-\alpha p} \\
&\qquad\times\Big|\int_s^t\int_{\mathbb{Y}}\|\tilde{G}(r,u_n(r),y)(g_n(r,y)-1)\|\vartheta(\mathrm{d}y)\,\mathrm{d}r\Big|^p\,\mathrm{d}s\,\mathrm{d}t \\
&\quad\leqslant 2C(q)C\left(1+\sup_{r\in[0,T]}\|u_n(r)\|^p\right)\int_0^T\int_0^t (t-s)^{-1-\alpha p}\,\mathrm{d}s\,\mathrm{d}t \\
&\quad\leqslant C(q,T)\left(1+\|\varsigma\|^p\right).
\end{aligned}
$$

From the above estimates of W_1-W_3, we can infer that $\{u_n\}_{n\geqslant 1}$ is relatively compact in $\mathcal{C}([0,T];D(A^{-\beta}))\cap L^2(0,T;\mathcal{H})$ with $\beta>1/2$ by using Aubin-Lions

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Lemma (see also [8, 38]). Then, there exists a subsequence, still denoted $\{u_n\}$, which converges to some element u in $\mathcal{C}([0,T];D(A^{-\beta}))\cap L^2(0,T;\mathcal{H})$. Finally, it remains to check that u is a solution to the following equations:

$$\mathrm{d}u(t)+B(u(t),u(t))\,\mathrm{d}t=\tilde{\Upsilon}_0(t,u(t))f(t)\,\mathrm{d}t \\ +\int_{\mathbb{Y}}\tilde{G}(t,u(t),y)(g(t,y)-1)\vartheta(\mathrm{d}y)\,\mathrm{d}t,\quad u(0)=\varsigma.$$

Similar to the proof of step 4 in proposition 6.3, we can prove the desired result. □

Proof of theorem 6.2.. According to theorem 2.4, using propositions 6.3 and 6.4, we can get the desired result of theorem 6.2. Then, we complete the proof. □

Appendix A.

In this section, we first recall the following classical Sobolev embeddings. Let D be a bounded domain in $\mathbb{R}^2$ and satisfy the cone condition. Then, we have (see e.g. [2, 8]):

$$\|u\|_q\leqslant C(q)\|u\|_{W^{1,2}}\text{ for }u\in W^{1,2}\text{ and }1\leqslant q<+\infty,\tag{A.1}$$

$$W^{2,1}\subset\mathcal{C}^0_B(D),\ W^{1,q}\subset\mathcal{C}^0_B(D)\text{ for }q\in(2,\infty).\tag{A.2}$$

Next, we recall the following result in [35] (see also [8, 12, 58]. For a given $q\in[2,\infty)$, there exists a constant C such that for every $u\in H^{1,q}$, it holds that

$$\|\nabla u\|_q\leqslant Cq\|\text{ curl }u\|_q\text{ for }q\in[2,\infty).\tag{A.3}$$

Moreover, for a given $q\in[2,\infty)$ and $r>0$, the operator B has a unique extension to a continuous bilinear operator from $H^{1,q}\times H^{1,q}$ to $H^{-r,q}$ and for all $u,v\in H^{1,q}$ resp. $\varphi,\psi\in D(A)$, it holds that

$$\|B(u,v)\|_{H^{-r,q}}\leqslant C\|u\|_{H^{1,q}}\|v\|_{H^{1,q}},\tag{A.4}$$

$$\langle B(u,v),v\rangle=0,\tag{A.5}$$

$$\langle\text{curl }B(\varphi,\varphi),\psi\rangle=\langle\varphi\cdot\nabla(\text{ curl }\varphi),\psi\rangle=\langle B(\varphi,\text{ curl }\varphi),\psi\rangle,\tag{A.6}$$

$$\langle\text{curl }B(u,v),\text{ curl }v|\text{curl }v|^{q-2}\rangle=0\text{ for all }u,v\in H^{2,q}\cap D(A).\tag{A.7}$$

Finally, if $q>2$, for all $u,v\in H^{1,q}$, it holds that

$$|B(u,v)|_H\leqslant C\|u\|_{H^{1,q}}\|v\|_{H^{1,2}}\text{ and }\|B(u,v)\|_q\leqslant C\|u\|_{H^{1,q}}\|v\|_{H^{1,q}}.\tag{A.8}$$

Here $C>0$ is a constant.

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