2

The weak currents

2.1 The weak currents and some of their properties

The effective weak interaction in Eq. (1.1) was motivated by nuclear β-decays. For many years this was the main theoretical framework for analyzing experiments. As new experimental discoveries became available, the form of the interaction was maintained but the current $J_\mu(x)$ was enlarged to incorporate the new observations. At the end of the sixties the charged current $J_\mu^\dagger(x)$ included a leptonic and a hadronic term,

$$J_\mu^\dagger = l_\mu^\dagger(x) + h_\mu^\dagger(x). \tag{2.1}$$

The leptonic part of the current is

$$l_\mu^\dagger(x) = \overline{\Psi}_{\mathrm{e}}(x)\gamma_\mu(1-\gamma_5)\Psi_{\nu_{\mathrm{e}}}(x) + \overline{\Psi}_\mu(x)\gamma_\mu(1-\gamma_5)\Psi_{\nu_\mu}(x), \tag{2.2}$$

with the first term corresponding to the electron and its neutrino and the second term to the muon and its neutrino. Its space-time structure has a vector part analogous to the electromagnetic current and an axial part introduced after the discovery of parity violation. A direct calculation using the currents in (2.2) gives the μ-decay spectrum, which is in good agreement with experiment. It also gives the decay rate of the muon as

$$\Gamma(\mu \to \mathrm{e} + \nu_{\mathrm{e}} + \bar{\nu}_\mu) = \frac{G_\mu m_\mu^5}{192\pi^3}. \tag{2.3}$$

From the observed decay rate and the mass of the muon the constant G_μ is determined to be

$$G_\mu = (1.166\,32 \pm 0.000\,04) \times 10^{-5}\,\mathrm{GeV}^{-2}. \tag{2.4}$$

This determination includes the effects of radiative corrections, which in the electroweak theory are finite and can be calculated precisely.

https://doi.org/10.1017/9781009402378.003 Published online by Cambridge University Press

The hadronic current consists of several parts determined by detailed analyses of hadron decays. For instance, the decay of a neutron, $n \to p + e^- + \nu_e$, is well described by the matrix element

$$\langle p|J_\mu^\dagger|n\rangle = \langle p|V_\mu^\dagger|n\rangle - \langle p|A_\mu^\dagger|n\rangle, \tag{2.5}$$

where we can decompose the matrix elements in terms of form factors. Lorentz invariance gives the general expressions

$$\langle p|V_\mu^\dagger|n\rangle = \bar{u}(p')\left(g_V\gamma_\mu + f_V\frac{(p+p')_\mu}{2M} + h_V\frac{q_\mu}{2M}\right)u(p) \tag{2.6}$$

and

$$\langle p|A_\mu^\dagger|n\rangle = \bar{u}(p')\left(g_A\gamma_\mu\gamma_5 + f_A\frac{\mathrm{i}\sigma_{\mu\nu}q^\nu}{2M}\gamma_5 + h_A\frac{q_\mu}{2M}\gamma_5\right)u(p), \tag{2.7}$$

where p_μ and p'_μ are the momenta of the neutron and proton, respectively, with $q_\mu = p'_\mu - p_\mu$. The functions g_V, f_V, and h_V are vector form factors describing the effects of strong interactions in the hadrons. Similarly g_A, f_A, and h_A are axial form factors. General symmetries, like charge symmetry and time-reversal, limit the form factors and demand that $h_V = f_A = 0$ (see Marshak *et al.*, 1969, p. 314). At zero momentum transfer, the vector form factor g_V was precisely determined and it is strikingly close to 1, while g_A is about -1.23. An explanation was proposed, namely that the strangeness-conserving part of $V_\mu^\dagger$ has the isospin content

$$V_\mu^\dagger = V_\mu^1 + \mathrm{i}V_\mu^2 =: V_\mu^+ \quad \text{and} \quad A_\mu^\dagger = A_\mu^1 + \mathrm{i}A_\mu^2 =: A_\mu^+, \tag{2.8}$$

where 1 and 2 denote the first and second components of isospin. This means that the charges

$$T^i = \int V_0^i(x)\mathrm{d}^3x \tag{2.9}$$

are the same isospin generators as those occurring in the strong interactions and are therefore conserved. This rule is called the conserved-vector-current (CVC) hypothesis. The T^i form an algebra that closes under commutation relations

$$\left[T^i, T^j\right] = \mathrm{i}\varepsilon^{ijk}T_k. \tag{2.10}$$

As a consequence, the commutator of T^+ with T^- produces the third component of isospin. In the late sixties T^3 had not yet been observed to mediate transitions with the strength G; there was no weak neutral current. But such an operator already existed in the electromagnetic current. The electromagnetic current consisted of

https://doi.org/10.1017/9781009402378.003 Published online by Cambridge University Press

two parts,

$$J_\mu^{\mathrm{em}}(x) = V_\mu^3(x) + \frac{1}{\sqrt{3}} V_\mu^8(x), \tag{2.11}$$

with $V_\mu^3(x)$ being the third component of isospin and V_μ^8 an iso-scalar current transforming as the eighth component of SU(3). It is evident that there is a relation between the weak and the electromagnetic currents, since the vector part of the weak current and the isovector part of the electromagnetic current form an isotriplet. The form of the interaction in (1.1) defines a universal coupling for leptonic, semi-leptonic, and non-leptonic decays. Once the coupling constant G has been determined, as in (2.4) from the muon decay, it can be used to translate the isotriplet hypothesis into relations between electromagnetic and weak matrix elements. In Section 11.3 we give a consequence of the isotriplet hypothesis and the cross section for neutrino–neutron quasi-elastic scattering. Expressions for the currents in terms of quark fields are given in Chapter 3.

Since V_μ^+ is an isospin current, its matrix elements at zero momentum transfer are simply given by Clebsch–Gordan coefficients. The strength g_V is determined in nuclear β-decay as well as in the elementary decays

$$\begin{aligned} \pi^+ &\longrightarrow \pi^0 + \mathrm{e}^+ + \nu_\mathrm{e}, \\ \mathrm{n} &\longrightarrow \mathrm{p} + \mathrm{e}^- + \bar{\nu}_\mathrm{e}. \end{aligned} \tag{2.12}$$

In all these cases the charge current connects states with the same isospin T, but different components T_3. At zero momentum transfer the relevant matrix element is

$$\langle I, I_3 + 1 | V_\mu^+(0) | I, I_3 \rangle = 1, \tag{2.13}$$

for $I = \frac{1}{2}$. The value of g_V is extracted from β-decay and its value is found (see Equation (9.28)) to be

$$g_\mathrm{V} = 0.9740 \pm 0.0003 \pm 0.0015. \tag{2.14}$$

This precise value includes radiative corrections, so its deviation from unity is significant.

Is the small difference of 2.6% a drawback of the theory or is there another component of the current? The discrepancy was explained by the observation that the hadronic current $V_\mu^\pm$ does not generate only the isospin group SU(2), but contains other pieces responsible for strangeness-changing decays, like

$$\begin{aligned} \Lambda^0 &\longrightarrow \mathrm{p} + \mathrm{e}^- + \bar{\nu}_\mathrm{e}, \\ \mathrm{K}^+ &\longrightarrow \pi^0 + \mathrm{e}^+ + \nu_\mathrm{e}. \end{aligned} \tag{2.15}$$

https://doi.org/10.1017/9781009402378.003 Published online by Cambridge University Press

Thus the hadronic current is the sum of a $\Delta S = 0$ term and a $\Delta S = 1$ term,

$$V_\mu^+ = \cos\theta_c\, V_\mu^{\Delta S=0} + \sin\theta_c\, V_\mu^{\Delta S=1}. \tag{2.16}$$

The two terms are interpreted as two components of the current orthogonal to each other and connected through the mixing angle θ_c. The first term contains the isospin current that appears in (2.8),

$$V_\mu^{\Delta S=0} = V_\mu^1 + \mathrm{i}V_\mu^2. \tag{2.17}$$

The second component produces strangeness-changing transitions and it has, in SU(3), the form

$$V_\mu^{\Delta S=1} = V_\mu^4 + \mathrm{i}V_\mu^5. \tag{2.18}$$

The matrix elements of $V_\mu^{\Delta S=0}$ and $V_\mu^{\Delta S=1}$ can be estimated accurately. The conclusion from numerous experimental estimates gives the mixing angle

$$\sin\theta_c = 0.220 \pm 0.002. \tag{2.19}$$

In this way universality is restored, since the sum of the squares of the hadronic couplings reproduces the coupling observed in muon decay. In addition the discrepancy of g_V from 1 is understood. The angle θ_c is called the Cabibbo angle. The appearance of the Cabbibo angle will become more evident in the context of the Cabibbo–Kobayashi–Maskawa matrix, which enters the full Lagrangian of the weak interaction.

Finally, we mention one more difference between electromagnetic and weak interactions. The electromagnetic amplitude for the reaction $e^+e^- \to \mu^+\mu^-$ has the amplitude

$$\mathcal{M} = \mathrm{i}e^2 J^\mu \frac{g_{\mu\nu}}{q^2} J^{\nu\dagger}, \tag{2.20}$$

with an explicit photon propagator and the product of two currents, like

$$J_\mu = l_\mu^{\mathrm{em}} + J_\mu^{\mathrm{em}}. \tag{2.21}$$

The hadronic current was discussed in Chapter 1 and the leptonic current has a term for each charged lepton like

$$l_\mu^{\mathrm{em}} = \overline{\Psi}_l \gamma_\mu \Psi_l. \tag{2.22}$$

On comparing (2.20) with (1.1), we note that the propagator is missing in (1.1). It should have been there in the form

$$g^2 \Delta_{\mu\nu} = g^2 \frac{-\mathrm{i}g_{\mu\nu}}{q^2 - M_W^2}, \tag{2.23}$$

https://doi.org/10.1017/9781009402378.003 Published online by Cambridge University Press

if the weak interaction were mediated by the exchange of a particle of mass M_{W} and coupling strength g. At very low energies, however, at which most of the decays take place, $q^2 \ll M_{\mathrm{W}}^2$ and

$$g^2 \Delta_{\mu\nu} \longrightarrow \mathrm{i} g_{\mu\nu} \frac{g^2}{M_{\mathrm{W}}^2} = \mathrm{i} g_{\mu\nu} \frac{G}{\sqrt{2}}. \tag{2.24}$$

Thus the form in (1.1) is indeed a very good approximation.

2.2 The partially conserved axial current

A second property of the weak currents deals with approximations that are possible in matrix elements of the axial current. The charged pions decay weakly into $\mu\bar{\nu}_{\mu}$ pairs with a hadronic matrix element

$$\langle 0|A_{\mu}^{\pm}(x)|\pi^{\pm}(q)\rangle = \mathrm{i} f_{\pi}(q^2) q_{\mu} \mathrm{e}^{-\mathrm{i}qx}. \tag{2.25}$$

Here q_{μ} is the four-momentum of the pion and f_{π} defines the decay coupling constant. The form of the matrix is dictated by Lorentz invariance. The coupling $f_{\pi}(q^2 = m_{\pi}^2)$ is measured with the pion on the mass shell. We can take the divergence of this matrix element and obtain

$$q^{\mu} \langle 0|A_{\mu}^{\pm}(x)|\pi^{\pm}\rangle = \mathrm{i} f_{\pi} q^2 \mathrm{e}^{-\mathrm{i}qx}. \tag{2.26}$$

We conclude from this relation that the axial current is not conserved, because neither f_{π} nor m_{π} is zero. However, it may be approximately conserved because $q^2 = m_{\pi}^2$ is a small number relative to the mass squared of all other hadrons. Thus, for many low-energy processes that involve the axial current with four-momentum q_{μ}, it is possible to replace the divergence of the axial current by the pion field

$$\partial^{\mu} A_{\mu}^{i} = f_{\pi} m_{\pi}^2 \phi^{i}, \tag{2.27}$$

and, in addition, after we have extracted the pion propagator, the reduced matrix element is a slowly varying function of q^2 provided that $q^2 \lesssim m_{\pi}^2$. Equation (2.27) is an operator relation and holds for all matrix elements. We must be careful, however, to replace the pion field by its source $j_{\pi}^{i} = (\square^2 + m^2)\phi^{i}$ and substitute for the pion–nucleon vertex the coupling

$$\langle p| j_{\pi^+} |n\rangle = \mathrm{i}\sqrt{2} g_{\pi\mathrm{NN}} \bar{u}(p') \gamma_5 u(p). \tag{2.28}$$

Several applications have established that the matrix elements of the axial current and its divergence can be treated this way. We shall describe here an application of this procedure to the matrix element in β-decay, which leads to a remarkable relation known as the Goldberger–Treiman relation. We present the derivation in

https://doi.org/10.1017/9781009402378.003 Published online by Cambridge University Press

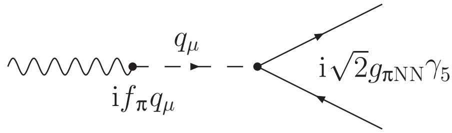

Figure 2.1. Direct coupling of the axial current to a particle of zero mass.

two ways in order to emphasize that the second method is based on an underlying symmetry. Consider the matrix element

$$\langle p|A_\mu^+|n\rangle = \bar{u}(p')\left(g_\mathrm{A}(q^2)\gamma_\mu\gamma_5 + h_\mathrm{A}(q^2)\frac{q_\mu}{2M}\gamma_5\right)u(p). \tag{2.29}$$

Taking the divergence of both sides of this equation gives

$$\langle p|\partial^\mu A_\mu^+|n\rangle = \left(2Mg_\mathrm{A}(q^2) + q^2\frac{h_\mathrm{A}(q^2)}{2M}\right)\mathrm{i}\bar{u}(p')\gamma_5 u(p). \tag{2.30}$$

On the other hand, from (2.27)

$$\begin{aligned}\langle p|\partial^\mu A_\mu^+|n\rangle &= f_\pi m_\pi^2\langle p|\phi^+|n\rangle = f_\pi m_\pi^2\frac{1}{-q^2+m_\pi^2}\langle p|j_\pi^+|n\rangle \\ &= f_\pi\frac{m_\pi^2}{-q^2+m_\pi^2}\mathrm{i}\sqrt{2}g_{\pi\mathrm{NN}}\bar{u}(p')\gamma_5 u(p).\end{aligned} \tag{2.31}$$

Taking the limit $q^2 \to 0$ with $m_\pi^2 \neq 0$ in the last two equations, we obtain

$$\sqrt{2}Mg_\mathrm{A} = g_{\pi\mathrm{NN}}f_\pi. \tag{2.32}$$

This is the Goldberger–Treiman relation. For the experimental values of the coupling constants it holds at the 10% level. It is a remarkable relation, relating the pion–nucleon coupling constant to two couplings of weak interactions.

There is a second way of looking at partially conserved axial current (PCAC). The meaning of PCAC is that the actual world is not far from the limit in which the axial currents are conserved at the expense of having zero-mass pions ($m_\pi = 0$, $f_\pi \neq 0$). In this approach we can still define f_π and g_A through Eqs. (2.25) and (2.29). Because the axial current is now conserved, Eq. (2.30) becomes

$$2Mg_\mathrm{A}(q^2) + q^2\frac{h_\mathrm{A}(q^2)}{2M} = 0. \tag{2.33}$$

In the limit of $q^2 \to 0$ the second term of Eq. (2.29) does not vanish but contributes the amplitude

$$\mathrm{i}f_\pi q_\mu\frac{\mathrm{i}}{q^2}\,\mathrm{i}\sqrt{2}g_{\pi\mathrm{NN}}\bar{u}(p')\gamma_5 u(p) \tag{2.34}$$

https://doi.org/10.1017/9781009402378.003 Published online by Cambridge University Press

shown by the diagram in Fig. 2.1. The amplitude has a pole at $q^2 = 0$ and its divergence gives

$$q^2 \frac{h_A(q^2)}{2M} = -\sqrt{2} f_\pi g_{\pi NN} + \text{terms proportional to } q^2, \tag{2.35}$$

which, together with (2.33), gives again the Goldberger–Treiman relation. This demonstrates that the form factor $h_A(q^2)$ is dominated at small momentum transfers by the pion pole.

2.3 Regularities among the forces

The subjects covered in the first two chapters represent basic topics developed long before the electroweak theory. They strongly suggest that the weak force is not an isolated phenomenon, but one intimately connected with the other forces of nature. The isotriplet hypothesis clearly states that the isovector part of the electromagnetic current and the vector part of the weak current for $\Delta S = 0$ transitions form an isospin triplet. In addition, it states that the charges $T^{\pm}$ are the same generators as those of strong isospin. We note that operators of the three types of interactions are related. The isotriplet hypothesis also posed a problem: that of explaining why the neutral member of the multiplet did not occur in the weak interactions by itself, but only through electromagnetism. This question was answered with the discovery of weak neutral currents.

The hypothesis of PCAC relates couplings of the weak interactions to the pion–nucleon coupling constant through the Goldberger–Treiman relation. In another application, PCAC combined with equal-time commutation relations, it is possible to calculate the deviation of g_A from 1 as an integral over the pion–nucleon cross sections.

Consequences of PCAC hold at the 10%–20% level. They are understood to hold because the mass of the pion is small in comparison with the masses of other hadrons. That is, there is an underlying symmetry, which is broken by the small mass of the pion. The previous remarks provide a strong motivation to search for a closer connection of the weak, electromagnetic, and perhaps the strong interactions. The successful theory which unifies the weak and electromagnetic forces is studied in the following chapters. The electroweak theory is so far in excellent agreement with experiment. It made many predictions that have been confirmed by experimental data. Finally, the reader should keep in mind that the theory must also provide a natural explanation of the empirical rules described so far and others to be described in the following chapters.

https://doi.org/10.1017/9781009402378.003 Published online by Cambridge University Press

Problems for Chapters 1 and 2

1. The scattering of particles

$$a + b \longrightarrow c + d$$

is described by the amplitude

$$f(\theta) = \frac{1}{k}\sum_l (2l+1)\frac{(\eta_l e^{2i\delta_l} - 1)}{2i} P_l(\cos\theta),$$

where η_l and δ_l are real functions and k is the magnitude of the momentum of particle a or b in the center-of-mass system. δ_l is the phase shift and η_l is introduced to describe inelastic scattering: for elastic scattering $\eta_l = 1$ and for inelastic scattering $\eta_l < 1$.

(a) Prove the optical theorem and show that

$$\sigma_{\text{tot}} = \frac{2\pi}{k^2}\sum_l (2l+1)[1 - \eta_l \cos(2\delta_l)].$$

(b) Show that, for elastic scattering,

$$\sigma_{\text{el}} = \frac{4\pi}{k^2}\sum_l (2l+1)\left|\frac{\eta_l e^{2i\delta_l} - 1}{2i}\right|^2.$$

(c) Show that, from (a) and (b), it follows that

$$\sigma_{\text{tot}} = \frac{\pi}{k^2}\sum_l (2l+1)(1 - \eta_l^2).$$

2. Using the result from Problem 1 (part (c)), show that the cross section for the reaction $\nu_\mu + e^- \to \mu^- + \nu_e$ is limited by

$$\sigma(\nu_\mu + e^- \to \mu^- + \nu_e) \leq \frac{\pi}{2E_{\text{cm}}^2},$$

where E_{cm} is the energy of the ν_μ or the e^- in the center-of-mass frame. Take into account that it is an $l = 0$ scattering and that there is a spin factor of $(2s + 1)$.

3. From the Hermiticity of the electromagnetic current, show that $F_1(q^2)$ and $F_2(q^2)$ are real.
4. From time-reversal invariance, show that $F_1(q^2)$ and $F_2(q^2)$ are real.
5. By considering the non-relativistic limit of the Pauli interaction,

$$\frac{1}{2}\mu\overline{\Psi}\sigma_{\mu\nu}\Psi F^{\mu\nu}, \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu,$$

give a physical interpretation of the term containing $F_2(q^2)$. Express $F_2(0)$ in terms of the proton's anomalous magnetic moment.

Reference

Marshak, R. E., Riazuddin, and Ryan, C. P. (1969), *Theory of Weak Interactions in Particle Physics* (New York, Interscience)

https://doi.org/10.1017/9781009402378.003 Published online by Cambridge University Press

Select bibliography

For CVC and V-A interactions:

Feynman, R. P., and Gell-Mann, M. (1958), *Phys. Rev*. **109**, 1938
Gerschtein, S. S., and Zeldovich, J. R. (1955), *JETP* **29**, 698
Marshak, R. E., and Sudarshan, E. C. G. (1958), *Phys. Rev*. **109**, 1860

For PCAC:

Adler, S. L., and Dashen, R. F. (1968), *Current Algebras* (New York, W. A. Benjamin Inc.)
Cabibbo, N. (1963), *Phys. Rev. Lett*. **10**, 531
Cheng, T. P., and Li, L. F. (1984), *Gauge Theory of Elementary Particle Physics* (Oxford, Clarendon Press)
Goldberger, M. L., and Treiman, S. B. (1958), *Phys. Rev*. **110**, 1178

https://doi.org/10.1017/9781009402378.003 Published online by Cambridge University Press