Canad. Math. Bull. Vol. **18** (1), 1975

A COUNTER-EXAMPLE TO COHERENCE IN CARTESIAN CLOSED CATEGORIES

BY

M. E. SZABO

0. **Introduction.** It follows from [3] that all morphisms of free closed categories on finite discrete categories are components of natural or "generalized" natural transformations, and from [8] that all hom-sets of such categories are finite. The purpose of this paper is to show that neither statement remains true if the categories are also assumed to be *cartesian.*

1. **Cartesian closed categories.** For our present purposes, we define a cartesian closed category to be a list $\langle \mathbf{K}, T, \wedge, \supset, \Phi, \Psi, \Omega\rangle$ consisting of a category $\mathbf{K}$, a distinguished object T of $\mathbf{K}$, a bifunctor $\wedge:\mathbf{K}\times\mathbf{K}\to\mathbf{K}$, a bifunctor $\supset:\mathbf{K}^{op}\times\mathbf{K}\to\mathbf{K}$, and natural bijections Φ, Ψ, and Ω with the following components:

(i) $\Phi(A):[A, T]\cong\{*\}$ for each $A\in \mathrm{Obj}(\mathbf{K})$;

(ii) $\Psi(A, B, C):[A, B\wedge C]\cong[A, B]\times[A, C]$ for each $A, B, C\in\mathrm{Obj}(\mathbf{K})$;

(iii) $\Omega(A, B, C):[A\wedge B, C]\cong[B, A\supset C]$ for each $A, B, C\in\mathrm{Obj}(\mathbf{K})$,

where $\mathrm{Obj}(\mathbf{K})$ stands for the class of objects of $\mathbf{K}$, $[A, B]$ denotes $\mathrm{Hom}_{\mathbf{K}}(A, B)$, and $\{*\}$ is a fixed one-point set.

By the *free* cartesian closed category on a category $\mathbf{C}$ we mean the unique (up to isomorphism) category $F(\mathbf{C})$ with the property that for any functor $\theta:\mathbf{C}\to U(\mathbf{D})$, where $U(\mathbf{D})$ is the underlying category of a cartesian closed category $\mathbf{D}$, there exists a unique functor $\theta':F(\mathbf{C})\to\mathbf{D}$ which preserves the cartesian closed structure *exactly*, i.e., $\theta'(A\wedge B)=\theta'(A)\wedge\theta'(B)$, etc. $F(\mathbf{C})$ can be explicitly constructed using the methods of [4, 5]. Its existence already follows from Freyd's Adjoint Functor Theorem. For a discussion of this theorem and for all undefined concepts of this paper the reader is referred to [7].

2. **The counter-example.** It is sufficient for our purposes to take $\mathbf{C}=\mathbf{1}$, where $\mathbf{1}$ is a discrete one-object category whose only object will be denoted by "0". Let

$$\varepsilon(A, B) = \Omega^{-1}(A, A\supset B, B)\,(1(A\supset B)): A\wedge(A\supset B)\to B,$$

$$\delta(A) = \psi^{-1}(A, A, A)(\langle 1(A), 1(A)\rangle): A\to A\wedge A,$$

and let

$$\alpha(A, B, C): A\wedge(B\wedge C)\to(A\wedge B)\wedge C$$

Received by the editors December 6, 1972 and, in revised form, August 24, 1973.

https://doi.org/10.4153/CMB-1975-020-1 Published online by Cambridge University Press

be the unique isomorphism induced by Φ between the co-representing objects $A \wedge (B \wedge C)$ and $(A \wedge B) \wedge C$ of the functor $[X, A]\times[X, B]\times[X, C]: F(\mathbf{1})\to Sets$, for $A, B, C \in \mathrm{Obj}(F(\mathbf{1}))$, and consider the following inductively defined sequence of morphisms of $F(\mathbf{1})$:

(i)

$$\begin{array}{ccl}
0\wedge(0\supset 0) & \xrightarrow{1(0)\wedge\delta(0\supset 0)} & 0\wedge((0\supset 0)\wedge(0\supset 0)) \\
 & & \downarrow \alpha(0,0\supset 0,0\supset 0) \\
 & & (0\wedge(0\supset 0))\wedge(0\supset 0) \\
 & \searrow{\scriptstyle \omega_1(0)} & \downarrow \varepsilon(0,0)\wedge 1(0\supset 0) \\
 & & 0\wedge(0\supset 0) \\
 & & \downarrow \varepsilon(0,0) \\
 & & 0
\end{array};$$

(ii)

$$\begin{array}{ccl}
0\wedge(0\supset 0) & \xrightarrow{1(0)\wedge\delta(0\supset 0)} & 0\wedge((0\supset 0)\wedge(0\supset 0)) \\
 & & \downarrow 1(0)\wedge(1(0\supset 0)\wedge\delta(0\supset 0)) \\
 & & 0\wedge((0\supset 0)\wedge((0\supset 0)\wedge(0\supset 0))) \\
 & & \downarrow \alpha(0,0\supset 0,(0\supset 0)\wedge(0\supset 0)) \\
 & & (0\wedge(0\supset 0))\wedge((0\supset 0)\wedge(0\supset 0)) \\
 & & \downarrow \varepsilon(0,0)\wedge 1((0\supset 0)\wedge(0\supset 0)) \\
 & \searrow{\scriptstyle \omega_2(0)} & 0\wedge((0\supset 0)\wedge(0\supset 0)) \\
 & & \downarrow \alpha(0,0\supset 0,0\supset 0) \\
 & & (0\wedge(0\supset 0))\wedge(0\supset 0) \\
 & & \downarrow \varepsilon(0,0)\wedge 1(0\supset 0) \\
 & & 0\wedge(0\supset 0) \\
 & & \downarrow \varepsilon(0,0) \\
 & & 0
\end{array}$$

etc.

We claim that $\omega_i(0)\neq\omega_j(0)$ if $i\neq j$. Since $F(\mathbf{1})$ is free, it is sufficient to find a convenient cartesian closed category in which these inequalities hold and where they can be easily calculated. The category of sets and functions is the obvious choice. Let 0 correspond to the natural number object $\mathbb{N}$, in which case $0\supset 0$ corresponds to the function set $\mathbb{N}^{\mathbb{N}}$, and $\wedge$ stands for the cartesian product of sets. Let $\langle a, f\rangle \in \mathbb{N}\times\mathbb{N}^{\mathbb{N}}$, with "$f$" denoting the successor function. Then it is clear that $\omega_i(\mathbb{N})(\langle a, f\rangle)=f^{i+1}(a)=a+(i+1)$. Hence $\omega_i(\mathbb{N})=\omega_j(\mathbb{N})$ iff $i=j$.

Next we claim that the morphisms $\omega_i(0)$ are not components of natural or

https://doi.org/10.4153/CMB-1975-020-1 Published online by Cambridge University Press

"generalized" natural transformations. We assume familiarity with the usual meaning of "naturality". A transformation connecting functors in several variables is natural if it is natural in each variable. In this sense δ and α are natural transformations, whereas ε is not. However, by abandoning the requirement of the functorality in A of the expression $A \wedge (A \supset B)$, Eilenberg and Kelly in [1] exploited the commutativity properties of ε, and admitted it as a transformation connecting the trifunctor $1_{F(\mathbf{1})} \wedge (1_{F(\mathbf{1})} \supset 1_{F(\mathbf{1})})$ (whose object function is defined by $1_{F(\mathbf{1})} \wedge (1_{F(\mathbf{1})} \supset 1_{F(\mathbf{1})})(A, B, C) = A \wedge (B \supset C)$) and the identity functor $1_{F(\mathbf{1})}$, since its components make the following diagrams commute for all $A, A', B, B' \in \mathrm{Obj}(F(\mathbf{1}))$:

(i)
$$\begin{array}{ccc} B \wedge (B \supset A) & \xrightarrow{1(B)\wedge(1(B)\supset f)} & B \wedge (B \supset A') \\ {\scriptstyle \varepsilon(B,A)}\downarrow & & \downarrow{\scriptstyle \varepsilon(B,A')} \\ A & \xrightarrow{\quad f \quad} & A' \end{array}$$

(ii)
$$\begin{array}{ccc} B \wedge (B' \supset A) & \xrightarrow{g\wedge 1(B'\supset A)} & B' \wedge (B' \supset A) \\ \downarrow{\scriptstyle 1(B)\wedge(g\supset 1(A))} & & \downarrow{\scriptstyle \varepsilon(B',A)} \\ B \wedge (B \supset A) & \xrightarrow{\varepsilon(B,A)} & A \end{array} .$$

In [1], ε and similar transformations were called "generalized" natural transformations.

The ω_i connect objects of the form $A \wedge (B \supset C)$ and D, and by [1] are therefore candidates for generalized natural transformations between the functors $1_{F(\mathbf{1})} \wedge (1_{F(\mathbf{1})} \supset 1_{F(\mathbf{1})})$ and $1_{F(\mathbf{1})}$. However, the definition of generalized naturality requires that diagrams (i) and (ii) commute for all $A, A', B, B' \in \mathrm{Obj}(F(\mathbf{1}))$ with ω_i in place of ε, and since the ω_i are defined iff $A = A' = B = B'$, they are therefore not generalized natural. We might try to extend the notion of generality further by requiring only that the following special cases (i′) and (ii′) of (i) and (ii) commute for all $A \in \mathrm{Obj}(F(\mathbf{1}))$:

(i′)
$$\begin{array}{ccc} A \wedge (A \supset A) & \xrightarrow{1(A)\wedge(1(A)\supset h)} & A \wedge (A \supset A) \\ {\scriptstyle \omega_i(A)}\downarrow & & \downarrow{\scriptstyle \omega_i(A)} \\ A & \xrightarrow{\quad h \quad} & A \end{array}$$

(ii′)
$$\begin{array}{ccc} A \wedge (A \supset A) & \xrightarrow{g\wedge(1(A\supset A))} & A \wedge (A \supset A) \\ \downarrow{\scriptstyle 1(A)\wedge(g\supset 1(A))} & & \downarrow{\scriptstyle \omega_i(A)} \\ A \wedge (A \supset A) & \xrightarrow{\omega_i(A)} & A \end{array} .$$

However, an easy calculation shows that under the previous interpretation in the category of sets, neither diagram commutes.

3. **Conclusion.** A class of morphisms in a category is said to be *coherent* if

https://doi.org/10.4153/CMB-1975-020-1 Published online by Cambridge University Press

all diagrams whose edges are taken from this class, commute. Although it is well-known that by virtue of the existence of usually two projections $A \wedge A \rightrightarrows A$, and the usually distinct symmetry and identity automorphisms of $A \wedge A$, for example, the class of morphisms of $F(\mathbf{1})$ is not coherent, these counter-examples can be accounted for by designating the different morphisms as components of distinct natural or generalized natural transformations. The present example shows that for many classes of morphisms of $F(\mathbf{1})$ this cannot be done, and that in the case of *cartesian* closed categories, naturality criteria for commutativity analogous to those developed in [2, 3, 6] for closed categories frequently fail.

References

1. S. Eilenberg and G. M. Kelly, *A Generalization of the Functorial Calculus*, J. Algebra **3** (1966), 366–375.
2. G. M. Kelly, *On MacLane's Conditions for Coherence of Natural Associativities, Commutativities, etc.*, J. Algebra **4** (1964), 397–402.
3. G. M. Kelly and S. MacLane, *Coherence in Closed Categories*, J. Pure and Applied Algebra **1** (1971), 97–140.
4. J. Lambek, *Deductive Systems and Categories I*, Math. Systems Theory **2** (1968), 287–318.
5. J. Lambek, *Deductive Systems and Categories II*, Lecture Notes in Mathematics **86**, Springer, Berlin, 1969, 76–122.
6. S. MacLane, *Natural Associativity and Commutativity*, Rice Univ. Stud. **49** (1963), 28–46.
7. S. MacLane, *Categories for the Working Mathematician*, Springer, New York, 1971.
8. M. E. Szabo, *The Logic of Closed Categories*, Notre Dame Journal of Formal Logic, (to appear).

S.G.W. Department of Mathematics
Concordia University
Montreal, Quebec

https://doi.org/10.4153/CMB-1975-020-1 Published online by Cambridge University Press