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The Ekedahl–Oort stratification and the semi-module stratification

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Abstract. In this paper, we compare the $\mathbb{J}$-stratification (or the semi-module stratification) and the Ekedahl–Oort stratification of affine Deligne–Lusztig varieties in the superbasic case. In particular, we classify the cases where the $\mathbb{J}$-stratification gives a refinement of the Ekedahl–Oort stratification, which include many interesting cases such that the affine Deligne-Lusztig variety admits a simple geometric structure.

1 Introduction

Affine Deligne–Lusztig varieties were introduced by Rapoport [34], which play an important role in understanding geometric and arithmetic properties of Shimura varieties. The uniformization theorem by Rapoport and Zink [33] allows us to describe the Newton strata of Shimura varieties in terms of Rapoport–Zink spaces, whose underlying spaces are special cases of affine Deligne–Lusztig varieties.

Let F be a non-Archimedean local field with finite residue field $\mathbb{F}_q$ of prime characteristic p, and let L be the completion of the maximal unramified extension of F. Let σ denote the Frobenius automorphism of L/F. Further, we write $\mathcal{O}$ (resp. $\mathcal{O}_F$) for the valuation ring of L (resp. F). Finally, we denote by ϖ a uniformizer of F (and L) and by v_L the valuation of L such that $v_L(\varpi) = 1$.

Let G be an unramified connected reductive group over $\mathcal{O}_F$. Let $B \subset G$ be a Borel subgroup and $T \subset B$ a maximal torus in B, both defined over $\mathcal{O}_F$. For $\mu, \mu' \in X_*(T)$ (resp. $X_*(T)_{\mathbb{Q}}$), we write $\mu' \leq \mu$ if $\mu - \mu'$ is a nonnegative integral (resp. rational) linear combination of positive coroots. For a cocharacter $\mu \in X_*(T)$, let ϖ^μ be the image of $\varpi \in \mathbb{G}_m(F)$ under the homomorphism $\mu: \mathbb{G}_m \to T$.

Set $K = G(\mathcal{O})$. We fix a dominant cocharacter $\mu \in X_*(T)_+$ and $b \in G(L)$. Then the affine Deligne–Lusztig variety $X_\mu(b)$ is the locally closed reduced $\overline{\mathbb{F}}_q$-subscheme of the affine Grassmannian $\mathcal{G}r = G(L)/K$ defined as

$$X_\mu(b) = \{xK \in \mathcal{G}r \mid x^{-1}b\sigma(x) \in K\varpi^\mu K\}.$$

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The closed affine Deligne–Lusztig variety is the closed reduced $\overline{\mathbb{F}}_q$-subscheme of $\mathcal{G}r$ defined as

$$X_{\leq\mu}(b) = \bigcup_{\mu'\leq\mu} X_{\mu'}(b).$$

Both $X_\mu(b)$ and $X_{\leq\mu}(b)$ are locally of finite type in the equal characteristic case and locally perfectly of finite type in the mixed characteristic case (cf. [19, Corollary 6.5], [18, Lemma 1.1]). Finally, the affine Deligne–Lusztig varieties $X_\mu(b)$ and $X_{\leq\mu}(b)$ carry a natural action (by left multiplication) by the σ-centralizer of b

$$J_b(F) = \{g \in G(L) \mid g^{-1}b\sigma(g) = b\}.$$

The geometric properties of affine Deligne–Lusztig varieties have been studied by many people. For example, the non-emptiness criterion and the dimension formula are already known for the affine Deligne–Lusztig varieties in the affine Grassmannian (see [8], [42] and [17]). Let $B(G)$ denote the set of σ-conjugacy classes of $G(L)$. Thanks to Kottwitz [28], a σ-conjugacy class $[b] \in B(G)$ is uniquely determined by two invariants: the Kottwitz point $\kappa(b) \in \pi_1(G)/((1-\sigma)\pi_1(G))$ and the Newton point $\nu_b \in X_*(T)_{\mathbb{Q},+}$. Set $B(G,\mu) = \{[b] \in B(G) \mid \kappa(b) = \kappa(\varpi^\mu), \nu_b \leq \mu^\diamond\}$, where $\mu^\diamond \in X_*(T)_{\mathbb{Q},+}$ denotes the σ-average of μ. Then $X_\mu(b) \neq \emptyset$ if and only if $[b] \in B(G,\mu)$. If this is the case, then we have

$$\dim X_\mu(b) = \langle \rho, \mu - \nu_b\rangle - \frac{1}{2}\operatorname{def}(b),$$

where ρ is the half sum of positive roots and $\operatorname{def}(b)$ is the defect of b. Moreover, the parametrization problem of the set of irreducible components $\operatorname{Irr} X_\mu(b)$ is also known. Let $\widehat{G}$ be the Langlands dual of G defined over $\overline{\mathbb{Q}}_l$ with $l \neq p$. Surprisingly, there exists a natural bijection between $J_b(F)\backslash \operatorname{Irr} X_\mu(b)$ and a certain weight space of the crystal basis $\mathbb{B}_\mu$ of the irreducible $\widehat{G}$-module V_μ of highest weight μ. This is conjectured by Chen and Zhu, and proved in general by Nie [32] and Zhou-Zhu [47].

Via the relationship to Shimura varieties, or more directly to Rapoport–Zink spaces, the results on the geometry of affine Deligne–Lusztig varieties have numerous applications to number theory (e.g., the Kudla-Rapoport program [29], Zhang's Arithmetic Fundamental Lemma [46], ...). Many of these applications make use of the special cases where $X_{\leq\mu}(b)$ admits a simple description. The fully Hodge–Newton decomposable case, introduced by Görtz, He and Nie [13], is one of such cases. They proved that (G,μ) is fully Hodge–Newton decomposable if and only if $X_{\leq\mu}(\tau_\mu)$ is naturally a union of (classical) Deligne–Lusztig varieties (in fact, they studied the cases with arbitrary parahoric level). This stratification is the so-called weak Bruhat-Tits stratification, a stratification indexed in terms of the Bruhat-Tits building of $J_b(F)$ (which exists only in the fully Hodge–Newton decomposable case). The case of Coxeter type is a special case of this case such that each Deligne-Lusztig variety appearing in this stratification is of Coxeter type (cf. [14, Section 2.3]). In this case, we drop the "weak" above. For example, the cases of Coxeter type include the case for certain unitary groups of signature $(1, n-1)$ studied in [44] by Vollaard and Wedhorn, which has been used in [29] and [46].

To give a conceptual way to explain the relationship between the geometry of affine Deligne–Lusztig varieties and the Bruhat-Tits building of $J_b(F)$ indicated by above examples, Chen and Viehmann [2] introduced the $\mathbb{J}$-stratification, where $\mathbb{J}$ stands for $J_b(F)$. The $\mathbb{J}$-strata are locally closed subsets of $\mathcal{G}r$. By intersecting each $\mathbb{J}$-stratum with $X_{\leq\mu}(b)$, we obtain the $\mathbb{J}$-stratification of $X_{\leq\mu}(b)$ (see Section 2.4 for details). In [9], Görtz showed that the Bruhat-Tits stratification coincides with the $\mathbb{J}$-stratification. In fact the Bruhat-Tits stratification is a refinement of the Ekedahl–Oort stratification (see Section 2.2 for the latter). So the $\mathbb{J}$-stratification is also a refinement of the Ekedahl–Oort stratification when (G,μ) is of Coxeter type. This does not hold in general even if μ is minuscule. See [2, Example 4.1] for a counterexample in the case $G = \mathrm{GL}_9$. Therefore, the cases when the $\mathbb{J}$-stratification is a refinement of the Ekedahl–Oort stratification should be special cases, which are of particular interest.

Usually it seems very difficult to study the $\mathbb{J}$-stratification. However, in the case that $G = \mathrm{GL}_n$ and b is superbasic (i.e., $\kappa(b) \in \mathbb{Z}$ is coprime to n), the $\mathbb{J}$-stratification coincides with a stratification by semi-modules [2, Proposition 3.4]. The notion of semi-modules was first considered by de Jong and Oort [3] (see Section 3.1) for minuscule cocharacters. Later Viehmann [42] introduced a notion of extended semi-modules for arbitrary cocharacters, which generalizes the notion of semi-modules. It played a crucial role to prove the dimension formula (for split groups) and the Chen-Zhu conjecture mentioned above. This is because for these problems, we can reduce the general case to the case that $G = \mathrm{GL}_n$ and b is superbasic.

The aim of this paper is to compare the Ekedahl–Oort stratification and the semi-module stratification (for $G = \mathrm{GL}_n$). To state the main results, we need some notation. Let W_0 be the (finite) Weyl group of T in G and let $\tilde{W}$ be the Iwahori-Weyl group of T in G. Then $\tilde{W} = X_*(T) \rtimes W_0$. We denote the projection $\tilde{W} \to W_0$ by p. For $\mu \in X_*(T)_+$, we denote by $\mathrm{Adm}(\mu)$ the admissible subset of $\tilde{W}$. Let ${}^S\mathrm{Adm}(\mu)$ be a certain subset of $\mathrm{Adm}(\mu)$, which is the index set of the Ekedahl–Oort stratification of $X_{\leq\mu}(\tau_\mu)$ (see Section 2.2). We fix (a representative in $G(L)$ of a) length 0 element $\tau_\mu \in \tilde{W}$ whose σ-conjugacy class in $G(L)$ is the unique basic element in $B(G,\mu)$. Finally, let $\mathrm{LP}(w) \subseteq W_0$ be the length positive elements for w (see Section 2.5).

Theorem A (See Theorem 7.2) *Let $G = \mathrm{GL}_n$ and let $\mu \in X_*(T)_+$. Assume that τ_μ is superbasic. Then the following assertions are equivalent.*

(i) *The $\mathbb{J}$-stratification (or the semi-module stratification) of $X_{\leq\mu}(\tau_\mu)(\neq \emptyset)$ gives a refinement of the Ekedahl–Oort stratification.*
(ii) *For any $w \in {}^S\mathrm{Adm}(\mu)$ whose corresponding Ekedahl–Oort stratum is nonempty, there exists $v \in \mathrm{LP}(w)$ such that $v^{-1}p(w)v$ is a Coxeter element.*
(iii) *The cocharacter μ has one of the following forms modulo $\mathbb{Z}\omega_n$:*

$$\omega_1, \quad \omega_{n-1}, \qquad (n \geq 1),$$
$$\omega_2, \quad 2\omega_1, \quad \omega_{n-2}, \quad 2\omega_{n-1}, \qquad (\text{odd } n \geq 3),$$
$$\omega_2 + \omega_{n-1}, \quad 2\omega_1 + \omega_{n-1} \quad \omega_1 + \omega_{n-2}, \quad \omega_1 + 2\omega_{n-1}, \qquad (n \geq 3),$$
$$\omega_3, \quad \omega_{n-3}, \qquad (n = 7, 8),$$
$$3\omega_1, \quad 3\omega_{n-1}, \qquad (n = 4, 5),$$
$$\omega_1 + \omega_2, \quad \omega_3 + \omega_4, \qquad (n = 5),$$

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$$4\omega_1, \quad \omega_1 + 3\omega_2, \quad 4\omega_2, \quad 3\omega_1 + \omega_2, \qquad (n = 3),$$
$$m\omega_1 \text{ with } m \text{ odd}, \qquad (n = 2).$$

Here, ω_k denotes the cocharacter of the form $(1, \ldots, 1, 0, \ldots, 0)$ in which 1 is repeated k times. Moreover, if one of the above conditions holds, then each $\mathbb{J}$-stratum is universally homeomorphic to an affine space.

See Section 2.4 for the reason why we choose τ_μ. In fact, this choice is the reasonable one suggested in [2, Remark 2.1], which is unique in this case.

Although the cocharacters ω_1 and ω_{n-1} are of Coxeter type for any n, the cocharacters $2\omega_1$ and ω_2 are of Coxeter type only when $n = 2$ and $n = 4$ respectively (cf. [14, Theorem 1.4]). In Theorem A, these two cocharacters are no longer exceptional cases. Note also that the condition (ii) works in more general setting. In [38], we study this condition for GL_n without the superbasic assumption. It turns out that if μ satisfies (ii), then the $\mathbb{J}$-stratification of $X_{\leq\mu}(\tau_\mu)$ gives a refinement of the Ekedahl–Oort stratification, and each $\mathbb{J}$-stratum is universally homeomorphic to the product of a classical Deligne–Lusztig variety and an affine space. This simple description can be considered as a natural generalization of the Bruhat-Tits stratification. Moreover, in a joint work [37] with Schremmer and Yu, we proved that (ii) implies a simple geometric structure on each Ekedahl–Oort stratum of $X_{\leq\mu}(\tau_\mu)$ for general G. In fact, the condition (ii) for GL_n is also a generalization of Coxeter type [37, Theorem 4.12]. So Theorem A tells us that the two conditions which contain the cases of Coxeter type are actually equivalent at least in the superbasic case.

If μ is minuscule and $\operatorname{ch} F = 0$, then $X_\mu(\tau_\mu) (= X_{\leq\mu}(\tau_\mu))$ for GL_n is the perfection of the special fiber of the Rapoport–Zink space attached to (GL_n, μ, τ_μ) (cf. [14, Section 5]). These Rapoport–Zink spaces are moduli spaces of p-divisible groups, which have been studied in [43]. Especially in the superbasic case, each $\mathbb{J}$-stratum of $X_\mu(\tau_\mu)$ is known to be isomorphic to an affine space (before perfection). However, even in this case, there is no good description of the closure of each $\mathbb{J}$-stratum in general. On the other hand, it turned out in [38] that if μ is a minuscule cocharacter appearing in the list (iii) above, then each $\mathbb{J}$-stratum of $X_\mu(\tau_\mu)$ can be written as a certain union of $\mathbb{J}$-strata. It is also worth mentioning that the condition (i) in Theorem A is essential to describe this union explicitly because we need to attach $w \in {}^S\mathrm{Adm}(\mu)$ to each $\mathbb{J}$-stratum in a natural way (cf. [38, Section 2.3]).

In [1], Chen-Tong compared the Newton stratification and the Harder–Narashimhan stratification of the flag variety attached to (G, μ) under the assumption that μ is minuscule. As a result, they showed that the former gives a refinement of the latter if and only if (G, μ) is weakly fully Hodge–Newton decomposable [1, Definition 2.4]. Recently, Schremmer informed the author that there is an upcoming work with He and Viehmann which also aims at generalizing the fully Hodge–Newton decomposable case. For a pair (G, μ), they define a nonnegative rational number $\operatorname{depth}(G, \mu)$. Then it is known that (G, μ) is fully Hodge–Newton decomposable if and only if $\operatorname{depth}(G, \mu) \leq 1$ (cf. [13, Definition 3.2]). They classified the cases where $1 < \operatorname{depth}(G, \mu) < 2$. The classifications of these works have similarities, and most cocharacters in Theorem A appear in these works (see also [38, Section 1]). Moreover, the nice stratification in [38] suggests that these cases would be new cases such that

$X_{\leq\mu}(\tau_\mu)$ admits a simple description (as already predicted in [1, Remark 2.16]). Thus, for general G, both (i) and (ii) are also reasonable conditions to find such simple cases which would have many applications as the Bruhat-Tits stratification.

It is worth mentioning that there are some other (G,μ) such that the corresponding basic affine Deligne–Lusztig variety admits a certain simple description. For example, the works by Fox-Imai [7] (see also [6]) and Trentin [41] are such cases. Interestingly, both cases have $\mathrm{depth}(G,\mu) = 2$. It is also interesting to compare the $\mathbb{J}$-stratification and the Ekedahl–Oort stratification in these cases because the result will be useful to find new simple cases.

Cyclic semi-modules are certain simple elements in the set of extended semi-modules. It is easy to see that if there exists a noncyclic semi-module for μ, then the semi-module stratification of $X_\mu(\tau_\mu)$ never gives a refinement of the Ekedahl–Oort stratification (Corollary 3.10). Along the way of proving Theorem A, we also prove the following classification theorem, which ensures that there exists a noncyclic semi-module in many cases.

Theorem B (See Theorem 4.17) *Every top extended semi-module (the semi-module whose corresponding stratum is top dimensional) for μ is cyclic if and only if μ has one of the following forms modulo $\mathbb{Z}\omega_n$:*

(i) ω_i *with* $1 \leq i \leq n-1$ *such that i is coprime to n.*
(ii) $\omega_1 + \omega_i$ *or* $\omega_{n-1} + \omega_{n-i}$ *with* $1 \leq i \leq n-1$ *such that* $i+1$ *is coprime to n.*
(iii) $(nr+i)\omega_1$ *or* $(nr+i)\omega_{n-1}$ *with* $r \geq 0$ *and* $1 \leq i \leq n-1$ *such that i is coprime to n.*
(iv) $(nr+i-j)\omega_1 + \omega_j$ *or* $(nr+i-j)\omega_{n-1} + \omega_{n-j}$ *with* $r \geq 1$, $2 \leq j \leq n-1$ *and* $1 \leq i \leq n-1$ *such that i is coprime to n.*

The key ingredient of the proof of Theorem B is an explicit construction of top extended semi-modules from crystal bases via the natural map in the Chen-Zhu conjecture, which was established in [40] by the author. This method is a completely new way of studying the affine Deligne–Lusztig varieties. Since the Chen-Zhu conjecture holds for arbitrary G, it is an interesting question in general to investigate the affine Deligne–Lusztig varieties by crystal bases.

The paper is organized as follows. In Section 2, we introduce the affine Deligne–Lusztig variety and stratifications of it. We also recall the length positive elements and the non-emptiness criterion of the affine Deligne–Lusztig variety in the affine flag variety. In Section 3 and Section 4, we recollect known results on semi-modules and crystal bases respectively. Also in Section 4, we prove Theorem B using combinatorics on Young tableaux. In Sections 5 and 6, we examine the semi-module stratification and the Ekedahl–Oort stratification respectively by an explicit calculation of semi-modules and elements in ${}^S\mathrm{Adm}(\mu)$. In particular, using the non-emptiness criterion mentioned above, we show that Theorem A (ii) does not hold for many μ. Finally, in Section 7, we prove the main theorem, combining Theorem B and the results in Section 5 and Section 6.

2 Preliminaries

Keep the notations in Section 1.

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2.1 Notation

Let $\Phi = \Phi(G, T)$ denote the set of roots of T in G. We denote by Φ_+ (resp. Φ_-) the set of positive (resp. negative) roots distinguished by B. Let Δ be the set of simple roots and $\Delta^\vee$ be the corresponding set of simple coroots. Let $X_*(T)$ be the set of cocharacters, and let $X_*(T)_+$ be the set of dominant cocharacters.

The Iwahori-Weyl group $\tilde{W}$ is defined as the quotient $N_{G(L)}T(L)/T(\mathcal{O})$. This can be identified with the semi-direct product $W_0 \ltimes X_*(T)$, where W_0 is the finite Weyl group of G. We denote the projection $\tilde{W} \to W_0$ by p. We have a length function $\ell\colon \tilde{W} \to \mathbb{Z}_{\geq 0}$ given as

$$\ell(w_0\varpi^\lambda) = \sum_{\alpha\in\Phi_+, w_0\alpha\in\Phi_-} |\langle\alpha,\lambda\rangle + 1| + \sum_{\alpha\in\Phi_+, w_0\alpha\in\Phi_+} |\langle\alpha,\lambda\rangle|,$$

where $w_0 \in W_0$ and $\lambda \in X_*(T)$.

Let $S \subset W_0$ denote the subset of simple reflections, and let $\tilde{S} \subset \tilde{W}$ denote the subset of simple affine reflections. We often identify Δ and S. The affine Weyl group W_a is the subgroup of $\tilde{W}$ generated by $\tilde{S}$. Then we can write the Iwahori-Weyl group as a semi-direct product $\tilde{W} = W_a \rtimes \Omega$, where $\Omega \subset \tilde{W}$ is the subgroup of length 0 elements. Moreover, $(W_a, \tilde{S})$ is a Coxeter system. We denote by $\leq$ the Bruhat order on $\tilde{W}$. For any $J \subseteq \tilde{S}$, let ${}^J\tilde{W}$ be the set of minimal length representatives for the cosets in $W_J\backslash\tilde{W}$, where W_J denotes the subgroup of $\tilde{W}$ generated by J.

Let $w \in \tilde{W}$. There exists a positive integer k such that $w^k = \varpi^\lambda$ for some $\lambda \in X_*(T)$. We set $\nu_w = \lambda/k \in X_*(T)_\mathbb{Q}$. This is independent of the choice of k.

For $w \in W_a$, we denote by $\mathrm{supp}(w) \subseteq \tilde{S}$ the set of simple affine reflections occurring in every (equivalently, some) reduced expression of w. Note that $\tau \in \Omega$ acts on $\tilde{S}$ by conjugation. We define the σ-support $\mathrm{supp}_\sigma(w\tau)$ of $w\tau$ as the smallest $\tau\sigma$-stable subset of $\tilde{S}$ which contains $\mathrm{supp}(w)$.

For $w, w' \in \tilde{W}$ and $s \in \tilde{S}$, we write $w \xrightarrow{s}_\sigma w'$ if $w' = sw\sigma(s)$ and $\ell(w') \leq \ell(w)$. We write $w \to_\sigma w'$ if there is a sequence $w = w_0, w_1, \ldots, w_k = w'$ of elements in $\tilde{W}$ such that for any i, $w_{i-1} \xrightarrow{s_i}_\sigma w_i$ for some $s_i \in S$. If $w \to_\sigma w'$ and $w' \to_\sigma w$, we write $w \approx_\sigma w'$.

For $\alpha \in \Phi$, let $U_\alpha \subseteq G$ denote the corresponding root subgroup. We set

$$I = T(\mathcal{O}) \prod_{\alpha\in\Phi_+} U_\alpha(\varpi\mathcal{O}) \prod_{\beta\in\Phi_-} U_\beta(\mathcal{O}) \subseteq G(L),$$

which is called the standard Iwahori subgroup associated to the triple $T \subset B \subset G$.

In the case $G = \mathrm{GL}_n$, we will use the following description. Let T be the torus of diagonal matrices, and we choose the subgroup of upper triangular matrices B as Borel subgroup. Let χ_{ij} be the character $T \to \mathbb{G}_m$ defined by $\mathrm{diag}(t_1, t_2, \ldots, t_n) \mapsto t_i t_j^{-1}$. Then we have $\Phi = \{\chi_{ij} \mid i \neq j\}$, $\Phi_+ = \{\chi_{ij} \mid i < j\}$, $\Phi_- = \{\chi_{ij} \mid i > j\}$ and $\Delta = \{\chi_{i,i+1} \mid 1 \leq i < n\}$. Through a natural isomorphism $X_*(T) \cong \mathbb{Z}^n$, $X_*(T)_+$ can be identified with the set $\{(m_1, \ldots, m_n) \in \mathbb{Z}^n \mid m_1 \geq \cdots \geq m_n\}$. The finite Weyl group is the symmetric group of degree n. Let us write $s_1 = (1\,2), s_2 = (2\,3), \ldots, s_{n-1} = (n-1\;n)$. Set $s_0 = \varpi^{\chi_{1,n}^\vee}(1\,n)$, where $\chi_{1,n}$ is the unique highest root. Then $S = \{s_1, s_2, \ldots, s_{n-1}\}$ and $\tilde{S} = S \cup \{s_0\}$. The Iwahori subgroup $I \subset K$ is the inverse image of the lower triangular matrices under the projection $K \to G(\overline{\mathbb{F}}_q)$ induced by $\varpi \mapsto 0$. Set $\tau = \begin{pmatrix} 0 & \varpi \\ 1_{n-1} & 0 \end{pmatrix}$. We

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often regard τ as an element of $\tilde{W}$, which is a generator of $\Omega \cong \mathbb{Z}$. Note that $b \in \mathrm{GL}_n(L)$ is superbasic if and only if $[b] = [\tau^m]$ in $B(\mathrm{GL}_n)$ for some m coprime to n.

2.2 Affine Deligne–Lusztig varieties

For $w \in \tilde{W}$ and $b \in G(L)$, the affine Deligne–Lusztig variety $X_w(b)$ in the affine flag variety $G(L)/I$ is defined as

$$X_w(b) = \{xI \in G(L)/I \mid x^{-1}b\sigma(x) \in IwI\}.$$

For $\mu \in X_*(T)_+$ and $b \in G(L)$, the affine Deligne–Lusztig variety $X_\mu(b)$ in the affine Grassmannian $\mathcal{G}r = G(L)/K$ is defined as

$$X_\mu(b) = \{xK \in \mathcal{G}r \mid x^{-1}b\sigma(x) \in K\varpi^\mu K\}.$$

The closed affine Deligne–Lusztig variety is the closed reduced $\overline{\mathbb{F}}_q$-subscheme of $\mathcal{G}r$ defined as

$$X_{\preceq\mu}(b) = \bigcup_{\mu' \preceq \mu} X_{\mu'}(b).$$

Left multiplication by $g^{-1} \in G(L)$ induces an isomorphism between $X_\mu(b)$ and $X_\mu(g^{-1}b\sigma(g))$. Thus, the isomorphism class of the affine Deligne–Lusztig variety only depends on the σ-conjugacy class of b. Moreover, we have $X_\mu(b) = X_{\mu+\lambda}(\varpi^\lambda b)$ for each central $\lambda \in X_*(T)$.

The admissible subset of $\tilde{W}$ associated to μ is defined as

$$\mathrm{Adm}(\mu) = \{w \in \tilde{W} \mid w \leq \varpi^{w_0\mu} \text{ for some } w_0 \in W_0\}.$$

Note that $\mathrm{Adm}(\mu') \subseteq \mathrm{Adm}(\mu)$ if $\mu' \preceq \mu$. Indeed if $w \leq \varpi^{w_0\mu'}$ and $\mu' \preceq \mu$, then $w \leq \varpi^{w_0\mu}$ by [16, Lemma 4.5]. Set ${}^S\mathrm{Adm}(\mu) = \mathrm{Adm}(\mu) \cap {}^S\tilde{W}$. Then by [11, Theorem 3.2.1] (see also [15, Section 2.5]), we have

$$X_{\preceq\mu}(b) = \bigsqcup_{w \in {}^S\mathrm{Adm}(\mu)} \pi(X_w(b)),$$

where $\pi\colon G(L)/I \to G(L)/K$ is the projection. This is the so-called Ekedahl–Oort stratification.

For any $w \in {}^S\tilde{W}$, set

$$Z(w) := \{w_0 \in W_0 \mid w_0 w = w\sigma(w_0)\}.$$

Lemma 2.1 *Let $\varpi^\mu y \in {}^S\tilde{W}$ with μ dominant and $y \in W_0$. Assume that $Z(\varpi^\mu y) = \{1\}$. Then the projection map $\pi\colon X_{\varpi^\mu y}(b) \to X_\mu(b)$ is injective.*

Proof The proof is similar to [23, Lemma 5.4]. We may assume that $X_{\varpi^\mu y}(b) \neq \varnothing$. Let $gI, g'I \in X_{\varpi^\mu y}(b)$ such that $\pi(gI) = \pi(g'I)$. Then $g'^{-1}g \in K$ and hence $g'^{-1}g \in IxI$ for some $x \in W_0$. Since $(g'^{-1}g)(g^{-1}b\sigma(g)) = (g'^{-1}b\sigma(g'))(\sigma(g'^{-1}g))$, we have $(IxI)(I\varpi^\mu yI) \cap (I\varpi^\mu yI)(I\sigma(x)I) \neq \varnothing$. Note that $(IxI)(I\varpi^\mu yI) = Ix\varpi^\mu yI$ because $\varpi^\mu y \in {}^S\tilde{W}$. This implies that $x\varpi^\mu y = \varpi^\mu y\sigma(x)$. By our assumption, we must have $x = 1$ and hence $g'^{-1}g \in I$ as desired. ∎

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Example 2.2 Let $G = \mathrm{GL}_n$ and let $\varpi^\mu y \in {}^S\tilde{W}$ with μ dominant and $y \in W_0$. If y is an n-cycle and $\{s_1, s_{n-1}\} \nsubseteq Z(\varpi^\mu)$, then we have $Z(\varpi^\mu y) = \{1\}$. Indeed, for any $x \in W_0$, $x\varpi^\mu y = \varpi^\mu yx$ implies that $xyx^{-1} = y$ and $x \in Z(\varpi^\mu)$. Thus, $x = y^k$ for some $0 \le k \le n-1$ and $y^k\mu = \mu$. Since $\{s_1, s_{n-1}\} \nsubseteq Z(\varpi^\mu)$, we must have $k = 0$.

2.3 Deligne–Lusztig reduction method

The following Deligne–Lusztig reduction method was established in [10, Corollary 2.5.3].

Proposition 2.3 *Let $w \in \tilde{W}$ and let $s \in \tilde{S}$ be a simple affine reflection. If* $\mathrm{ch}(F) > 0$*, then the following two statements hold for any $b \in G(L)$.*

(i) *If $\ell(sw\sigma(s)) = \ell(w)$, then there exists a $J_b(F)$-equivariant universal homeomorphism $X_w(b) \to X_{sw\sigma(s)}(b)$.*
(ii) *If $\ell(sw\sigma(s)) = \ell(w) - 2$, then there exists a decomposition $X_w(b) = X_1 \sqcup X_2$ such that*
   - *X_1 is open and there exists a $J_b(F)$-equivariant morphism $X_1 \to X_{sw}(b)$, which is the composition of a Zariski-locally trivial $\mathbb{G}_m$-bundle and a universal homeomorphism.*
   - *X_2 is closed and there exists a $J_b(F)$-equivariant morphism $X_2 \to X_{sw\sigma(s)}(b)$, which is the composition of a Zariski-locally trivial $\mathbb{A}^1$-bundle and a universal homeomorphism.*

   If $\mathrm{ch}(F) = 0$*, then the above statements still hold by replacing $\mathbb{A}^1$ and $\mathbb{G}_m$ by $\mathbb{A}^{1,\mathrm{pfn}}$ and $\mathbb{G}_m^{\mathrm{pfn}}$ respectively.*

The following result is proved in [22, Theorem 2.10], which allows us to reduce the study of $X_w(b)$ for any w, via the Deligne–Lusztig reduction method, to the study of $X_w(b)$ for w of minimal length in its σ-conjugacy class.

Theorem 2.4 *For each $w \in \tilde{W}$, there exists an element w' which is of minimal length inside its σ-conjugacy class such that $w \to_\sigma w'$.*

Following [23, Section 3.4], we construct the reduction trees for w by induction on $\ell(w)$.

The vertices of the trees are elements of $\tilde{W}$. We write $x \rightharpoonup y$ if $x, y \in \tilde{W}$ and there exists $x' \in \tilde{W}$ and $s \in \tilde{S}$ such that $x \approx_\sigma x'$, $\ell(sx'\sigma(s)) = \ell(x') - 2$ and $y \in \{sx', sx'\sigma(s)\}$. These are the (oriented) edges of the trees. A reduction tree of w is a tree with these vertices and edges whose unique starting point is w and whose end points are of minimal length in its σ-conjugacy class of $\tilde{W}$.

The existence of a (not necessarily unique) reduction tree of w can be proved as follows. If w is of minimal length in its σ-conjugacy class of $\tilde{W}$, then the reduction tree for w consists of a single vertex w and no edges. Assume that w is not of minimal length and that a reduction tree is given for any $z \in \tilde{W}$ with $\ell(z) < \ell(w)$. By Theorem 2.4, there exist w' and $s \in \tilde{S}$ with $w \approx_\sigma w'$ and $\ell(sw'\sigma(s)) = \ell(w') - 2$. By our assumption, there exist reduction trees of sw' and $sw'\sigma(s)$. Then a reduction tree of w consists of the given reduction trees of sw' and $sw'\sigma(s)$ and the edges $w \rightharpoonup sw'$ and $w \rightharpoonup sw'\sigma(s)$.

Let $\mathcal{T}$ be a reduction tree of w. Recall that an end point of $\mathcal{T}$ is a vertex in $\mathcal{T}$ of minimal length. A reduction path in $\mathcal{T}$ is a path $\underline{p}\colon w \rightharpoonup w_1 \rightharpoonup \cdots \rightharpoonup w_n$, where w_n

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is an end point of $\mathcal{T}$. Set $\mathrm{end}(\underline{p}) = w_n$. We say that $x \rightharpoonup y$ is of type I (resp. II) if $\ell(x) - \ell(y) = 1$ (resp. $\ell(x) - \ell(y) = 2$). For any reduction path $\underline{p}$, we denote by $\ell_I(\underline{p})$ (resp. $\ell_{II}(\underline{p})$) the number of type I (resp. II) edges in $\underline{p}$. We write $X_{\underline{p}}$ for a locally closed subscheme of $X_w(b)$ which is $J_b(F)$-equivariant universally homeomorphic to an iterated fibration of type $(\ell_I(\underline{p}), \ell_{II}(\underline{p}))$ over $X_{\mathrm{end}(\underline{p})}(b)$.

Let $B(\tilde{W}, \sigma)$ be the set of σ-conjugacy classes in $\tilde{W}$. Let $\Psi\colon B(\tilde{W}, \sigma) \to B(G)$ be the map sending $[w] \in B(\tilde{W}, \sigma)$ to $[\dot{w}] \in B(G)$, where $\dot{w} \in G(L)$ is a lift of w. It is known that this map is well-defined and surjective, see [21, Theorem 3.7]. By [23, Proposition 3.9], we have the following description of $X_w(b)$.

Proposition 2.5 *Let $w \in \tilde{W}$ and $\mathcal{T}$ be a reduction tree of w. For any $b \in G(L)$, there exists a decomposition*

$$X_w(b) = \bigsqcup_{\substack{\underline{p} \text{ is a reduction path in } \mathcal{T};\\ \Psi(\mathrm{end}(\underline{p}))=[b]}} X_{\underline{p}}.$$

In the case that $G = \mathrm{GL}_n$ and $b = \tau^m$ with m coprime to n, we can count the number of top irreducible components and rational points of $X_w(b)^0 = \{gI \in X_w(b) \mid \kappa(g) = v_L(\det(g)) = 0\}$ using the reduction tree for w. By [22, Proposition 3.5], the σ-conjugacy class of τ^m in $\tilde{W}$ is the unique element in $B(\tilde{W}, \sigma)$ which maps to $[\tau^m] \in B(G)$ under Ψ. Note also that τ^m is the unique minimal length element in its σ-conjugacy class. We define a polynomial as

$$F_{w,b} \coloneqq \sum_{\underline{p}} (\mathbf{q}-1)^{\ell_I(\underline{p})} \mathbf{q}^{\ell_{II}(\underline{p})} \in \mathbb{N}[\mathbf{q}-1],$$

where $\underline{p}$ runs over all the reduction paths in $\mathcal{T}$ with $\mathrm{end}(\underline{p}) = \tau^m$.

Proposition 2.6 *Assume that $G = \mathrm{GL}_n$ and $b = \tau^m$ with m coprime to n. Let $w \in \tilde{W}$ and let $\mathcal{T}$ be a reduction tree of w. Then the number of top irreducible components of $X_w(b)^0$ is equal to the leading coefficient of $F_{w,b}$ (as a polynomial in $\mathbf{q}-1$). Moreover, we have*

$$|X_w(b)^{0,\sigma}| = F_{w,b}|_{\mathbf{q}=q}.$$

Proof Note that each $J_b(F)$-orbit of an irreducible component of $X_w(b)$ can be represented by an irreducible component of $X_w(b)^0$. Moreover, it is known that the stabilizer in $J_b(F)$ is a parahoric subgroup (cf. [47, Proposition 3.1.4]), i.e., $J_b(F) \cap I = \{g \in J_b(F) \mid \kappa(g) = 0\}$. Then the statement follows from [23, Theorem 3.4 and Proposition 3.5] and [24, Corollary 4.4]. ∎

Remark 2.7 The polynomials $F_{w,b}$ are called *class polynomials*. However, the definition above is an ad hoc one. See [23, Section 3] for the definition in general and the connection to reduction trees.

2.4 The $\mathbb{J}$-stratification

For any $g, h \in G(L)$, let $\mathrm{inv}(g, h)$ denote the relative position, i.e., the unique dominant cocharacter such that $g^{-1}h \in K\varpi^{\mathrm{inv}(g,h)}K$. By definition, two elements $gK, hK \in$

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$G(L)/K$ lie in the same $\mathbb{J}$-stratum if and only if for all $j \in J_b(F)$, $\mathrm{inv}(j,g) = \mathrm{inv}(j,h)$. Clearly, this does not depend on the choice of g, h. By [2, Proposition 2.11], the $\mathbb{J}$-strata are locally closed in $\mathcal{G}r$. By intersecting each $\mathbb{J}$-stratum with $X_\mu(b)$ (resp. $X_{\leq\mu}(b)$), we obtain the $\mathbb{J}$-stratification of $X_\mu(b)$ (resp. $X_{\leq\mu}(b)$).

As explained in [2, Remark 2.1], the $\mathbb{J}$-stratification heavily depends on the choice of b in its σ-conjugacy class. So we need to fix a specific representative to compare the $\mathbb{J}$-stratification on $X_\mu(b)$ (or $X_{\leq\mu}(b)$) to another stratification. It is pointed out in loc. cit that if $[b]$ is a basic class in $B(G,\mu)$, then a reasonable choice of b is the unique length 0 element τ_μ. Also, for any $w \in \tilde{W}$, the $J_{\dot{w}}(F)$-stratification is independent of the choice of a lift in $G(L)$. See [9, Lemma 2.5].

In the case where $G = \mathrm{GL}_n$ and $b = \tau^m$ with m coprime to n, there is a group-theoretic way to describe the $\mathbb{J}$-stratification, which we will call the semi-module stratification. Indeed, by [2, Remark 3.1 and Proposition 3.4], the $\mathbb{J}$-stratification on $\mathcal{G}r$ coincides with the stratification

$$G(L)/K = \bigsqcup_{\lambda \in X_*(T)} I\varpi^\lambda K/K.$$

So in this case, each $\mathbb{J}$-stratum of $X_\mu(b)$ (resp. $X_{\leq\mu}(b)$) coincides with $X_\mu^\lambda(b)$ (resp. $X_{\leq\mu}^\lambda(b)$) for some $\lambda \in X_*(T)$, where $X_\mu^\lambda(b) = X_\mu(b) \cap I\varpi^\lambda K/K$ (resp. $X_{\leq\mu}^\lambda(b) = X_{\leq\mu}(b) \cap I\varpi^\lambda K/K$). Set $J_b(F)^0 = J_b(F) \cap K = J_b(F) \cap I$. Note that $\tau X_\mu^\lambda(b) = X_\mu^{\tau\lambda}(b)$ and $J_b(F)/J_b(F)^0 = \{\tau^k J_b(F)^0 \mid k \in \mathbb{Z}\}$. Thus,

$$J_b(F)X_\mu^\lambda(b) = \bigsqcup_{k\in\mathbb{Z}} X_\mu^{\tau^k\lambda}(b) \quad \text{and} \quad J_b(F)X_{\leq\mu}^\lambda(b) = \bigsqcup_{k\in\mathbb{Z}} X_{\leq\mu}^{\tau^k\lambda}(b).$$

See Section 3.1 for the precise definition of (extended) semi-modules. As we will explain in Section 3.2, the set $\{\lambda \in X_*(T) \mid X_\mu^\lambda(b) \neq \varnothing\}$ can be regarded as semi-modules for μ. Let $w_{\max}$ be the longest element in W_0. Then we have

$$\{\lambda \in X_*(T) \mid X_{-w_{\max}\mu}^\lambda(b^{-1}) \neq \varnothing\} = \{-w_{\max}\lambda \in X_*(T) \mid X_\mu^\lambda(b) \neq \varnothing\}.$$

Indeed it is easy to check that the image of $X_\mu^\lambda(b)$ under the automorphism of $\mathcal{G}r$ by $gK \mapsto w_{\max}^t g^{-1}K$ is $X_{-w_{\max}\mu}^{-w_{\max}\lambda}(b^{-1})$. This gives the description of "dual" semi-modules for μ.

2.5 Length positive elements

We denote by δ^+ the indicator function of the set of positive roots, i.e.,

$$\delta^+ \colon \Phi \to \{0,1\}, \quad \alpha \mapsto \begin{cases} 1 & (\alpha \in \Phi_+) \\ 0 & (\alpha \in \Phi_-). \end{cases}$$

Note that any element $w \in \tilde{W}$ can be written in a unique way as $w = x\varpi^\mu y$ with μ dominant, $x, y \in W_0$ such that $\varpi^\mu y \in {}^S\tilde{W}$. We have $p(w) = xy$ and $\ell(w) = \ell(x) + \langle\mu, 2\rho\rangle - \ell(y)$. We define the set of *length positive* elements by

$$\mathrm{LP}(w) = \{v \in W_0 \mid \langle v\alpha, y^{-1}\mu\rangle + \delta^+(v\alpha) - \delta^+(xyv\alpha) \geq 0 \text{ for all} \alpha \in \Phi_+\}.$$

Then we always have $y^{-1} \in \mathrm{LP}(w)$. Indeed y satisfies the condition that $\langle \alpha, \mu \rangle \geq \delta^+(-y^{-1}\alpha)$ for all$\alpha \in \Phi_+$. Since $\delta^+(\alpha) + \delta^+(-\alpha) = 1$, we have

$$\langle y^{-1}\alpha, y^{-1}\mu \rangle + \delta^+(y^{-1}\alpha) - \delta^+(x\alpha) = \langle \alpha, \mu \rangle - \delta^+(-y^{-1}\alpha) + \delta^+(-x\alpha) \geq 0.$$

Lemma 2.8 *For any $w = x\varpi^\mu y \in \tilde{W}$ as above, we define*

$$\Phi_w := \{\alpha \in \Phi_+ \mid \langle \alpha, \mu \rangle - \delta^-(y^{-1}\alpha) + \delta^-(x\alpha) = 0\}.$$

Here δ^- denotes the indicator function of the set of negative roots. Then we have

$$y\,\mathrm{LP}(w) = \{r^{-1} \in W_0 \mid r(\Phi_+ \backslash \Phi_w) \subset \Phi_+ \text{ or equivalently, } r^{-1}\Phi_+ \subset \Phi_+ \cup -\Phi_w\}.$$

Proof Let $r \in W_0$ such that $r(\Phi_+ \backslash \Phi_w) \subset \Phi_+$. Let $\alpha \in \Phi_+$. If $r^{-1}\alpha \in \Phi_+$, then we can check that $y^{-1}r^{-1} \in \mathrm{LP}(w)$ similarly as the case $r = 1$ above. If $r^{-1}\alpha \in \Phi_-$, then we must have $r^{-1}\alpha \in -\Phi_w$. Since $\delta^-(-\alpha) = \delta^+(\alpha)$, it follows that

$$\begin{aligned}&\langle y^{-1}r^{-1}\alpha, y^{-1}\mu \rangle + \delta^+(y^{-1}r^{-1}\alpha) - \delta^+(xr^{-1}\alpha)\\ &= -\left(\langle -r^{-1}\alpha, \mu \rangle - \delta^-(-y^{-1}r^{-1}\alpha) + \delta^-(-xr^{-1}\alpha)\right) = 0.\end{aligned}$$

Thus, $y^{-1}r^{-1} \in \mathrm{LP}(w)$. This shows $\{r^{-1} \in W_0 \mid r(\Phi_+ \backslash \Phi_w) \subset \Phi_+\} \subseteq y\,\mathrm{LP}(w)$.

Let $v \in \mathrm{LP}(w)$ and let $\alpha \in \Phi_+$. If $yv\alpha \in \Phi_-$, then

$$\langle -yv\alpha, \mu \rangle - \delta^-(-v\alpha) + \delta^-(-xyv\alpha) = -(\langle v\alpha, y^{-1}\mu \rangle + \delta^+(v\alpha) - \delta^+(xyv\alpha)) \leq 0.$$

On the other hand, by the characterization of y above, we have

$$\langle -yv\alpha, \mu \rangle - \delta^-(-v\alpha) + \delta^-(-xyv\alpha) = \langle -yv\alpha, \mu \rangle - \delta^+(v\alpha) + \delta^+(xyv\alpha) \geq 0.$$

Thus, $\langle -yv\alpha, \mu \rangle - \delta^-(-v\alpha) + \delta^-(-xyv\alpha) = 0$ and hence $yv\alpha \in -\Phi_w$. This shows $y\,\mathrm{LP}(w) \subseteq \{r^{-1} \in W_0 \mid r(\Phi_+ \backslash \Phi_w) \subset \Phi_+\}$. The proof is finished. ∎

The notion of length positive elements is defined by Schremmer [35]. The description of $\mathrm{LP}(w)$ in Lemma 2.8 is due to Lim [30].

We say that the Dynkin diagram of G is σ-connected if it cannot be written as a union of two proper σ-stable subdiagrams that are not connected to each other. The following theorem is a refinement of the non-emptiness criterion in [12], which is conjectured by Lim [30] and proved by Schremmer [36, Proposition 5].

Theorem 2.9 *Assume that the Dynkin diagram of G is σ-connected. Let $b \in G(L)$ be a basic element with $\kappa(b) = \kappa(\dot{w})$. Then $X_w(b) = \varnothing$ if and only if both of the following two conditions are satisfied:*

(i) *$|W_{\mathrm{supp}_\sigma(w)}|$ is not finite.*
(ii) *There exists $v \in \mathrm{LP}(w)$ such that $\mathrm{supp}_\sigma(\sigma^{-1}(v)^{-1}p(w)v) \subsetneq S$.*

Remark 2.10 If $\kappa(b) \neq \kappa(\dot{w})$, then $X_w(b) = \varnothing$.

Remark 2.11 Let $w \in \tilde{W}, w_0 \in W_0$ and let $J \subseteq \Delta$ such that $J = \sigma(J)$. Then we say that w is a (J, w_0, σ)-*alcove element* if the following conditions are both satisfied:

(1) $w_0^{-1}w\sigma(w_0) \in \tilde{W}_J := X_*(T) \rtimes W_J$, and
(2) For any $\alpha \in w_0(\Phi_+ \backslash \Phi_J)$, $U_\alpha \cap {}^wI \subseteq U_\alpha \cap I$, where Φ_J denotes the root system generated by J.

In [36, Proposition 5], the condition (ii) in Theorem 2.9 is written as (ii)' There exist $J \subsetneq \Delta$ and $w_0 \in W_0$ such that w is a (J, w_0, σ)-alcove element.

The equivalence of (ii) and (ii)' follows from [30, Lemmas 3.7 and 3.9] (see also [37, Definition 2.3] and the comment right after it).

In the case $G = \mathrm{GL}_n$, there exists a length-preserving automorphism ς of $\tilde{W}$ defined as

$$w_0 \varpi^{\lambda} \mapsto w_{\max} w_0 w_{\max}^{-1} \varpi^{-w_{\max}\lambda}, \quad w_0 \in W_0,\ \lambda \in X_*(T).$$

Note that $\varsigma(\tau^m) = \tau^{-m}$, $\varsigma(s_0) = s_0$ and $\varsigma(s_i) = s_{n-i}$ for $1 \le i \le n-1$. Let $w = x\varpi^{\mu} y$ be as above. For any $\alpha \in \Phi_+$ and $v \in \mathrm{LP}(w)$, we have

$$\begin{aligned}&\langle \varsigma(v)(-w_{\max}\alpha), \varsigma(y^{-1})(-w_{\max}\mu)\rangle + \delta^+(\varsigma(v)(-w_{\max}\alpha)) - \delta^+(\varsigma(xy)\varsigma(v)(-w_{\max}\alpha))\\ &= \langle v\alpha, y^{-1}\mu\rangle + \delta^+(v\alpha) - \delta^+(xyv\alpha) \ge 0.\end{aligned}$$

Thus, $\mathrm{LP}(\varsigma(w)) = \varsigma(\mathrm{LP}(w)) = w_{\max}\,\mathrm{LP}(w) w_{\max}^{-1}$. In particular, there exists $v \in \mathrm{LP}(w)$ such that $v^{-1}p(w)v$ is a Coxeter element if and only if the same is true for $\varsigma(w)$ and $\mathrm{LP}(\varsigma(w))$.

3 Semi-modules

From now and until the end of this paper, we set $G = \mathrm{GL}_n$ and $b = \tau^m$ with m coprime to n. For $\mu \in X_*(T)_+$, let $\mu(i)$ denotes the i-th entry of μ. Then $[\tau^m] \in B(G, \mu)$ if and only if $m = \mu(1) + \cdots + \mu(n)$. We assume this from now. Also, without loss of generality, we may and will assume that $\mu(n) = 0$. Recall that $w_{\max}$ is the longest element in W_0.

3.1 Extended semi-modules

Here we recall the definition of extended semi-modules in a combinatorial way from [42]. Note that although we choose the subgroup of upper triangular matrices B as a Borel subgroup in this paper, the fixed Borel subgroup in [42] is the subgroup of *lower* triangular matrices.

Definition 3.1 A *semi-module* for m, n is a subset $A \subset \mathbb{Z}$ that is bounded below and satisfies $m + A \subset A$ and $n + A \subset A$. Set $\bar{A} = A \backslash (n + A)$. The semi-module A is called normalized if $\sum_{a \in \bar{A}} a = \frac{n(n-1)}{2}$.

For a semi-module A, there exists a unique $\mu' \in \mathbb{N}^n$ satisfying the following condition: Let $a_0 = \min \bar{A}$ and let inductively $a_i = a_{i-1} + m - \mu'(i)n$ for $i = 1, \ldots, n$. Then $a_0 = a_n$ and $\{a_0, a_1, \ldots, a_{n-1}\} = \bar{A}$. We call μ' the *type* of A.

Lemma 3.2 *There is a bijection between the set of normalized semi-modules for m, n and the set of possible types $\mu' \in \mathbb{N}^n$ with $\nu_b \preceq w_{\max}\mu'$.*

Proof This is [42, Lemma 3.3]. ∎

Definition 3.3 An *extended semi-module* (A, φ) for $\mu \in X_*(T)_+$ is a normalized semi-module A for m, n together with a function $\varphi: \mathbb{Z} \to \mathbb{N} \cup \{-\infty\}$ satisfying the following properties:

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(1) $\varphi(a) = -\infty$ if and only if $a \notin A$.
(2) $\varphi(a+n) \geq \varphi(a) + 1$ for all $a \in \mathbb{Z}$.
(3) $\varphi(a) \leq \max\{k \mid a + m - kn \in A\}$ for all $a \in A$. If $b \in A$ for all $b \geq a$, then the two sides are equal.
(4) There is a decomposition of A into disjoint union of sequences $a^1_j, \ldots, a^n_j$ with $j \in \mathbb{N}$ and the following properties:
    (a) $\varphi(a^l_{j+1}) = \varphi(a^l_j) + 1$.
    (b) If $\varphi(a^l_j + n) = \varphi(a^l_j) + 1$, then $a^l_{j+1} = a^l_j + n$. Otherwise $a^l_{j+1} > a^l_j + n$.
    (c) The n-tuple $(\varphi(a^l_0))$ is a permutation of μ.

An extended semi-module such that the equality holds in (3) for all $a \in A$ is called *cyclic*.

For any $\lambda \in X_*(T)$, we denote by λ_{dom} the dominant conjugate of λ. Let μ' be the type of a semi-module for m, n. Let φ be a function such that (1) and the equation in (3) hold. Then it is easy to check that (A, φ) is a cyclic semi-module for μ'_{dom}. In general, the following lemma holds.

Lemma 3.4 *Let (A, φ) be an extended semi-module for μ and let μ' be the type of A. Then $\mu'_{\mathrm{dom}} \leq \mu$ and (A, φ) is cyclic if and only if $\mu' \in W_0\mu$. In particular, if μ is minuscule, then all extended semi-modules for μ are cyclic.*

Proof See [42, Lemma 3.6 and Corollary 3.7]. See also [17, Lemma 5.9]. ∎

Let $e_0, \ldots, e_{n-1}$ be the standard basis of L^n. Then the lattice $\mathcal{O}^n$ is generated by $e_0, \ldots, e_{n-1}$. For $i \in \mathbb{Z}$, we define e_i by $e_{i+n} = \varpi e_i$. Note that we have $\tau e_i = e_{i+1}$ for any i. In the sequel, we identify $\mathcal{G}r$ and $\{M \subset L^n \text{ lattice}\}$ by $gK \mapsto g\mathcal{O}^n$.

Let $X_\mu(b)^0$ be a $\overline{\mathbb{F}}_q$-subscheme of $X_\mu(b)$ defined as $X_\mu(b)^0 = \{gK \in X_\mu(b) \mid \kappa(g) = 0\}$. We associate to $M \in X_\mu(b)^0$ an extended semi-module for μ. Let $v \in L^n$. Then we can write $v = \sum_{i\in\mathbb{Z}}[\alpha_i]e_i$ with $\alpha_i \in \overline{\mathbb{F}}_q$ and $\alpha_i = 0$ for sufficiently small i. Here, $[\alpha_i]$ denotes the Teichmüller lift of α_i if $\operatorname{ch} F = 0$ and $[\alpha_i] = \alpha_i$ if $\operatorname{ch} F > 0$. Let

$$\mathcal{I}\colon L^n \backslash \{0\} \to \mathbb{Z}, \quad v \mapsto \min\{i \mid \alpha_i \neq 0\}.$$

For $M \in \mathcal{G}r$, we define the set

$$A(M) = \{\mathcal{I}(v) \mid v \in M \backslash \{0\}\}.$$

It is easy to check that if $M \in X_\mu(b)^0$, then $A(M)$ is a normalized semi-module for m, n. We also define $\varphi(M)\colon \mathbb{Z} \to \mathbb{N} \cup \{-\infty\}$ by

$$a \mapsto \begin{cases} \max\{k \mid \exists v \in M\backslash\{0\} \text{ with } \mathcal{I}(v) = a, \varpi^{-k} b\sigma(v) \in M\} & (a \in A(M)) \\ -\infty & (a \notin A(M)). \end{cases}$$

Lemma 3.5 *Let $M \in X_\mu(b)^0$. Then $(A(M), \varphi(M))$ is an extended semi-module for μ.*

Proof See [42, Lemma 4.1]. ∎

For an extended semi-module (A, φ) for μ, let

$$S_{A,\varphi} = \{M \mid A(M) = A, \varphi(M) = \varphi\} \subset \mathcal{G}r.$$

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Lemma 3.6 *The set $S_{A,\varphi}$ is a locally closed subscheme of $X_\mu(b)^0$.*

Proof See [42, Lemma 4.2]. ∎

Let $\mathbb{A}_\mu$ be the set of extended semi-modules for μ. Set $\mathbb{A}_\mu^{\mathrm{top}} = \{(A,\varphi) \in \mathbb{A}_\mu \mid \dim S_{A,\varphi} = \dim X_\mu(b)\}$. By Proposition 3.7 below, $J_b(F)\backslash \operatorname{Irr} X_\mu(b)$ is parametrized by $\mathbb{A}_\mu^{\mathrm{top}}$. In the sequel, we also use the symbol $\mathbb{A}$ to denote the affine space as usual. We hope our notation will not cause confusions.

For an extended semi-module (A,φ) for μ, let

$$\mathcal{V}(A,\varphi) = \{(a,c) \in A \times A \mid c > a, \varphi(a) > \varphi(c) > \varphi(a-n)\}.$$

Proposition 3.7 *Let (A,φ) be an extended semi-module for μ. There exists a nonempty open subscheme $U_{A,\varphi} \subseteq \mathbb{A}^{|\mathcal{V}(A,\varphi)|}$ and a morphism $U_{A,\varphi} \to S_{A,\varphi}$ which is bijective on $\overline{\mathbb{F}}_q$-valued points. In particular, $S_{A,\varphi}$ is irreducible and of dimension $|\mathcal{V}(A,\varphi)|$. Moreover, if (A,φ) is a cyclic extended semi-module, then $U_{A,\varphi} = \mathbb{A}^{|\mathcal{V}(A,\varphi)|}$.*

Proof See [42, Theorem 4.3]. ∎

Here we briefly describe $U_{A,\varphi}$ and the map $U_{A,\varphi} \to S_{A,\varphi}$. For any $x \in \overline{\mathbb{F}}_q^{|\mathcal{V}(A,\varphi)|} = \mathbb{A}^{|\mathcal{V}(A,\varphi)|}$, we denote the coordinate of x by $x_{a,c}$. We associate to every x a set of elements $\{v(a) \in L^n \mid a \in A\}$ which satisfies the following equations.

If $a = \max \bar{A}$, then

$$v(a) = e_a + \sum_{(a,c)\in\mathcal{V}(A,\varphi)} [x_{a,c}]v(c).$$

For any other element $a \in \bar{A}$, we want

$$v(a) = v' + \sum_{(a,c)\in\mathcal{V}(A,\varphi)} [x_{a,c}]v(c),$$

where $v' = \varpi^{-\varphi(a')} b\sigma(v(a'))$ for a' being minimal satisfying $a' + m - \varphi(a')n = a$. For $a \in n + A$, we want

$$v(a) = \varpi v(a-n) + \sum_{(a,c)\in\mathcal{V}(A,\varphi)} [x_{a,c}]v(c).$$

Here $[x_{a,c}]$ denotes the Teichmüller lift of $x_{a,c}$ if $\operatorname{ch} F = 0$ and $[x_{a,c}] = x_{a,c}$ if $\operatorname{ch} F > 0$. The set $\{v(a) \in L^n \mid a \in A\}$ is uniquely determined by the equations above. Hence, the map $\mathbb{A}^{|\mathcal{V}(A,\varphi)|} \to \mathcal{G}r, x \mapsto \langle v(a)\rangle_{a\in A}$ is well-defined. By applying σ on the above equations for x, we can easily check that this map is compatible with the action of σ, i.e., $\sigma(x) \coloneqq (x_{a,c}^q)$ maps to $\sigma\langle v(a)\rangle_{a\in A}$. Let $U_{A,\varphi}$ be the preimage of $S_{A,\varphi}$ under this map. Then $S_{A,\varphi}$ and hence $U_{A,\varphi}$ are stable under σ (because $\sigma(b) = b$). In particular, we have $|S_{A,\varphi}^\sigma| = |U_{A,\varphi}^\sigma|$. So if (A,φ) is cyclic, then $|S_{A,\varphi}^\sigma| = q^{|\mathcal{V}(A,\varphi)|}$. Although not needed in this paper, it is also worth mentioning that if (A,φ) is noncyclic, then $S_{A,\varphi}$ is never universally homeomorphic to an affine space.

Proposition 3.8 *If (A,φ) is noncyclic, then $|S_{A,\varphi}^\sigma| < q^{|\mathcal{V}(A,\varphi)|}$. In particular, $S_{A,\varphi}$ is never universally homeomorphic to an affine space.*

Proof Let $x \in \mathbb{A}^{|\mathcal{V}(A,\varphi)|}$. Note that if $x_{a,c} = 0$ for all $(a,c) \in \mathcal{V}(A,\varphi)$, then $v(a) = e_a$ for all $a \in A$. Set $M = \langle e_a\rangle_{a\in A}$. Then it is easy to check that $(A(M),\varphi(M))$ is a cyclic semi-module for the dominant conjugate of the type of $A(M)$. So if (A,φ) is not cyclic, then $M \notin S_{A,\varphi}$ and hence $|S^{\sigma}_{A,\varphi}| = |U^{\sigma}_{A,\varphi}| < q^{|\mathcal{V}(A,\varphi)|}$. The last statement follows from [4, Propositions 4.1.12 and 8.1.11 (ii)]. ∎

3.2 The stratification by extended semi-modules

For any $\lambda \in X_*(T)$, set $A^{\lambda} = \{(i-1)+\lambda(i)n+kn \mid 1 \le i \le n, k \in \mathbb{N}\}$. It is easy to check that for a lattice $M \in I\varpi^{\lambda}K/K$, we have $A(M) = A^{\lambda}$. Thus, we have the following lemma, which relates the semi-module stratification to the stratification by extended semi-modules.

Lemma 3.9 *Let $\lambda \in X_*(T)$ with $\lambda(1)+\cdots+\lambda(n) = 0$. Then $X^{\lambda}_{\mu}(b) \neq \varnothing$ if and only if there exists an extended semi-module (A^{λ},φ) for μ. If this is the case, we have*

$$X^{\lambda}_{\mu}(b) = \bigsqcup_{\varphi} S_{A^{\lambda},\varphi},$$

where φ runs over all the functions $\mathbb{Z} \to \mathbb{N} \cup \{-\infty\}$ such that the pair of A^{λ} and the function is an extended semi-module for μ.

For $\lambda \in X_*(T)$ with $X^{\lambda}_{\mu}(b) \neq \varnothing$, let $1 \le i_0 \le n$ such that $(i_0-1)+\lambda(i_0)n = \min \overline{A^{\lambda}}$. Let $1 \le m_0 < n$ be the residue of m modulo n, and let $\lambda_{b,\mathrm{dom}}$ be $((\lfloor \frac{m}{n} \rfloor + 1)^{(m_0)}, \lfloor \frac{m}{n} \rfloor^{(n-m_0)})$. Then

$$\begin{aligned}&(i_0-1)+\lambda(i_0)n+m-(\lambda(i_0)+\lambda_{b,\mathrm{dom}}(c^m(i_0))-\lambda(c^m(i_0)))n\\&= c^m(i_0)-1+\lambda(c^m(i_0))n \in \overline{A^{\lambda}},\end{aligned}$$

where $c = s_1\cdots s_{n-1}$. Repeating the same argument, we can check that the type of A^{λ} is a conjugate of $b\lambda-\lambda = c^m\lambda+\lambda_{b,\mathrm{dom}}-\lambda$. By Lemma 3.4, an extended semi-module (A^{λ},φ) for μ is cyclic if and only if $b\lambda-\lambda \in W_0\mu$.

Corollary 3.10 *Let $\mu \in X_*(T)_+$. If there exists a noncyclic semi-module for μ, then the semi-module stratification of $X_{\le\mu}(b)$ is not a refinement of the -Oort stratification.*

Proof Let (A^{λ},φ) be a noncyclic semi-module for μ. Then we have $(b\lambda-\lambda)_{\mathrm{dom}} \prec \mu$ by Lemma 3.4. On the other hand, there always exists a cyclic semi-module (A^{λ},φ') for $(b\lambda-\lambda)_{\mathrm{dom}}$. By Lemma 3.9, $X^{\lambda}_{\le\mu}(b)$ intersects both $X_{\mu}(b)$ and $X_{(b\lambda-\lambda)_{\mathrm{dom}}}(b)$. This implies that $X^{\lambda}_{\le\mu}(b)$ is not contained in any set of the form $\pi(X_w(b))$ with $w \in \tilde{W}$, which finishes the proof. ∎

For $\mu = (\mu(1),\dots,\mu(n-1),0) \in X_*(T)_+$, set $\mu^* = (\mu(1),\mu(1)-\mu(n-1),\dots,\mu(1)-\mu(2),0)$ and $b^* = \tau^{n\mu(1)-m}$. If (A^{λ},φ) is an extended semi-module for μ, then there exists $\varphi'\colon\mathbb{Z} \to \mathbb{N} \cup \{-\infty\}$ such that $(A^{-w_{\max}\lambda},\varphi')$ is an extended semi-module for μ^* (see Section 2.4). Clearly, $b\lambda-\lambda \in W_0\mu$ if and only if $b^*(-w_{\max}\lambda)+w_{\max}\lambda \in -W_0\mu^*$. Thus, we have the following lemma.

Lemma 3.11 *There exists a noncyclic extended semi-module for μ if and only if the same is true for μ^*.*

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3.3 The minuscule case

In this subsection, we treat the minuscule case. Consider G^d with a Frobenius automorphism $\sigma_\bullet$ given by

$$(g_1, g_2, \dots, g_d) \mapsto (g_2, \dots, g_d, \sigma(g_1)).$$

For $\mu_\bullet = (\mu_1, \dots, \mu_d) \in X_*(T)_+^d$ and $b_\bullet = (1, \dots, 1, b) \in G^d(L)$ with $b \in G(L)$, we define $X_{\mu_\bullet}(b_\bullet) \subset \mathcal{G}r^d = G^d(L)/K^d$ as

$$X_{\mu_\bullet}(b_\bullet) = \{x_\bullet K^d \in \mathcal{G}r^d \mid x_\bullet^{-1} b_\bullet \sigma_\bullet(x_\bullet) \in K^d \varpi^{\mu_\bullet} K^d\}.$$

Let us denote by $\operatorname{Irr} X_{\mu_\bullet}(b_\bullet)$ the set of irreducible components of $X_{\mu_\bullet}(b_\bullet)$. Through the identification $J_b(F) \cong J_{b_\bullet}(F)$ given by $g \mapsto (g, \dots, g)$, this set is equipped with an action of $J_b(F)$.

For minuscule $\mu_\bullet \in X_*(T)_+^d$ and $b_\bullet = (1, \dots, 1, b) \in G^d(L)$, we define

$$\mathcal{A}_{\mu_\bullet}^{\mathrm{top}} := \{\lambda_\bullet \in X_*(T)^d \mid \dim X_{\mu_\bullet}^{\lambda_\bullet}(b_\bullet) = \dim X_{\mu_\bullet}(b_\bullet)\}.$$

Here, $X_{\mu_\bullet}^{\lambda_\bullet}(b_\bullet)$ denotes $X_{\mu_\bullet}(b_\bullet) \cap I^d \varpi^{\lambda_\bullet} K^d / K^d$. For $\lambda_\bullet, \lambda'_\bullet \in \mathcal{A}_{\mu_\bullet}^{\mathrm{top}}$, we write $\lambda_\bullet \sim \lambda'_\bullet$ if $\lambda_\bullet = \tau^k \lambda'_\bullet = (\tau^k \lambda'_1, \dots, \tau^k \lambda'_d)$ for some $k \in \mathbb{Z}$. Let $\mathbb{A}_{\mu_\bullet}^{\mathrm{top}}$ denote the set of equivalence classes with respect to $\sim$, and let $[\lambda_\bullet] \in \mathbb{A}_{\mu_\bullet}^{\mathrm{top}}$ denote the equivalence class represented by $\lambda_\bullet \in \mathcal{A}_{\mu_\bullet}^{\mathrm{top}}$. Then $J_b(F) \backslash \operatorname{Irr} X_{\mu_\bullet}(b_\bullet)$ is parametrized by $\mathbb{A}_{\mu_\bullet}^{\mathrm{top}}$ as follows.

Proposition 3.12 *Assume that $\mu_\bullet \in X_*(T)_+^d$ is minuscule. Then the map $\lambda_\bullet \mapsto \overline{X_{\mu_\bullet}^{\lambda_\bullet}(b_\bullet)}$ induces a bijection*

$$\mathbb{A}_{\mu_\bullet}^{\mathrm{top}} \cong J_b(F) \backslash \operatorname{Irr} X_{\mu_\bullet}(b_\bullet).$$

Proof See [18, Proposition 1.6]. Note that we have $\operatorname{Stab}_{J_b(F)}(X_{\mu_\bullet}^{\lambda_\bullet}(b_\bullet)) = J_b(F)^0$. ■

We also define

$$\mathcal{A}_{\mu_\bullet}^{j} := \{\lambda_\bullet \in X_*(T)^d \mid \dim X_{\mu_\bullet}^{\lambda_\bullet}(b_\bullet) = j\},$$

for $1 \leq j \leq \dim X_{\mu_\bullet}(b_\bullet)$. We can similarly consider the equivalence relation $\sim$ as above. If $d = 1$, then $\mathbb{A}_\mu^j := \mathcal{A}_\mu^j / \sim$ can be identified with (extended) semi-modules for μ whose corresponding stratum has dimension j, see Lemma 3.4 and Lemma 3.9.

Proposition 3.13 *Set $\mu = \omega_i$. Then we always have $|\mathbb{A}_\mu^{\mathrm{top}}| = |\mathbb{A}_\mu^0| = 1$. If $i = 2, n-2$, then $|\mathbb{A}_\mu^j| = 1$ for all $0 \leq j \leq \dim X_\mu(b)$. If $i = 3, n-3$, then $|\mathbb{A}_\mu^{\dim X_\mu(b) - 1}| = 2$.*

Proof We can easily check the equalities in the proposition using [18, Theorem 4.16] (cf. [3, Remark 6.16]), which gives a combinatorial way of computing $|\mathbb{A}_\mu^j|$. In fact, all of the assertions except the last assertion follow from [43, Proposition 5.5]. ■

Example 3.14 We always have $\mathbb{A}_{\omega_i}^0 = \{[0]\}$.

4 Crystal bases

Keep the notations and assumptions in Section 3.

4.1 Crystals and young tableaux

In this subsection, we first recall the definition of $\widehat{G}$-crystals from [45, Definition 3.3.1].

Definition 4.1 A (normal) *$\widehat{G}$-crystal* is a finite set $\mathbb{B}$, equipped with a weight map $\mathrm{wt}: \mathbb{B} \to X_*(T)$, and operators $\tilde{e}_\alpha, \tilde{f}_\alpha: \mathbb{B} \to \mathbb{B} \cup \{0\}$ for each $\alpha \in \Delta$, such that

(i) for every $\mathbf{b} \in \mathbb{B}$, either $\tilde{e}_\alpha \mathbf{b} = 0$ or $\mathrm{wt}(\tilde{e}_\alpha \mathbf{b}) = \mathrm{wt}(\mathbf{b}) + \alpha^\vee$, and either $\tilde{f}_\alpha \mathbf{b} = 0$ or $\mathrm{wt}(\tilde{f}_\alpha \mathbf{b}) = \mathrm{wt}(\mathbf{b}) - \alpha^\vee$,
(ii) for all $\mathbf{b}, \mathbf{b}' \in \mathbb{B}$ one has $\mathbf{b}' = \tilde{e}_\alpha \mathbf{b}$ if and only if $\mathbf{b} = \tilde{f}_\alpha \mathbf{b}'$, and
(iii) if $\varepsilon_\alpha, \phi_\alpha: \mathbb{B} \to \mathbb{Z}$, $\alpha \in \Delta$ are the maps defined by

$$\varepsilon_\alpha(\mathbf{b}) = \max\{k \mid \tilde{e}_\alpha^k \mathbf{b} \neq 0\} \quad \text{and} \quad \phi_\alpha(\mathbf{b}) = \max\{k \mid \tilde{f}_\alpha^k \mathbf{b} \neq 0\},$$

then $\phi_\alpha(\mathbf{b}) - \varepsilon_\alpha(\mathbf{b}) = \langle \alpha, \mathrm{wt}(\mathbf{b}) \rangle$.

For a $\widehat{G}$-crystal $\mathbb{B}$, let $\mathbb{B}^* = \{\mathbf{b}^* \mid \mathbf{b} \in \mathbb{B}\}$ be the dual $\widehat{G}$-crystal. Setting $0^* = 0$, the maps are given by

$$\mathrm{wt}(\mathbf{b}^*) = -\mathrm{wt}(\mathbf{b}), \quad \tilde{e}_\alpha(\mathbf{b}^*) = (f_\alpha \mathbf{b})^*, \quad \text{and} \quad \tilde{f}_\alpha(\mathbf{b}^*) = (\tilde{e}_\alpha \mathbf{b})^*.$$

For $\lambda \in X_*(T)$, we denote by $\mathbb{B}(\lambda)$ the set of elements with weight λ for $\widehat{G}$, called the *weight space* with weight λ for $\widehat{G}$. Let $\mathbb{B}_1$ and $\mathbb{B}_2$ be two $\widehat{G}$-crystals. A morphism $\mathbb{B}_1 \to \mathbb{B}_2$ is a map of underlying sets compatible with wt, $\tilde{e}_\alpha$ and $\tilde{f}_\alpha$.

In the sequel, we write $\tilde{e}_i$ and $\tilde{f}_i$ (resp. ε_i and ϕ_i) instead of $\tilde{e}_{\chi_{i,i+1}}$ and $\tilde{f}_{\chi_{i,i+1}}$ (resp. $\varepsilon_{\chi_{i,i+1}}$ and $\phi_{\chi_{i,i+1}}$) for simplicity.

Example 4.2 Let $\mathbb{B}_\mu$ be the crystal basis of the irreducible $\widehat{G}$-module of highest weight $\mu \in X_*(T)_+$. Then $\mathbb{B}_\mu$ is a crystal. We call $\mathbb{B}_\mu$ a *highest weight crystal* of highest weight μ (cf. [45, Definition 3.3.1(3)]). There exists a unique element $\mathbf{b}_\mu \in \mathbb{B}_\mu$ satisfying $\tilde{e}_\alpha \mathbf{b}_\mu = 0$ for all α, $\mathrm{wt}(\mathbf{b}_\mu) = \mu$, and $\mathbb{B}_\mu$ is generated from $\mathbf{b}_\mu$ by the operators $\tilde{f}_\alpha$.

We give a realization of $\mathbb{B}_\mu$ by Young tableaux. This allows us to treat it in a combinatorial way.

Definition 4.3 A *Young diagram* is a collection of boxes arranged in left-justified rows with a weakly decreasing number of boxes in each row. For a dominant cocharacter $\mu \in X_*(T)_+$, we denote by Y_μ the Young diagram having $\mu(i)$ boxes in the ith row. A *skew Young diagram* is a diagram obtained by removing a smaller Young diagram from a larger one that contains it. For dominant cocharacters $\mu, \nu \in X_*(T)_+$ with $\nu(i) \leq \mu(i)$, we denote by $Y_{\mu/\nu}$ the skew Young diagram obtained by removing Y_ν from Y_μ.

Definition 4.4 A *tableau* is a (skew) Young diagram filled with numbers, one for each box. A *semi-standard tableau* is a tableau obtained from a (skew) Young diagram by filling the boxes with the numbers $1, 2, \ldots, n$ subject to the conditions

(i) the entries in each row are weakly increasing from left to right,
(ii) the entries in each column are strictly increasing from top to bottom.

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1	1	2	4
2	3	3	
4			

		2	4
	3	3	
4			

Let $K_{\mu/\nu}(\lambda)$ be the number of all semi-standard tableaux $\mathbf{b}$ of shape $Y_{\mu/\nu}$ such that the number of $\boxed{i}$ appearing in $\mathbf{b}$ is $\lambda(i)$ for $1 \leq i \leq n$. This is sometimes called the *Kostka number*. In Section 4.3, we need the following well-known result.

Proposition 4.5 *Let $\lambda, \lambda' \in X_*(T)_+$. If $\lambda \leq \lambda'$, then $K_{\mu/\nu}(\lambda') \leq K_{\mu/\nu}(\lambda)$. In particular, $K_{\mu/\nu}(\lambda') \neq 0$ implies $K_{\mu/\nu}(\lambda) \neq 0$.*

Proof See [5, Proposition 1.2] and the remark right after the proposition. ∎

We denote by $\mathcal{B}(Y)$ the set of all semi-standard tableaux of shape Y.

Theorem 4.6 *Let $\mu = (\mu(1), \ldots, \mu(n)) \in X_*(T)_+ \setminus \{0\}$ with $\mu(n) = 0$. Then $\mathcal{B}(Y_\mu)$ has a crystal structure. Moreover, the crystal $\mathcal{B}(Y_\mu)$ is isomorphic to $\mathbb{B}_\mu$.*

Proof This is [25, Theorems 7.3.6 and 7.4.1]. ∎

In the sequel, we identify $\mathbb{B}_\mu$ and $\mathcal{B}(Y)$ by Theorem 4.6. For a semi-standard tableau $\mathbf{b} \in \mathbb{B}_\mu$, let k_i denote the number of i's appearing in $\mathbf{b}$. Then the weight map wt on $\mathbb{B}_\mu$ is given by $\mathrm{wt}(\mathbf{b}) = (k_1, \ldots, k_n)$. The following result is an explicit description of the actions of $\tilde{e}_i$ and $\tilde{f}_i$ on $\mathbb{B}_\mu$.

Theorem 4.7 *The actions of $\tilde{e}_i$ and $\tilde{f}_i$ on $\mathbf{b} \in \mathbb{B}_\mu$ can be computed by following the steps below:*

(i) *In the Far-Eastern reading $\mathbf{b}_1 \otimes \cdots \otimes \mathbf{b}_N$ of $\mathbf{b}$, we identify i (resp. $\boxed{i+1}$) by + (resp. −) and neglect other boxes.*
(ii) *Let $u_i(\mathbf{b}) = u^1 u^2 \cdots u^l$ ($u^j \in \{\pm\}$) be the sequence obtained by (i). If there is "+−" in $u(\mathbf{b})$, then we neglect such a pair. We continue this procedure as far as we can.*
(iii) *Let $u_i(\mathbf{b})_{\mathrm{red}} = - \cdots - + \cdots +$ be the sequence obtained by (ii). Then $\tilde{e}_i$ changes the rightmost − in $u_i(\mathbf{b})_{\mathrm{red}}$ to +, and $\tilde{f}_i$ changes the leftmost + in $u_i(\mathbf{b})_{\mathrm{red}}$ to −. If there is no such − (resp. +), then $\tilde{e}_i \mathbf{b} = 0$ (resp. $\tilde{f}_i \mathbf{b} = 0$).*

Moreover, $\varepsilon_i(\mathbf{b})$ (resp. $\phi_i(\mathbf{b})$) is equal to the number of − (resp. +) in $u_i(\mathbf{b})_{\mathrm{red}}$.

Proof The first statement is [27, Theorem 3.4.2]. The second statement follows immediately from this. ∎

Next we recall the Weyl group action on crystals. Let $\mathbb{B}$ be a $\widehat{G}$-crystal. For any $1 \leq i \leq n-1$ and $\mathbf{b} \in \mathbb{B}$, we set

$$s_i \mathbf{b} = \begin{cases} \tilde{f}_i^{\langle \chi_{i,i+1}, \mathrm{wt}(\mathbf{b}) \rangle} \mathbf{b} & \text{if } \langle \chi_{i,i+1}, \mathrm{wt}(\mathbf{b}) \rangle \geq 0 \\ \tilde{e}_i^{-\langle \chi_{i,i+1}, \mathrm{wt}(\mathbf{b}) \rangle} \mathbf{b} & \text{if } \langle \chi_{i,i+1}, \mathrm{wt}(\mathbf{b}) \rangle \leq 0. \end{cases}$$

Then we have the obvious relation

$$\mathrm{wt}(s_i \mathbf{b}) = s_i(\mathrm{wt}(\mathbf{b})).$$

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By [26, Theorem 7.2.2], this extends to the action of the Weyl group W_0 on $\mathbb{B}$, which is compatible with the action on $X_*(T)$. For example, $w_{\max}\mathbf{b}_\mu \in \mathbb{B}_\mu$ has the *lowest* weight $w_{\max}\mu$. It is well-known that the dual of $\mathbb{B}_\mu$ is isomorphic to $\mathbb{B}_{-w_{\max}\mu}$ (see for example [25, Lemma 3.5.2]).

Lemma 4.8 *Let $w, w' \in W_0$ and $\mathbf{b} \in \mathbb{B}$. If $w(\mathrm{wt}(\mathbf{b})) = w'(\mathrm{wt}(\mathbf{b}))$, then $w\mathbf{b} = w'\mathbf{b}$.*

Proof This is [40, Lemma 3.10]. ∎

Let $\mathbf{b} \in \mathbb{B}(\lambda)$. If λ' is a conjugate of λ, i.e., there exists $w \in W_0$ such that $\lambda' = w\lambda$, then we call $w\mathbf{b}$ the conjugate of $\mathbf{b}$ with weight λ'. By Lemma 4.8, this does not depend on the choice of w.

Finally we consider the minuscule case. If $\mu \in X_*(T)_+$ is minuscule, then $\mathrm{wt}\colon \mathbb{B}_\mu \to X_*(T)$ gives an identification between $\mathbb{B}_\mu$ and the set of cocharacters which are conjugate to μ. Suppose $\mu_\bullet = (\mu_1, \dots, \mu_d) \in X_*(T)_+^d$ is minuscule. We can also identify $\mathbb{B}_{\mu_\bullet}^{\widehat{G}^d} := \mathbb{B}_{\mu_1} \times \cdots \times \mathbb{B}_{\mu_d}$ with the set of cocharacters in $X_*(T)^d$ which are conjugate to $\mu_\bullet$.

For $1 \le k < n$, let ω_k be the cocharacter of the form $(1, \dots, 1, 0, \dots, 0)$ in which 1 is repeated k times. Assume that each μ_i is equal to ω_{k_i} for some $1 \le k_i < n$ and $i \le j$ if and only if $k_i \le k_j$. In the rest of paper, we call such $\mu_\bullet$ *Far-Eastern*. If $\mu_\bullet$ is Far-Eastern, then $|\mu_\bullet| := \mu_1 + \cdots + \mu_d$ is dominant and its last entry is 0. Let $\mathrm{FE}\colon \mathbb{B}_{|\mu_\bullet|} \to \mathbb{B}_{\mu_\bullet}^{\widehat{G}^d}$ be a map defined by decomposing $\mathbf{b} \in \mathbb{B}_\mu$ into its columns from right to left. We call FE the Far-Eastern reading.

4.2 Construction of extended semi-modules

In this subsection, we recall from [40, Section 4.2] the way of constructing extended semi-modules. See [40, Section 4.3] for some examples of computation. Let $\mu_\bullet \in X_*(T)_+^d$ be a Far-Eastern cocharacter. Set $\mu = |\mu_\bullet|$.

Let λ_b denote the cocharacter whose i-th entry is $\lfloor \frac{im}{n} \rfloor - \lfloor \frac{(i-1)m}{n} \rfloor$. Set $\lambda_b^{\mathrm{op}} = w_{\max}\lambda_b$. For any $\mathbf{b} \in \mathbb{B}_\mu(\lambda_b)$, we denote by $\mathbf{b}^{\mathrm{op}}$ the conjugate of $\mathbf{b}$ with weight λ_b^{op}. Let $1 \le m_0 < n$ be the residue of m modulo n. Note that each entry of λ_b is $\lfloor \frac{m}{n} \rfloor$ or $\lfloor \frac{m}{n} \rfloor + 1$ and $\lambda_b(i) = \lambda_b(n+1-i)$ for any $2 \le i \le n-1$. Let $i_0 = 1 < i_1 < i_2 < \cdots < i_{m_0} = n$ be the integers such that $\lambda_b(i_1) = \lambda_b(i_2) = \cdots = \lambda_b(i_{m_0}) = \lfloor \frac{m}{n} \rfloor + 1$. Then

$$\lambda_b^{\mathrm{op}} = w'_{\max}\lambda_b, \quad \text{where} w'_{\max} = (s_{i_{m_0-1}} \cdots s_{n-1}) \cdots (s_{i_1} \cdots s_{i_2-1})(s_1 \cdots s_{i_1-1}).$$

Here $\lambda_b(i) = \lfloor \frac{m}{n} \rfloor$ (resp. $\lambda_b(i+1) = \lfloor \frac{m}{n} \rfloor$) if and only if $s_{i-1}s_i \le w'_{\max}$ (resp. $s_i s_{i+1} \le w'_{\max}$). By Lemma 4.8, it follows that $\mathbf{b}^{\mathrm{op}}$ can be computed by the action of the Coxeter element $w'_{\max}$. In this computation, each s_i acts as the action of $\tilde{e}_i$ because $\lfloor \frac{m}{n} \rfloor - (\lfloor \frac{m}{n} \rfloor + 1) = -1$. Therefore, if we write

$$\mathrm{FE}(\mathbf{b}) = (\mathbf{b}_1, \dots, \mathbf{b}_d)$$

then there exists $(w_1, \dots, w_d) \in W_0^d$ such that

$$\mathrm{FE}(\mathbf{b}^{\mathrm{op}}) = (w_1\mathbf{b}_1, \dots, w_d\mathbf{b}_d)$$

and each simple reflection appears exactly once in some $\mathrm{supp}(w_j)$.

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Lemma 4.9 *The tuple $(w_1, \dots, w_d) \in W_0^d$ as above is uniquely determined by* $\mathbf{b}$*. In particular,* $w(\mathbf{b}) \coloneqq w_1^{-1} \cdots w_d^{-1}$ *is a Coxeter element uniquely determined by* $\mathbf{b}$*.*

Proof This is [40, Lemma 4.3]. ∎

Set $w(\mathbf{b}) = w_1^{-1} \cdots w_d^{-1}$ and $\Upsilon(\mathbf{b}) = \{v \in W_0 \mid v^{-1} c^m v = w(\mathbf{b})\}$, where $c = s_1 s_2 \cdots s_{n-1}$. Clearly $|\Upsilon(\mathbf{b})| = n$.

For any $\mathbf{b}' \in \mathbb{B}_\mu$, set

$$\xi(\mathbf{b}') = (\varepsilon_1(\mathbf{b}') + \cdots + \varepsilon_{n-1}(\mathbf{b}'), \varepsilon_2(\mathbf{b}') + \cdots + \varepsilon_{n-1}(\mathbf{b}'), \dots, \varepsilon_{n-1}(\mathbf{b}'), 0).$$

Let λ_b^- be the anti-dominant conjugate of λ_b, and let $\mathbf{b}^-$ be the conjugate of $\mathbf{b}$ with weight λ_b^-. For any $\mathbf{b} \in \mathbb{B}_\mu(\lambda_b)$ and $v \in \Upsilon(\mathbf{b})$, we define $\xi_\bullet(\mathbf{b}, v) \in X_*(T)^d$ by

$$\xi_j(\mathbf{b}, v) = v\xi(v^{-1}\mathbf{b}^-) + \sum_{1 \le j' < j} v w_1^{-1} \cdots w_{j'-1}^{-1} \operatorname{wt}(\mathbf{b}_{j'}) \quad (1 \le j \le d).$$

Let $C \in \operatorname{Irr} X_\mu(b)^0$. By Proposition 3.7, $C = \overline{S_{A,\varphi}}$ for some $(A, \varphi) \in \mathbb{A}_\mu^{\mathrm{top}}$. On the other hand, by Proposition 3.12 and [32, Proposition 3.13], there exists a unique $\lambda_\bullet \in \mathcal{A}_{\mu_\bullet}^{\mathrm{top}}$ with $\lambda_1(1) + \cdots + \lambda_1(n) = 0$ such that $C = \mathrm{pr}(\overline{X_{\mu_\bullet}^{\lambda_\bullet}(b_\bullet)})$. Here $\mathrm{pr} : \mathcal{G}r^d \to \mathcal{G}r$ denotes the projection to the first factor. The following theorem is established in [40, Theorem 4.4] by the author.

Theorem 4.10 *We have $v_{\xi_j(\mathbf{b},v)} = v w_1^{-1} \cdots w_{j-1}^{-1}$ and $\xi_\bullet(\mathbf{b}, v) \in \mathcal{A}_{\mu_\bullet}^{\mathrm{top}}$. If v' is an element in $\Upsilon(\mathbf{b})$ different from v', then $\xi_\bullet(\mathbf{b}, v) \sim \xi_\bullet(\mathbf{b}, v')$. Let $\xi_\bullet^0(\mathbf{b})$ be the unique cocharacter in $[\xi_\bullet(\mathbf{b}, v)]$ such that $\xi_1^0(\mathbf{b})(1) + \cdots + \xi_1^0(\mathbf{b})(n) = 0$. Then for any $(A, \varphi) \in \mathbb{A}_\mu^{\mathrm{top}}$, there exists a unique $\mathbf{b} \in \mathbb{B}_\mu(\lambda_b)$ such that $\overline{S_{A,\varphi}} = \mathrm{pr}(\overline{X_{\mu_\bullet}^{\xi_\bullet^0(\mathbf{b})}(b_\bullet)})$.*

Proof This is [40, Theorem 4.4]. ∎

This correspondence between $\mathbb{A}_\mu^{\mathrm{top}}$ and $\mathbb{B}_\mu(\lambda_b)$ is compatible with the natural bijection in the Chen-Zhu conjecture constructed by Nie in [32].

Corollary 4.11 *Let $(A, \varphi) \in \mathbb{A}_\mu^{\mathrm{top}}$. Let $\mathbf{b} \in \mathbb{B}_\mu(\lambda_b)$ such that $\overline{S_{A,\varphi}} = \mathrm{pr}(\overline{X_{\mu_\bullet}^{\xi_\bullet^0(\mathbf{b})}(b_\bullet)})$. Then (A, φ) is cyclic if and only if*

$$\sum_{1 \le j \le d} w_1^{-1} \cdots w_{j-1}^{-1} \operatorname{wt}(\mathbf{b}_j) \in W_0 \mu.$$

Proof By Lemma 3.9, we have $A = A^{\xi_1^0(\mathbf{b})}$. Recall that (A, φ) is cyclic if and only if $b\xi_1^0(\mathbf{b}) - \xi_1^0(\mathbf{b}) \in W_0\mu$. Since $b\xi_1^0(\mathbf{b}) - \xi_1^0(\mathbf{b})$ is a conjugate of $b\xi_1(\mathbf{b}, v) - \xi_1(\mathbf{b}, v)$, this is also equivalent to $v^{-1} b\xi_1(\mathbf{b}, v) - v^{-1}\xi_1(\mathbf{b}, v) \in W_0\mu$. By Theorem 4.10,

$$v^{-1} b\xi_1(\mathbf{b}, v) - v^{-1}\xi_1(\mathbf{b}, v) = \sum_{1 \le j \le d} w_1^{-1} \cdots w_{j-1}^{-1} \operatorname{wt}(\mathbf{b}_j).$$

This finishes the proof. ∎

We say that an element $\mathbf{b} \in \mathbb{B}_\mu(\lambda_b)$ is cyclic if

$$\lambda(\mathbf{b}) \coloneqq \sum_{1 \le j \le d} w_1^{-1} \cdots w_{j-1}^{-1} \operatorname{wt}(\mathbf{b}_j) \in W_0\mu.$$

Now we give another interpretation of Lemma 3.11. Recall that $\mathbb{B}_\mu^*$ is isomorphic to $\mathbb{B}_{\mu^*}$. We denote by $\mathbf{b}^* \in \mathbb{B}_{\mu^*}$ the dual of $\mathbf{b} \in \mathbb{B}_\mu$. Note that we have $(w\mathbf{b})^* = w\mathbf{b}^*$ for any $w \in W_0$. So if $\mathbf{b} \in \mathbb{B}_\mu(\lambda_b)$, then $\mathbf{b}^{\mathrm{op}*} = w_{\max}\mathbf{b}^* \in \mathbb{B}_{\mu^*}(\lambda_{b^*})$.

Lemma 4.12 *We have $\lambda(\mathbf{b}^{\mathrm{op}*}) = -w(\mathbf{b})^{-1}\lambda(\mathbf{b}) + (d,\dots,d)$. In particular, $\mathbf{b} \in \mathbb{B}_\mu(\lambda_b)$ is cyclic if and only if $\mathbf{b}^{\mathrm{op}*} \in \mathbb{B}_{\mu^*}(\lambda_{b^*})$ is cyclic.*

Proof Note that if $(\mu_1,\dots,\mu_d)$ is Far-Eastern, then $(\mu_d^*,\dots,\mu_1^*)$ is Far-Eastern. So if we write

$$\mathrm{FE}(\mathbf{b}) = \mathbf{b}_1 \otimes \cdots \otimes \mathbf{b}_d \quad \text{and} \quad \mathrm{FE}(\mathbf{b}^{\mathrm{op}}) = w_1\mathbf{b}_1 \otimes \cdots \otimes w_d\mathbf{b}_d,$$

in $\mathbb{B}_{\mu_1} \otimes \cdots \otimes \mathbb{B}_{\mu_d}$, then we have

$$\mathrm{FE}(\mathbf{b}^*) = \mathbf{b}_d^* \otimes \cdots \otimes \mathbf{b}_1^* \quad \text{and} \quad \mathrm{FE}(\mathbf{b}^{\mathrm{op}*}) = w_d\mathbf{b}_d^* \otimes \cdots \otimes w_1\mathbf{b}_1^*,$$

in $\mathbb{B}_{\mu_d^*} \otimes \cdots \otimes \mathbb{B}_{\mu_1^*}$. Thus $w(\mathbf{b}^{\mathrm{op}*}) = w_d \cdots w_1 = w(\mathbf{b})^{-1}$, $\Upsilon(\mathbf{b}^{\mathrm{op}*}) = \Upsilon(\mathbf{b})$ and

$$\begin{aligned}\lambda(\mathbf{b}^{\mathrm{op}*}) &= \mathrm{wt}(w_d\mathbf{b}_d^*) + w_d\,\mathrm{wt}(w_{d-1}\mathbf{b}_{d-1}^*) + \cdots + w_d\cdots w_2\,\mathrm{wt}(w_1\mathbf{b}_1^*)\\ &= -w(\mathbf{b})^{-1}\lambda(\mathbf{b}) + (d,\dots,d),\end{aligned}$$

as desired. ■

4.3 Noncyclic semi-standard tableaux

The goal of this section is to specify the dominant cocharacters μ such that every $\mathbf{b} \in \mathbb{B}_\mu(\lambda_b)$ is cyclic. Set $d = \mu(1)$.

Lemma 4.13 *Assume that $n \geq 3$. We have $d \geq 2\lfloor \frac{m}{n} \rfloor + \lfloor \frac{2m_0}{n} \rfloor + 1$ or $d \geq 2\lfloor \frac{nd-m}{n} \rfloor + \lfloor \frac{2(n-m_0)}{n} \rfloor + 1$.*

Proof It suffices to show that $d \leq 2\lfloor \frac{m}{n} \rfloor + \lfloor \frac{2m_0}{n} \rfloor$ is equivalent to $d \geq 2\lfloor \frac{nd-m}{n} \rfloor + \lfloor \frac{2(n-m_0)}{n} \rfloor + 1$. Note that $\lfloor \frac{m}{n} \rfloor = \frac{m-m_0}{n}$, $\lfloor \frac{nd-m}{n} \rfloor = \frac{nd-m-(n-m_0)}{n}$. So $d \leq 2\lfloor \frac{m}{n} \rfloor + \lfloor \frac{2m_0}{n} \rfloor$ is equivalent to $(n-2)d \leq 2(m-d-m_0) + n\lfloor \frac{2m_0}{n} \rfloor$, and $d \geq 2\lfloor \frac{nd-m}{n} \rfloor + \lfloor \frac{2(n-m_0)}{n} \rfloor + 1$ is equivalent to $(n-2)d \leq 2(m-d-m_0) + n(1 - \lfloor \frac{2(n-m_0)}{n} \rfloor)$. Then the assertion follows from the fact that $\lfloor \frac{2m_0}{n} \rfloor = 0$ (resp. 1) if and only if $\lfloor \frac{2(n-m_0)}{n} \rfloor = 1$ (resp. 0). ■

Lemma 4.14 *Assume that $n \geq 3$. Let $\mu \in X_*(T)_+$ such that $d \geq 2\lfloor \frac{m}{n} \rfloor + \lfloor \frac{2m_0}{n} \rfloor + 1$, $\mu(2) \geq 2$ and $\lfloor \frac{m}{n} \rfloor \geq 2$. Then $\mathbb{B}_\mu(\lambda_b)$ contains at least one noncyclic element.*

Proof First we consider the case $n = 3$. In this case, we have $2 \leq \mu(2) \leq \lfloor \frac{m}{n} \rfloor$ because $\mu(3) = 0$. Let $\mathbf{b}$ be the unique element in $\mathbb{B}_\mu(\lambda_b)$ whose second row contains exactly one $\boxed{3}$. Then $w(\mathbf{b}) = s_2 s_1$ and $s_1 \in \mathrm{supp}(w_{d-\lfloor \frac{m}{n} \rfloor})$.

1	⋯	⋯	1	2	3	⋯	3
2	⋯	2	3				

Since $2 \leq \mu(2) \leq \lfloor \frac{m}{n} \rfloor$, we have

$$w_1^{-1}\cdots w_{d-\mu(2)}^{-1}\,\mathrm{wt}(\mathbf{b}_{d-\mu(2)+1}) = (0,1,1) \quad \text{and} \quad w_1^{-1}\cdots w_{d-1}^{-1}\,\mathrm{wt}(\mathbf{b}_d) = (1,0,1).$$

Thus $\lambda(\mathbf{b}) \notin W_0\mu$ because $\mu(n) = 0$. This proves the case $n = 3$.

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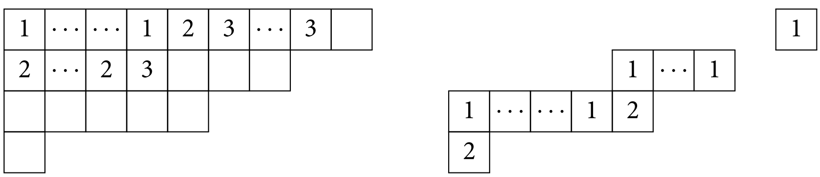
In the rest of the proof, we assume that $n \geq 4$. Let λ be a conjugate of λ_b such that $(\lambda(1), \lambda(2), \lambda(3)) = (\lfloor \frac{m}{n} \rfloor, \lfloor \frac{m}{n} \rfloor + \lfloor \frac{2m_0}{n} \rfloor, \lfloor \frac{m}{n} \rfloor + 1)$ and $\lambda(4) \geq \cdots \geq \lambda(n)$. Set

$$\mu_0 = \left(3\lfloor \frac{m}{n} \rfloor + \lfloor \frac{2m_0}{n} \rfloor + 1 - \min\{\mu(2), \lfloor \frac{m}{n} \rfloor\}, \min\{\mu(2), \lfloor \frac{m}{n} \rfloor\}, 0, \dots, 0\right) \in X_*(T)_+,$$

and $\lambda_0 = (\lambda(1), \lambda(2), \lambda(3), 0, \dots, 0) \in X_*(T)$. Note that we have $\mu(1) + \mu(2) \geq 3\lfloor \frac{m}{n} \rfloor + \lfloor \frac{2m_0}{n} \rfloor + 1$. Indeed if $\mu(1) + \mu(2) \leq 3\lfloor \frac{m}{n} \rfloor + \lfloor \frac{2m_0}{n} \rfloor$, then by $\mu(1) \geq 2\lfloor \frac{m}{n} \rfloor + \lfloor \frac{2m_0}{n} \rfloor + 1$, we have $\mu(2) \leq \lfloor \frac{m}{n} \rfloor - 1$. This implies $\mu(3) + \cdots + \mu(n-1) \leq (n-3)(\lfloor \frac{m}{n} \rfloor - 1)$, or equivalently $3\lfloor \frac{m}{n} \rfloor + n + m_0 - 3 \leq \mu(1) + \mu(2)$, which is a contradiction. Thus Y_μ contains Y_{μ_0}.

Let $\mathbf{b}_0$ be the unique element in $\mathbb{B}_{\mu_0}(\lambda_0)$ whose second row contains exactly one $\boxed{3}$. We will show that there exists $\mathbf{b}' \in \mathbb{B}_\mu(\lambda)$ that contains $\mathbf{b}_0$. It is easy to check that $\mu(n-1) \leq \lfloor \frac{m}{n} \rfloor$ and $\mu(n-2) \leq \mu_0(1)$. So each column in Y_{μ/μ_0} has at most $n-3$ boxes. By filling each column with the numbers $1, \dots, n-3$ so that the entries are starting with 1 and increasing by one from top to bottom, we obtain a skew Young tableau of shape Y_{μ/μ_0}. Let k_i be the number of $\boxed{i}$ in this tableau. Clearly we have $k_1 \geq \cdots \geq k_{n-3}$.



By $(\lambda(4), \dots, \lambda(n)) \leq (k_1, \dots, k_{n-3})$ and Proposition 4.5, there exists at least one skew Young tableau of shape Y_{μ/μ_0} such that the number of $\boxed{i}$ is $\lambda(i+3)$ for each $1 \leq i \leq n-3$. By replacing $1, \dots, n-3$ by $4, \dots, n$ respectively, we obtain a skew Young tableau of shape Y_{μ/μ_0} such that the number of $\boxed{i}$ is $\lambda(i)$ for each $4 \leq i \leq n$. Let $\mathbf{b}'$ be the tableau obtained by joining $\mathbf{b}_0$ and this skew tableau. Clearly we have $\mathbf{b}' \in \mathbb{B}_\mu(\lambda)$, which shows our claim.

Let $\mathbf{b}' \in \mathbb{B}_\mu(\lambda)$ containing $\mathbf{b}_0$, and let $\mathbf{b} \in \mathbb{B}_\mu(\lambda_b)$ be the conjugate of $\mathbf{b}'$. Then $s_2 s_1 \leq w(\mathbf{b})$ and $s_1 \in \mathrm{supp}(w_{d-\lfloor \frac{m}{n} \rfloor})$. Let $k(\mathbf{b}')$ be the number of $\boxed{4}$ in the second row of $\mathbf{b}'$. If $k(\mathbf{b}') < \lfloor \frac{m}{n} \rfloor$, then we have

$$(w_1^{-1} \cdots w_{d-\min\{\mu(2), \lfloor \frac{m}{n} \rfloor\}}^{-1} \mathrm{wt}(\mathbf{b}_{d-\min\{\mu(2), \lfloor \frac{m}{n} \rfloor\}+1}))(2) = 1,$$

and

$$(w_1^{-1} \cdots w_{d-1}^{-1} \mathrm{wt}(\mathbf{b}_d))(2) = 0.$$

Thus, $\lambda(\mathbf{b}) \notin W_0 \mu$ and hence $\mathbf{b}$ is noncyclic. If $k(\mathbf{b}') \neq 0$, then $\lambda(\mathbf{b})(1) = \lfloor \frac{m}{n} \rfloor - 1$.

Assume that $\mu(3) < \lfloor \frac{m}{n} \rfloor - 1$. Then $\mathbf{b}$ is always noncyclic by the above argument.

Assume that $\mu(3) \geq 2$. Let $\mathbf{b}'_1 = \boxed{j}$ be the leftmost box in the third row of $\mathbf{b}'$, and let $\mathbf{b}'_2 = \boxed{j'}$ be the box right to $\mathbf{b}'_1$. Clearly $4 \leq j \leq j'$.

1	$\cdots$	$\cdots$	1	2	3	$\cdots$	3	
2	$\cdots$	2	3	4	$\cdots$	4		
j	j'							

Then in $\mathbf{b}'$, all $\boxed{j\text{–}1}$ are in the first or second row. Since the number of $\boxed{j}$ in the first or second row is less than $\mathrm{wt}(\mathbf{b}')(j-1)$, there exists at least one $\boxed{j\text{–}1}$ such that there is no box beneath it or the number in the box beneath it is greater than j. So the tableau obtained by replacing $\mathbf{b}'_1$ by the rightmost one among such $\boxed{j\text{–}1}$ is semi-standard. Repeating the same argument, we may assume $j = 4$. Similarly, if $\lfloor \frac{m}{n} \rfloor \geq 3$, we may also assume $j' = 4$. Indeed if $j' \geq 6$ and the leftmost column in $\mathbf{b}'$ contains $\boxed{j'\text{–}1}$ but does not contain $\boxed{j'}$, we replace $\mathbf{b}'_2$ by this $\boxed{j'\text{–}1}$. In other cases, by $\lfloor \frac{m}{n} \rfloor \geq 3$, there exists at least one $\boxed{j'\text{–}1}$ such that there is no box beneath it or the number in the box beneath it is greater than j', and we replace $\mathbf{b}'_2$ by the rightmost $\boxed{j'\text{–}1}$ among such $\boxed{j'\text{–}1}$. Then the obtained tableau is semi-standard. Thus, if $\lfloor \frac{m}{n} \rfloor \geq 3$, there exists $\mathbf{b}'$ containing $\mathbf{b}_0$ such that $k(\mathbf{b}') < \lfloor \frac{m}{n} \rfloor$, which is noncyclic by the above argument. If $\lfloor \frac{m}{n} \rfloor = 2$ and $n = 4$, then $\mathbf{b}$ is noncyclic because $k(\mathbf{b}') < 2$. If $\lfloor \frac{m}{n} \rfloor = 2$ and $n \geq 5$, we may also assume $j' = 4$ and hence $\mathbf{b}$ is noncyclic unless the third row of $\mathbf{b}'$ contains three $\boxed{5}$. If $\lfloor \frac{m}{n} \rfloor = 2$, $n \geq 5$ and the third row of $\mathbf{b}'$ contains three $\boxed{5}$, then

$$(w_1^{-1} \cdots w_{d-2}^{-1}\,\mathrm{wt}(\mathbf{b}_{d-1}))(4) = 1 \quad \text{and} \quad (w_1^{-1} \cdots w_{d-1}^{-1}\,\mathrm{wt}(\mathbf{b}_d))(4) = 0.$$

Thus, $\lambda(\mathbf{b}) \notin W_0\mu$ and hence $\mathbf{b}$ is noncyclic.

1	1	2	3	3				
2	3	4	4					
4	5	5	5					

Assume that $\lfloor \frac{m}{n} \rfloor = 2$ and $\mu(3) = 1$. By the same argument as above, we may assume that the leftmost column of $\mathbf{b}'$ contains $\boxed{4}$. So $\mathbf{b}$ is noncyclic when $\lambda(4) = 2$. If $\mu(1) > 5 + \lfloor \frac{2m_0}{n} \rfloor$, we may assume that the first row of $\mathbf{b}'$ also contains $\boxed{4}$. This can be checked easily as above using $\mu(3) = 1$. Thus, if $\mu(1) > 5 + \lfloor \frac{2m_0}{n} \rfloor$, we obtain a noncyclic $\mathbf{b}$.

1	1	2	3	3	4			
2	3	4						
4								

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If $\mu(1) = 5 + \lfloor \frac{2m_0}{n} \rfloor$, then we have $n = 4$ or 5. More precisely, we have

$$\mu = (6,4,1,0), (5,5,1,1,0), (6,5,1,1,0), (6,6,1,0,0), \text{ or } (6,6,1,1,0),$$

and $\mathbf{b}'$ contains one of the following smaller Young tableaux when $\lambda(4) = 3$.

$$\begin{array}{|c|c|c|c|c|}\hline 1 & 1 & 2 & 3 & 3 \\ \hline 2 & 3 & 4 & 4 \\ \cline{1-4} 4 \\ \cline{1-1}\end{array} \qquad \begin{array}{|c|c|c|c|c|c|}\hline 1 & 1 & 2 & 2 & 3 & 3 \\ \hline 2 & 3 & 4 & 4 \\ \cline{1-4} 4 \\ \cline{1-1}\end{array}$$

We can easily check that $\mathbf{b}$ is noncyclic in every case.

Putting things together, we have proved the lemma. ∎

Lemma 4.15 *Assume that $n \geq 4$. Let $\mu \in X_*(T)_+$ such that $d \geq 3 + \lfloor \frac{2m_0}{n} \rfloor$, $\mu(2) \geq 2$ and $\lfloor \frac{m}{n} \rfloor = 1$. Then $\mathbb{B}_\mu(\lambda_b)$ contains at least one noncyclic element.*

Proof Let λ be a conjugate of λ_b such that $(\lambda(1), \lambda(2), \lambda(3)) = (\lambda_b(1), \lambda_b(2), \lambda_b(3))$ and $\lambda(4) \geq \cdots \geq \lambda(n)$. Assume that $(\lambda_b(1), \lambda_b(2), \lambda_b(3)) = (1,2,2)$ and $\mu(2) \geq 3$. Similarly as the proof of Lemma 4.14, we can easily show that there exists $\mathbf{b}' \in \mathbb{B}_\mu(\lambda)$ containing the following smaller Young tableau.

$$\begin{array}{|c|c|c|c|}\hline 1 & 2 & 3 & 4 \\ \hline 2 & 3 & 4 \\ \cline{1-3}\end{array}$$

Let $\mathbf{b} \in \mathbb{B}_\mu(\lambda_b)$ be the conjugate of $\mathbf{b}'$. If $\mu(3) < 2$, then $\mathbf{b}$ is noncyclic because $\lambda(\mathbf{b})(2) = 2$. If $\mu(3) \geq 2$, then similarly as the proof of Lemma 4.14, we may assume that the second row of $\mathbf{b}'$ does not contain $\boxed{5}$. In this case, the conjugate $\mathbf{b} \in \mathbb{B}_\mu(\lambda_b)$ of $\mathbf{b}'$ is noncyclic because

$$(w_1^{-1} \cdots w_{d-3}^{-1} \operatorname{wt}(\mathbf{b}_{d-2}))(3) = 1 \quad \text{and} \quad (w_1^{-1} \cdots w_{d-1}^{-1} \operatorname{wt}(\mathbf{b}_d))(3) = 0.$$

Assume that $(\lambda_b(1), \lambda_b(2), \lambda_b(3)) = (1,2,2)$ and $\mu(2) = 2$. Then there exists $\mathbf{b}' \in \mathbb{B}_\mu(\lambda)$ containing one of the following smaller Young tableaux.

$$\begin{array}{|c|c|c|c|c|}\hline 1 & 2 & 3 & 3 & 4 \\ \hline 2 & 4 \\ \cline{1-2}\end{array} \qquad \begin{array}{|c|c|c|c|}\hline 1 & 2 & 3 & 4 \\ \hline 2 & 4 \\ \cline{1-2} 3 \\ \cline{1-1}\end{array}$$

It is easy to check that the conjugate $\mathbf{b} \in \mathbb{B}_\mu(\lambda_b)$ of $\mathbf{b}'$ is noncyclic.

Assume that $(\lambda_b(1), \lambda_b(2), \lambda_b(3)) \neq (1,2,2)$. Then there exists $\mathbf{b}' \in \mathbb{B}_\mu(\lambda)$ containing one of the following smaller Young tableaux.

$$\begin{array}{|c|c|c|c|}\hline 1 & 2 & 3 & 4 \\ \hline 2 & 4 \\ \cline{1-2}\end{array} \qquad \begin{array}{|c|c|c|c|}\hline 1 & 3 & 3 & 4 \\ \hline 2 & 4 \\ \cline{1-2}\end{array} \qquad \begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 & 4 \\ \cline{1-2}\end{array}$$

Let $\mathbf{b} \in \mathbb{B}_\mu(\lambda_b)$ be the conjugate of $\mathbf{b}'$. Since $\lambda(\mathbf{b})(1) = 1$, $\mathbf{b}$ is noncyclic if $\mu(3) = 0$. If $\mu(3) \geq 2$, then similarly as the proof of Lemma 4.14, we may assume that the second row of $\mathbf{b}'$ does not contain $\boxed{5}$. In this case, $\mathbf{b}$ is noncyclic because

$$(w_1^{-1} \cdots w_{d-2}^{-1} \operatorname{wt}(\mathbf{b}_{d-1}))(3) = 1 \quad \text{and} \quad (w_1^{-1} \cdots w_{d-1}^{-1} \operatorname{wt}(\mathbf{b}_d))(3) = 0.$$

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If $\mu(3) = 1$ and $\mu(1) > 3 + \lfloor \frac{2m_0}{n} \rfloor$, then we may also assume that the second row of $\mathbf{b}'$ does not contain $\boxed{5}$ and hence $\mathbf{b}$ is noncyclic. If $\mu(3) = 1$ and $\mu(1) = 3 + \lfloor \frac{2m_0}{n} \rfloor$, then we may assume that the leftmost column of $\mathbf{b}'$ contains $\boxed{5}$. We can easily check that $\mathbf{b}$ is noncyclic by an easy calculation.

1	2	3	4
2	4	5	
5			

1	3	3	4
2	4	5	
5			

1	3	4
2	4	5
5		

This finishes the proof. ∎

Lemma 4.16 *Assume that $n \geq 5$. Let $\mu \in X_*(T)_+$ such that $\lfloor \frac{m}{n} \rfloor = 0$. If (1) $\mu(2) \geq 2$ or (2) $d \geq 3, \mu(2) = 1$, then $\mathbb{B}_\mu(\lambda_b)$ contains at least one noncyclic element.*

Proof Let $1 < i_1 < i_2 < \cdots < i_{m_0} = n$ be the integers such that $\lambda_b(i_1) = \lambda_b(i_2) = \cdots = \lambda_b(i_{m_0}) = 1$. Let $\mathbf{b}$ be the Young tableau in $\mathbb{B}_\mu(\lambda_b)$ obtained by filling Y_μ with $i_1, \ldots, i_{m_0}$ from top to bottom, starting from the leftmost column.

i_1	i_{k+1}	$\cdots$	i_m
i_2	i_{k+2}	$\vdots$	
$\vdots$	$\vdots$		
i_k			

If (1) holds, then $\mathbf{b}$ is noncyclic because

$$\mathrm{wt}(\mathbf{b}_1)(i_m) = 1 \quad \text{and} \quad (w_1^{-1} \cdots w_{d-1}^{-1} \mathrm{wt}(\mathbf{b}_d))(i_m) = 0.$$

Let $k = \max\{i \mid \mu(i) \neq 0\}$. If (2) holds, then the Young tableau $\mathbf{c} \in \mathbb{B}_\mu(\lambda_b)$ obtained by replacing $\boxed{i_k}$ by $\boxed{i_{k+1}}$ in $\mathbf{b}$ is noncyclic because $\lambda(\mathbf{c})(i_k) = 2$.

i_1	i_k	i_{k+2}	$\cdots$	i_m
$\vdots$				
i_{k-1}				
i_{k+1}				

This finishes the proof. ∎

Theorem 4.17 *Every $\mathbf{b} \in \mathbb{B}_\mu(\lambda_b)$ is cyclic if and only if μ has one of the following forms:*

(i) *ω_i with $1 \leq i \leq n-1$ such that i is coprime to n.*
(ii) *$\omega_1 + \omega_i$ or $\omega_{n-1} + \omega_{n-i}$ with $1 \leq i \leq n-1$ such that $i+1$ is coprime to n.*
(iii) *$(nr+i)\omega_1$ or $(nr+i)\omega_{n-1}$ with $r \geq 0$ and $1 \leq i \leq n-1$ such that i is coprime to n.*
(iv) *$(nr+i-j)\omega_1 + \omega_j$ or $(nr+i-j)\omega_{n-1} + \omega_{n-j}$ with $r \geq 1$, $2 \leq j \leq n-1$ and $1 \leq i \leq n-1$ such that i is coprime to n.*

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Proof It is easy to check that every $\mathbf{b} \in \mathbb{B}_\mu(\lambda_b)$ is cyclic if μ is one of the cocharacters in (i), (ii), (iii) and (iv). It remains to show that if μ does not belong to the list above, then $\mathbb{B}_\mu(\lambda_b)$ contains at least one noncyclic element. By Lemmas 4.12 and 4.13, we may assume that $d \geq 2\lfloor \frac{m}{n} \rfloor + \lfloor \frac{2m_0}{n} \rfloor + 1$. Then this follows from Lemmas 4.14, 4.15 and 4.16. ∎

Remark 4.18 Even if every top extended semi-module for μ is cyclic, there might be a noncyclic extended semi-module for μ. In fact, such cases exist, see Section 5.4.

5 The semi-module stratification

Keep the notations and assumptions in Section 3.

5.1 The semi-module stratification for ω_i

Recall that if μ is minuscule, then every extended semi-module is cyclic.

Lemma 5.1 *For any* $1 \leq j \leq \frac{n-3}{2}$ $(= \dim X_{\omega_2}(\tau^2))$, *we have*

$$\mathbb{A}^j_{\omega_2} = \begin{cases} \{[\chi^\vee_{2,n-1} + \chi^\vee_{4,n-3} + \cdots + \chi^\vee_{j,n-j+1}]\} & (j \text{ even}) \\ \{[\chi^\vee_{1,n} + \chi^\vee_{3,n-2} + \cdots + \chi^\vee_{j,n-j+1}]\} & (j \text{ odd}). \end{cases}$$

Proof By (the proof of) [43, Proposition 5.5], each normalized semi-module for 2, n is of the form $A_j = (2\mathbb{N} - j) \cup (\mathbb{N} + j + 1)$ for some $1 \leq j \leq \frac{n-3}{2}$. It is easy to check that

$$A_j = \begin{cases} A^{\chi^\vee_{2,n-1} + \chi^\vee_{4,n-3} + \cdots + \chi^\vee_{j,n-j+1}} & (j \text{ even}) \\ A^{\chi^\vee_{1,n} + \chi^\vee_{3,n-2} + \cdots + \chi^\vee_{j,n-j+1}} & (j \text{ odd}). \end{cases}$$

Let (A_j, φ_j) be the cyclic semi-module for ω_2. Then $n-2-j, n-1+j \in \bar{A}_j$ and $\varphi_j(n-2-j) = \varphi_j(n-1+j) = 1$. It is also easy to check that $|\mathcal{V}(A_j, \varphi_j)| = j$. This finishes the proof. ∎

Lemma 5.2 *Assume that* $n = 7$. *Then* $\dim X_{\omega_3}(\tau^3) = 3$ *and*

$$\mathbb{A}^1_{\omega_3} = \{[\chi^\vee_{1,7}]\}, \quad \mathbb{A}^2_{\omega_3} = \{[\chi^\vee_{1,6}], [\chi^\vee_{2,7}]\}, \quad \mathbb{A}^3_{\omega_3} = \{[\chi^\vee_{3,5}]\}.$$

Assume that $n = 8$. *Then* $\dim X_{\omega_3}(\tau^3) = 4$ *and*

$$\mathbb{A}^1_{\omega_3} = \{[\chi^\vee_{1,8}]\}, \quad \mathbb{A}^2_{\omega_3} = \{[\chi^\vee_{1,7}], [\chi^\vee_{2,8}]\},$$
$$\mathbb{A}^3_{\omega_3} = \{[\chi^\vee_{2,6}], [\chi^\vee_{3,7}]\}, \quad \mathbb{A}^4_{\omega_3} = \{[\chi^\vee_{1,8} + \chi^\vee_{4,5}]\}.$$

Proof Using Lemma 3.2, we can easily check the lemma by an easy calculation. ∎

5.2 The semi-module stratification for $\omega_1 + \omega_{n-2}$

Throughout this subsection, we set $\mu = \omega_1 + \omega_{n-2}$. Also we assume that $n \geq 4$.

Lemma 5.3 *Every extended semi-module for* μ *is cyclic. For any* $0 \leq j \leq n-2$ $(= \dim X_\mu(b))$, *we define* $\mathbb{A}^j_\mu$ *similarly as in Section* 3.3. *Then we have* $\mathbb{A}^0_\mu = \varnothing$ *and*

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$|\mathbb{A}_\mu^j| = j$. *More precisely, if j is odd, then $\mathbb{A}_\mu^j$ is equal to*

$$\{[\chi^\vee_{1,n-j+1}],[\chi^\vee_{1,n-j+3}+\chi^\vee_{2,n-j+2}],\dots,\\ [\chi^\vee_{1,n}+\chi^\vee_{2,n-1}+\cdots+\chi^\vee_{\frac{j+1}{2},n-\frac{j-1}{2}}],\dots,[\chi^\vee_{j-2,n}+\chi^\vee_{j-1,n-1}],[\chi^\vee_{j,n}]\},$$

and if j is even, then $\mathbb{A}_\mu^j$ is equal to

$$\{[\chi^\vee_{1,n-j+1}],[\chi^\vee_{1,n-j+3}+\chi^\vee_{2,n-j+2}],\dots,\\ [\chi^\vee_{1,n-1}+\chi^\vee_{2,n-2}+\cdots+\chi^\vee_{\frac{j}{2},n-\frac{j}{2}}],\dots,[\chi^\vee_{j-2,n}+\chi^\vee_{j-1,n-1}],[\chi^\vee_{j,n}]\}.$$

Proof Let (A,φ) be an extended semi-module for μ. Let μ' be the type of A. If (A,φ) is noncyclic, then by Lemma 3.4, $\mu'_{\mathrm{dom}} \prec \mu$, i.e., $\mu'_{\mathrm{dom}} = \omega_{n-1}$. By Lemma 3.2, we have $A = \{0,1,\dots,n-1,\dots\}$. By Definition 3.3 (3), $\varphi(a) = \max\{k \mid a+n-1-kn \in A\}$ for all $a \in A$. This contradicts to the assumption that (A,φ) is noncyclic. Thus, (A,φ) is cyclic.

Since μ' satisfies $\nu_b \leq w_{\max}\mu'$, it is easy to check that

$$w_{\max}\mu' = s_{l+1}\cdots s_{n-3}s_{n-2}s_{k-1}\cdots s_2 s_1\mu,$$

for some $1 \leq k \leq n-2$ and $k \leq l \leq n-2$. Let $\bar{A} = \{a_0, a_1\dots,a_{n-1}\}$ with $a_0 = \min\bar{A}$. Then we have $\varphi(a_0) = 0$, $\varphi(a_{n-l-1}) = 0$, $\varphi(a_{n-k}) = 2$ and $\varphi(a_i) = 1$ for $i \neq 0, n-l-1, n-k$. Thus,

$$\begin{aligned}\mathcal{V}(A,\varphi) =&\{(a_{n-k},a_{n-l-1}+n),(a_{n-k},a_{n-l}),(a_{n-k},a_{n-l+1}),\dots,(a_{n-k},a_{n-k-1})\}\\ &\cup\{(a_{n-k+1},a_{n-l-1}),(a_{n-k+2},a_{n-l-1}),\dots,(a_{n-1},a_{n-l-1})\},\end{aligned}$$

and $|\mathcal{V}(A,\varphi)| = l$. Then by Proposition 3.7, the description of $\mathbb{A}_\mu^l$ for each l in the lemma follows from direct computation. ∎

5.3 The semi-module stratification for $\omega_1 + \omega_{n-3}$

Throughout this subsection, we set $\mu = \omega_1 + \omega_{n-3}$. Also we assume that $n \geq 7$.

Lemma 5.4 *Every extended semi-module for μ is cyclic. For any $1 \leq j \leq \frac{3n-9}{2}$ ($= \dim X_\mu(b)$), we define $\mathbb{A}_\mu^j$ similarly as in Section 3.3. Then $|\mathbb{A}_\mu^{\frac{3n-9}{2}}| = n-3$ and $|\mathbb{A}_\mu^{\frac{3n-11}{2}}| \leq 2(n-4)$.*

Proof Using Lemma 5.1, we can show the first assertion similarly as the proof of Lemma 5.3. Indeed, for any semi-module A^λ in Lemma 5.1, there exists a unique φ such that (A^λ,φ) is an extended semi-module for some $\mu \in X_*(T)_+$. The equality $|\mathbb{A}_\mu^{\frac{3n-9}{2}}| = n-3$ follows from the Chen-Zhu conjecture.

Let (A,φ) be an extended semi-module for μ with type $\mu'(\in W_0\mu)$. Let $0 < k_1 < k_2$ be integers such that $\mu'(1) = \mu'(k_1+1) = \mu'(k_2+1) = 0$, and let l be an integer such that $\mu'(l+1) = 2$. Assume that $\nu_b \leq w_{\max}s_{k_2+1}\mu'$. Let (B,ψ) be an extended semi-module for μ with type $s_{k_2+1}\mu'$. Let $a_0 = \min\bar{A}$ (resp. $b_0 = \min\bar{B}$) and let inductively $a_i = a_{i-1}+n-2-\mu'(i)n$ (resp. $b_i = b_{i-1}+n-2-(s_{k_2+1}\mu')(i)n$) for $i = 1,\dots,n$. Then $a_0 = a_n$ (resp. $b_0 = b_n$) and $\{a_0,a_1,\dots,a_{n-1}\} = \bar{A}$ (resp. $\{b_0,b_1,\dots,b_{n-1}\} = \bar{B}$). We will show that if $l > k_2+1$ (resp. $l = k_2+1$), then $|\mathcal{V}(B,\psi)| \leq |\mathcal{V}(A,\varphi)|$ (resp. $|\mathcal{V}(B,\psi)| < |\mathcal{V}(A,\varphi)|-1$). Moreover, the equality does not hold if $k_2 - k_1 \leq 3$.

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Note that we have $\varphi(a_0) = \varphi(a_{k_1}) = \varphi(a_{k_2}) = 0, \varphi(a_l) = 2, \psi(b_0) = \psi(b_{k_1}) = \psi(b_{k_2+1}) = 0, \psi(b_l) = 2$. Note also that

$$\begin{aligned}\mathcal{V}(A,\varphi) =&\{(a,a') \mid a \in \bar{A} \text{with} \varphi(a) = 1, a' = a_{k_1} \text{or} a_{k_2}\}\\ &\sqcup\{(a_l,a') \mid a_l < a', \varphi(a') < 2\},\end{aligned}$$

and

$$\begin{aligned}\mathcal{V}(B,\psi) =&\{(b,b') \mid b \in \bar{B} \text{with} \psi(b) = 1, b' = b_{k_1} \text{or} b_{k_2+1}\}\\ &\sqcup\{(b_l,b') \mid b_l < b', \psi(b') < 2\}.\end{aligned}$$

Let $\mathcal{V}(A,\varphi)_1$ (resp. $\mathcal{V}(B,\psi)_1$) be the first subset in $\mathcal{V}(A,\varphi)$ (resp. $\mathcal{V}(B,\psi)$) above, and let $\mathcal{V}(A,\varphi)_2$ (resp. $\mathcal{V}(B,\psi)_2$) be its complement.

If $l > k_2 + 1$, then it follows that

$$b_k = \begin{cases} a_k + 1 & (k \neq k_2+1) \\ a_k + 1 - n & (k = k_2+1) \end{cases}, \quad \psi(b_k) = \begin{cases} \varphi(a_k) & (k \neq k_2, k_2+1) \\ 1-\varphi(a_k) & (k = k_2, k_2+1). \end{cases}$$

In particular, $b_{k_2+1} - 1 = a_{k_2} - 2$. So $|\mathcal{V}(B,\psi)_1| > |\mathcal{V}(A,\varphi)_1|$ implies that $|\mathcal{V}(B,\psi)_1| = |\mathcal{V}(A,\varphi)_1| + 1$ and $b_{k_2} < b_{k_1}$. By the fact $(a_l, a_{k_2+1}) \in \mathcal{V}(A,\varphi)_2$, we always have $|\mathcal{V}(B,\psi)_2| < |\mathcal{V}(A,\varphi)_2|$. Thus, $|\mathcal{V}(B,\psi)| \leq |\mathcal{V}(A,\varphi)|$. Moreover, if $k_2 - k_1 \leq 3$, then the equality does not hold because $b_{k_2} \geq b_{k_1}$.

If $l = k_2 + 1$, then it follows that

$$b_k = \begin{cases} a_k + 2 & (k \neq k_2+1) \\ a_k + 2 - 2n & (k = k_2+1) \end{cases}, \quad \psi(b_k) = \begin{cases} \varphi(a_k) & (k \neq k_2, k_2+1) \\ 2-\varphi(a_k) & (k = k_2, k_2+1). \end{cases}$$

In particular, $b_{k_2+1} - 2 = a_{k_2} - 2 - n$. By $v_b \leq w_{\max} s_{k_2+1}\mu'$, we have $k_2 \leq \frac{n-3}{2}$. Using this, we can easily check that $|\mathcal{V}(B,\psi)| < |\mathcal{V}(A,\varphi)_1|$ and $\mathcal{V}(A,\varphi)_2 = \{(a_{k_2+1}, a_{k_2} + n)\}$. Thus, $|\mathcal{V}(B,\psi)| < |\mathcal{V}(A,\varphi)| - 1$.

Assume that $v_b \leq w_{\max} s_{k_1+1}\mu'$. Let (C,χ) be an extended semi-module for μ with type $s_{k_1+1}\mu'$. Similarly as above, we can show that if $l \geq k_1 + 1$, then $|\mathcal{V}(C,\chi)| \leq |\mathcal{V}(A,\varphi)|$. Therefore, $|\mathcal{V}(A,\varphi)| \geq \frac{3n-11}{2}$ holds only if $k_2 = 2$ or $l > k_2 = 3$. From this and $|\mathbb{A}_\mu^{\frac{3n-9}{2}}| = n-3$, we obtain $|\mathbb{A}_\mu^{\frac{3n-11}{2}}| \leq 2(n-4)$. ■

5.4 The semi-module stratification for $\omega_1 + \omega_2, \omega_4 + \omega_{n-1}$

Lemma 5.5 *Assume that $n = 5$. Set $\mu = \omega_1 + \omega_2$. Then every extended semi-module for μ is cyclic. For any $1 \leq j \leq 3 (= \dim X_\mu(b))$, we define $\mathbb{A}_\mu^j$ similarly as in Section 3.3. Then*

$$\mathbb{A}_\mu^0 = \varnothing, \mathbb{A}_\mu^1 = \varnothing, \mathbb{A}_\mu^2 = \{\chi_{1,4}^\vee, \chi_{2,5}^\vee\}, \mathbb{A}_\mu^3 = \{\chi_{2,3}^\vee, \chi_{3,4}^\vee\}.$$

Proof The first assertion follows similarly as the proof of Lemma 5.3. The second assertion follows from direct computation. ■

Lemma 5.6 *Assume that $n = 7$ or 8. Let μ be $\omega_1 + \omega_2$ or $\omega_4 + \omega_{n-1}$. Then there exists a noncyclic extended semi-module for μ.*

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Proof As described in Lemma 5.2, there exists a unique top cyclic extended semi-module (A^λ, φ) for ω_3. We define $\varphi' : \mathbb{Z} \to \mathbb{N} \cup \{-\infty\}$ by setting

$$\varphi'(a) = \begin{cases} \varphi(a) & (a \neq 1) \\ 0 & (a = 1). \end{cases}$$

Then it is straightforward to check that (A^λ, φ') is a noncyclic extended semi-module for $\omega_1 + \omega_2$. The proof for $\omega_4 + \omega_{n-1}$ is similar. ■

6 The Ekedahl–Oort stratification

Keep the notations and assumptions in Section 3. For $\mu \in X_*(T)_+$, set

$$^S\mathrm{Adm}(\mu)_{\mathrm{cyc}} = \{w \in {}^S\mathrm{Adm}(\mu) \mid p(w) \mathrm{is} n\text{-cycle}\}.$$

By Theorem 2.9, $X_w(b) \neq \varnothing$ if $w \in {}^S\mathrm{Adm}(\mu)_{\mathrm{cyc}}$.

6.1 The Ekedahl–Oort stratification for ω_i

Throughout this subsection, we set $\mu = \omega_i$ and $c = s_i s_{i+1} \cdots s_{n-1} s_{i-1} \cdots s_2 s_1$. By [23, Theorem 2.7], we have $\dim X_{\varpi^\mu c}(b) = \dim X_\mu(b) = \langle \mu, \rho \rangle - \frac{n-1}{2}$.

Note that $|W_{\mathrm{supp}_\sigma(w)}|$ is finite if and only if $\mathrm{supp}_\sigma(w) \neq \tilde{S}$. Since τ^m acts transitively on $\tilde{S}$, $\mathrm{supp}_\sigma(w) \neq \tilde{S}$ if and only if $w \in \Omega$.

Lemma 6.1 *Assume that $n \geq 9$ and $4 \leq i \leq n-4$. Set $y = c s_i s_{i+1} s_{i-1} = (1\ i+1\ i+3\ i+4\ \cdots\ n\ i\ i-2\ \cdots\ 3\ 2)(i-1\ i+2)$. Then we have $\varpi^\mu y \in {}^S\mathrm{Adm}(\mu)$ and $X_{\varpi^\mu y}(b) \neq \varnothing$.*

Proof Under the assumption in the lemma, we have $\ell(\varpi^\mu y) = \langle \mu, 2\rho \rangle - \ell(y) (> 0)$ and hence $\varpi^\mu y \in {}^S\mathrm{Adm}(\mu)$ (cf. [31, (2.4.5)]). So, by Lemma 2.8 and Theorem 2.9, $X_{\varpi^\mu y}(b) \neq \varnothing$ is equivalent to saying $\mathrm{supp}(ryr^{-1}) \subsetneq S$ for any $r \in W_0$ such that $r(\Phi_+ \backslash \Phi_{\varpi^\mu y}) \subset \Phi_+$. It is easy to check that

$$\Phi_{\varpi^\mu y} = \Phi_{\{\chi_{1,2}, \chi_{2,3}, \ldots, \chi_{i,i+1}\}} \cup \Phi_{\{\chi_{i,i+1}, \chi_{i+1,i+2}, \ldots, \chi_{n-1,n}\}} \cup \{\chi_{i-2,i+2}, \chi_{i-1,i+2}, \chi_{i-1,i+3}\}.$$

In particular, we have $\chi_{1,i+2}, \chi_{i-1,n} \in \Phi_+ \backslash \Phi_{\varpi^\mu y}$. Note that we can decompose ryr^{-1} into disjoint cycles as

$$(r(1)\ r(i+1)\ r(i+3)\ r(i+4)\ \cdots\ r(n)\ r(i)\ r(i-2)\ \cdots\ r(3)\ r(2))(r(i-1)\ r(i+2)),$$

for any $r \in W_0$. So if $ryr^{-1} \in \bigcup_{J \subsetneq S} W_J$, then $(r(i-1)\ r(i+2)) = (1\ 2)$ or $(n-1\ n)$. This implies that $r\chi_{1,i+2}$ or $r\chi_{i-1,n}$ is negative and hence that r does not satisfy $r(\Phi_+ \backslash \Phi_{\varpi^\mu y}) \subset \Phi_+$. Thus, we have $X_{\varpi^\mu y}(b) \neq \varnothing$. ■

Lemma 6.2 *Assume that $n \geq 9$ and $i = 3$ (resp. $i = n-3$). Set $y = c s_3 s_4 s_5 s_6 s_2$ (resp. $y = c s_{n-3} s_{n-4} s_{n-5} s_{n-6} s_{n-2}$). Then we have $\varpi^\mu y \in {}^S\mathrm{Adm}(\mu)$ and $X_{\varpi^\mu y}(b) \neq \varnothing$.*

Proof We only treat the case $i = 3$. The proof for the case $i = n-3$ is similar.

The first assertion is easy. To show the second assertion, by Lemma 2.8 and Theorem 2.9, it suffices to check that $ryr^{-1} \notin \bigcup_{J \subsetneq S} W_J$ for any $r \in W_0$ such that $r(\Phi_+ \backslash \Phi_{\varpi^\mu y}) \subset \Phi_+$. By an explicit calculation, it follows that $\chi_{1,7}, \chi_{2,9} \in \Phi_+ \backslash \Phi_{\varpi^\mu y}$ and

$$ryr^{-1} = (r(1)\ r(4)\ r(6)\ r(8)\ r(9)\ \cdots\ r(n)\ r(3))(r(2)\ r(5)\ r(7)).$$

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If $ryr^{-1} \in \bigcup_{J \subsetneq S} W_J$, then $(r(2)\ r(5)\ r(7))$ is equal to $(1\ 2\ 3)$ or $(n-2\ n-1\ n)$. This implies that r does not satisfy $r(\Phi_+ \backslash \Phi_{\varpi^\mu y}) \subset \Phi_+$. Thus, we have $X_{\varpi^\mu y}(b) \neq \emptyset$. ■

Lemma 6.3 *Assume that $n \geq 9$ and $i = 3$ (resp. $i = n-3$). Let y be cs_is_{i-1} or cs_is_{i+1}. Then we have $\varpi^\mu y \in {}^S\mathrm{Adm}(\mu)$ and $X_{\varpi^\mu y}(b) \neq \emptyset$.*

Proof The proof is similar to the proof of Lemmas 6.1 and 6.2. Note that y is a n-cycle in this case. ■

Proposition 6.4 *Assume that $n \geq 9$ and $3 \leq i \leq n-3$. Then the semi-module stratification of $X_\mu(b)$ is not a refinement of the Ekedahl–Oort stratification.*

Proof First assume that $n \geq 9$ and $4 \leq i \leq n-4$. Let $\varpi^\mu y \in {}^S\tilde{W}$ be as in Lemma 6.1. Let $\mathcal{T}$ be a reduction tree of $\varpi^\mu y$. By Proposition 2.6, we have

$$|X_{\varpi^\mu y}(b)^{0,\sigma}| = \sum_{\underline{p}} (q-1)^{\ell_I(\underline{p})} q^{\ell_{II}(\underline{p})},$$

where $\underline{p}$ runs over all the reduction paths in $\mathcal{T}$ with $\mathrm{end}(\underline{p}) = \tau^m$. Set $d = \dim X_\mu(b) = \langle \mu, \rho \rangle - \frac{n-1}{2}$. Suppose that the semi-module stratification of $X_\mu(b)$ is a refinement of the Ekedahl–Oort stratification. Note that $Z(\varpi^\mu c) = Z(\varpi^\mu y) = \{1\}$. By Lemma 2.1, Proposition 2.3 and $\dim X_{\varpi^\mu c}(b) = d$, we have $\ell_I(\underline{p}) + \ell_{II}(\underline{p}) \leq \dim X_{\varpi^\mu y}(b) \leq d-1$ for any $\underline{p}$. On the other hand, we have $\ell_I(\underline{p}) + 2\ell_{II}(\underline{p}) = \ell(\varpi^\mu y) = 2d-3$. Thus, we have $\ell_I(\underline{p}) + \ell_{II}(\underline{p}) = d-1$ and $\ell_I(\underline{p}) = 1$ for any $\underline{p}$. It follows that

$$|\pi(X_{\varpi^\mu y}(b)^0)^\sigma| = |X_{\varpi^\mu y}(b)^{0,\sigma}| = k(q-1)q^{d-2},$$

where $k \geq 1$ is the number of irreducible components of $X_{\varpi^\mu y}(b)^0$. Again by Lemma 2.1 and the fact that each $S_{A,\varphi}$ is locally closed, we have $|\{(A,\varphi) \mid \dim S_{A,\varphi} = d-1, S_{A,\varphi} \subseteq \pi(X_{\varpi^\mu y}(b)^0)\}| = k$. By Lemma 3.4, it follows that $|\pi(X_{\varpi^\mu y}(b)^0)^\sigma| \geq kq^{d-1}$, which is a contradiction. This implies the proposition in this case.

Next assume that $n \geq 10$ and $i = 3, n-3$. Let $\varpi^\mu y \in {}^S\tilde{W}$ be as in Lemma 6.2. Suppose that the semi-module stratification of $X_\mu(b)$ is a refinement of the Ekedahl–Oort stratification. Similarly as above, we can check that

$$\dim X_{cs_is_{i-1}}(b) = X_{cs_is_{i+1}}(b) = d-1.$$

Note that $Z(\varpi^\mu c) = Z(\varpi^\mu cs_is_{i-1}) = Z(\varpi^\mu cs_is_{i+1}) = Z(\varpi^\mu y) = \{1\}$. By Lemma 2.1 and Proposition 3.13, we have $\dim X_{\varpi^\mu y}(b) \leq d-2$. Similarly as above, it follows that $|\pi(X_{\varpi^\mu y}(b)^0)^\sigma| = k(q-1)q^{d-3}$ and $|\pi(X_{\varpi^\mu y}(b)^0)^\sigma| \geq kq^{d-2}$. This is a contradiction, which finishes the proof. ■

The following proposition is the complement of Proposition 6.4.

Proposition 6.5 *We have*

$$\begin{aligned}
{}^S\mathrm{Adm}(\omega_1)_{\mathrm{cyc}} &= \{\tau\}, \\
{}^S\mathrm{Adm}(\omega_2)_{\mathrm{cyc}} &= \{\tau^2, s_0s_{n-1}\tau^2, s_0s_{n-1}s_{n-2}s_{n-3}\tau^2, \ldots, s_0s_{n-1}\cdots s_5s_4\tau^2\} && (n \geq 5), \\
{}^S\mathrm{Adm}(\omega_3)_{\mathrm{cyc}} &= \{\tau^3, s_0s_6\tau^3, s_0s_6s_1s_0\tau^3, s_0s_6s_5s_1\tau^3, s_0s_6s_5s_1s_0s_6\tau^3\} && (n = 7), \\
{}^S\mathrm{Adm}(\omega_3)_{\mathrm{cyc}} &= \{\tau^3, s_0s_1\tau^3, s_0s_7s_6s_5\tau^3, s_0s_7s_6s_1\tau^3, s_0s_7s_6s_5s_1s_0\tau^3, \\
&\qquad s_0s_7s_6s_1s_0s_7\tau^3, s_0s_7s_6s_5s_1s_0s_7s_6\tau^3\} && (n = 8).
\end{aligned}$$

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Let $\varpi^\mu y \in {}^S\tilde{W}$ be one of the elements above. Then there exists $v \in \mathrm{LP}(\varpi^\mu y)$ such that $v^{-1}yv$ is a Coxeter element. Moreover, $X_w(b) = \varnothing$ for any $w \in {}^S\mathrm{Adm}(\mu)\backslash{}^S\mathrm{Adm}(\mu)_{\mathrm{cyc}}$, and the semi-module stratification of $X_\mu(b)$ is a refinement of the Ekedahl–Oort stratification.

Proof The equalities in the proposition follow from easy calculations. For other statements, we only prove the case for ω_2. Other cases can be checked similarly.

Set $d = \frac{n-3}{2}$. For $0 \le j \le d$, we set $w_j = s_0 s_{n-1} \cdots s_{n-2j+1}\tau^2$. Then $\ell(w_j) = 2j$ and

$$p(w_j) = (1\,3\,5\,\cdots\,n-2j\;n-2j+1\,\cdots\,n\;2\;4\,\cdots\,n-2j-1).$$

Also it is easy to check that

$$\Phi_+\backslash\Phi_{w_j} = \{\chi_{1,n-2j+1}, \dots, \chi_{1,n-1}, \chi_{1,n}\}.$$

Clearly there exists $r \in W_0$ with $r(\Phi_+\backslash\Phi_{w_j}) \subset \Phi_+$ such that $rp(w_j)r^{-1}$ is a Coxeter element (cf. [40, Lemma 5.1]).

For an integer *j*, let $0 \le [j] < n$ denote its residue modulo n. For $a, b \in \mathbb{N}$ with $a - b \in 2\mathbb{Z}$, we define $t_{a,b} = s_{[b-2]} \cdots s_{[a+2]} s_{[a]}$. Set

$$w_{j,0} = w_j,\ w_{j,1} = t_{0,n-2j+1} w_j t_{0,n-2j+1}^{-1},\ w_{j,2} = t_{n-1,n-2j+2} w_{j,1} t_{n-1,n-2j+2}^{-1}, \dots, w_{j,j} = t_{n-j+1,n-j} w_{j,j-1} t_{n-j+1,n-j}^{-1}.$$

It is easy to check that the simple reflections in $t_{0,n-2j+1}, t_{n-1,n-2j+2}, \dots, t_{n-j+1,n-j}$ define

$$w_j = w_{j,0} \to_\sigma w_{j,1} = s_{n-1}s_{n-2}\cdots s_{n-2j+2}\tau^2 \to_\sigma w_{j,2} = s_{n-2}s_{n-3}\cdots s_{n-2j+3}\tau^2 \to_\sigma \cdots \to_\sigma w_{j,j} = \tau^2.$$

Let $\underline{p}_j$ be the reduction path (in a suitable reduction tree) defined by this reduction. Using Lemma 2.1, Propositions 2.5, 2.6 and 3.13, we can check that $X_{w_j}(\tau^2) = X_{\underline{p}_j}$ and $X_w(\tau^2) = \varnothing$ for any $w \in {}^S\mathrm{Adm}(\omega_2)\backslash{}^S\mathrm{Adm}(\omega_2)_{\mathrm{cyc}}$ by counting the number of rational points of $X_\mu(\tau^2)^0$ (note that $X_{\tau^2}(\tau^2)^0 = \{I\}$). It is easy to check that

$$\ell(t_{n-j+1,n-j}\cdots t_{n-1,n-2j+2}t_{0,n-2j+1}) = \ell(t_{n-j+1,n-j}) + \cdots + \ell(t_{n-1,n-2j+2}) + \ell(t_{0,n-2j+1}).$$

Thus by Proposition 2.3 (cf. [39, Section 3.3]), each element gI in $X_{w_j}(\tau^2)^0$ is contained in a Schubert cell associated to $t_{n-j+1,n-j}\cdots t_{n-1,n-2j+2}t_{0,n-2j+1}$. By Lemma 5.1, it follows that $\pi(X_{w_j}(b)^0)$ is equal to the unique semi-module stratum of dimension *j*. This shows that the semi-module stratification of $X_\mu(b)$ is a refinement of the Ekedahl–Oort stratification. ■

6.2 The Ekedahl–Oort stratification for $\omega_1 + \omega_{n-2}$

Throughout this subsection, we set $\mu = \omega_1 + \omega_{n-2}$. Also we assume that $n \ge 4$. Note that the unique dominant cocharacter μ' with $\mu' \prec \mu$ is $\mu' = \omega_{n-1}$. Clearly, we have ${}^S\mathrm{Adm}(\omega_{n-1})_{\mathrm{cyc}} = \{\tau^{n-1}\}$ and the semi-module stratification of $X_{\omega_{n-1}}(\tau^{n-1})$ is a refinement of the Ekedahl–Oort stratification.

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Proposition 6.6 *For any $1 \le j \le n-2(=\dim X_\mu(b))$, there exist exactly j elements of length $2j$ in ${}^S\mathrm{Adm}(\mu)^\circ_{\mathrm{cyc}} := {}^S\mathrm{Adm}(\mu)_{\mathrm{cyc}} \backslash \{\tau^{n-1}\}$. Let $\varpi^\mu y \in {}^S\tilde{W}$ be one of such elements. Then there exists $v \in \mathrm{LP}(\varpi^\mu y)$ such that $v^{-1}yv$ is a Coxeter element. Moreover, $X_w(b) = \emptyset$ for any $w \in {}^S\mathrm{Adm}(\mu) \backslash {}^S\mathrm{Adm}(\mu)_{\mathrm{cyc}}$, and the semi-module stratification of $X_\mu(b)$ is a refinement of the Ekedahl–Oort stratification.*

Proof We first prove by induction on n that there exist at least j elements of length $2j$ in ${}^S\mathrm{Adm}(\mu)^\circ_{\mathrm{cyc}}$, each of which has finite part y such that ryr^{-1} is a Coxeter element for some $r \in W_{\{s_2,\dots,s_{n-2}\}}$ satisfying $r(\Phi_+ \backslash \Phi_{\varpi^\mu y}) \subset \Phi_+$ (cf. Lemma 2.8). Note that if $y \in W_0$ satisfies

$$(*) \qquad y^{-1}(2) < y^{-1}(3) < \cdots < y^{-1}(n-2) \text{and} y^{-1}(n-1) < y^{-1}(n),$$

then by [39, Lemma 4.4], we have $\varpi^\mu y \in {}^S\mathrm{Adm}(\mu)$. In particular, since $\ell(\varpi^\mu) = 3n-5$, $\varpi^\mu y$ is an element of length $2j$ in ${}^S\mathrm{Adm}(\mu)^\circ_{\mathrm{cyc}}$ for any n-cycle y of length $3n-2j-5$. If $n=4$, then $s_1s_2s_3, s_2s_3s_1$ and $s_1s_2s_3s_1s_2$ are 4-cycles satisfying $(*)$. Moreover, $s_2(s_1s_2s_3s_1s_2)s_2 = s_1s_2s_3$ is a Coxeter element and $s_2(\Phi_+ \backslash \Phi_{\varpi^\mu s_1s_2s_3s_1s_2}) \subset \Phi_+$. So the claim is true for $n=4$.

Suppose that $n \ge 5$ and the claim is true for $n-1$. Let y be a $(n-1)$-cycle in $W_{\{s_1,s_2,\dots,s_{n-2}\}}$ such that $y^{-1}(2) < y^{-1}(3) < \cdots < y^{-1}(n-3)$ and $y^{-1}(n-2) < y^{-1}(n-1)$. Then $y' := s_1(1\ 2\ \cdots\ n)y(1\ 2\ \cdots\ n)^{-1}$ satisfies $(*)$ and $\ell(y') = \ell(y)+1$. So by the induction hypothesis, there exist at least $j-1$ elements in W_0 which are n-cycles of length $3n-2j-5$ satisfying $(*)$. Note that for any $r \in W_{\{s_2,\dots,s_{n-3}\}}$, we have $r'y'r'^{-1} = s_1(1\ 2\ \cdots\ n)ryr^{-1}(1\ 2\ \cdots\ n)^{-1}$, where $r' = (1\ 2\ \cdots\ n)r(1\ 2\ \cdots\ n)^{-1} \in W_{\{s_2,\dots,s_{n-2}\}}$. So again by the induction hypothesis, it is easy to verify that there exists $r \in W_{\{s_2,\dots,s_{n-3}\}}$ such that $r'y'r'^{-1}$ is a Coxeter element and $r'(\Phi_+ \backslash \Phi_{\varpi^\mu y'}) \subset \Phi_+$. Set $c = s_{n-2}s_{n-1}s_{n-3}\cdots s_2s_1$. It is easy to check that if n is odd (resp. even), then

$$c,\ cs_{n-2}s_{n-3},\ \dots,\ cs_{n-2}s_{n-3}\cdots s_2,\ cs_{n-2}s_{n-3}\cdots s_2s_3s_4,\ \dots,\ cs_{n-2}s_{n-3}\cdots s_2s_3s_4\cdots s_{n-2}s_{n-1}$$

$$(\text{resp. } c,\ cs_{n-2}s_{n-3},\ \dots,\ cs_{n-2}s_{n-3}\cdots s_3,\ cs_{n-2}s_{n-3}\cdots s_3s_2s_3,\ \dots,\ cs_{n-2}s_{n-3}\cdots s_3s_2s_3\cdots s_{n-2}s_{n-1}),$$

are n-cycles satisfying $(*)$. If y' is one of the elements above, then $\Phi_{\{\chi_{2,3},\dots,\chi_{n-2,n-1}\}} \cap \Phi_+ \subset \Phi_{\varpi^\mu y'}$ and there exists $r' \in W_{\{s_2,\dots,s_{n-2}\}}$ such that $r'y'r'^{-1}$ is a Coxeter element. Thus, the claim is also true for n. By induction, our claim is true for any $n \ge 4$.

Clearly $\nu_w = \nu_b$ for any $w \in {}^S\mathrm{Adm}(\mu)^\circ_{\mathrm{cyc}}$. Since $b = \tau^{n-1}$ is superbasic, the unique minimal length element in the σ-cojugacy class of w is τ^{n-1} (cf. [22, Proposition 3.5]). By Theorem 2.4, there exist a reduction tree $\mathcal{T}$ for w and a reduction path $\underline{p}$ in $\mathcal{T}$ such that $\mathrm{end}(\underline{p}) = \tau^{n-1}$ and $\ell_I(\underline{p}) = 0$. Thus by Lemma 2.1 and Proposition 2.6, $|\pi(X_w(b)^{0,\sigma})| \ge q^{\frac{\ell(w)}{2}}$ for any $w \in {}^S\mathrm{Adm}(\mu)^\circ_{\mathrm{cyc}}$. By the comparison of $|\sqcup_{w \in {}^S\mathrm{Adm}(\mu)^\circ_{\mathrm{cyc}}} \pi(X_w(b)^{0,\sigma})|$ and $|X_\mu(b)^{0,\sigma}|$, it follows from Lemma 5.3 and the claim we have shown above that there exist exactly j elements of length $2j$ in ${}^S\mathrm{Adm}(\mu)^\circ_{\mathrm{cyc}}$. Moreover, it follows that $\pi(X_w(b)^0)$ is irreducible of dimension $\frac{\ell(w)}{2}$ for any $w \in {}^S\mathrm{Adm}(\mu)^\circ_{\mathrm{cyc}}$ and that $X_w(b) = \emptyset$ for any $w \in {}^S\mathrm{Adm}(\mu) \backslash {}^S\mathrm{Adm}(\mu)_{\mathrm{cyc}}$.

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It remains to show that the semi-module stratification of $X_\mu(b)$ is a refinement of the Ekedahl–Oort stratification. We prove that for any $w \in {}^S\mathrm{Adm}(\mu)^\circ_{\mathrm{cyc}}$, there exists an extended semi-module (A^λ, φ) for μ such that $\pi(X_w(b)^0) = S_{A^\lambda,\varphi} (= X^\lambda_\mu(b)$ by Lemmas 3.9 and 5.3). We argue by induction on $\ell(w)$. If $\ell(w) = 2$, i.e., $w = \varpi^\mu c s_{n-2}s_{n-3}\cdots s_2 s_3 s_4 \cdots s_{n-2}s_{n-1} = s_0 s_{n-1}\tau^{n-1}$, then $w \to_\sigma s_0 w s_0 = \tau^{n-1}$. It easily follows from Theorem 2.9 that $X_{\tau^{n-1}s_0}(b) = \varnothing$. So by Proposition 2.3, we have $X_w(b)^0 = I s_0 I/I$ and hence $\pi(X_w(b)^0) = X_\mu^{\chi^\vee_{1,n}}(b)$.

Suppose that $\ell(w) \geq 4$ and the claim is true for any $w' \in {}^S\mathrm{Adm}(\mu)^\circ_{\mathrm{cyc}}$ with $\ell(w') < \ell(w)$. Since $\pi(X_w(b)^0)$ is irreducible of dimension $\frac{\ell(w)}{2}$, there exists a unique extended semi-module (A^λ, φ) for μ such that $\dim(\pi(X_w(b)^0) \cap S_{A^\lambda,\varphi}) = \frac{\ell(w)}{2}$. Also, $\pi(X_w(b)^0) \cap S_{A^\lambda,\varphi}$ is open in both $\pi(X_w(b)^0)$ and $S_{A^\lambda,\varphi}$. So the closure of $\pi(X_w(b)^0) \cap S_{A^\lambda,\varphi}$ in $X_\mu(b)$ is equal to both the closure of $\pi(X_w(b)^0)$ and $S_{A^\lambda,\varphi}$ in $X_\mu(b)$. By [20, Proposition 2.6] (see also [11, Section 3.3]), the closure of $\pi(X_w(b)^0)$ is contained in

$$\bigsqcup_{w' \in {}^S\mathrm{Adm}(\mu)^\circ_{\mathrm{cyc}},\, w' \leq_S w} \pi(X_{w'}(b)).$$

Here we write $w' \leq_S w$ if there exists $x \in W_0$ such that $xw'x^{-1} \leq w$. By the above description of the finite part of each element in ${}^S\mathrm{Adm}(\mu)^\circ_{\mathrm{cyc}}$, it is easily checked that if $w' \in {}^S\mathrm{Adm}(\mu)^\circ_{\mathrm{cyc}}$ and $\ell(w) = \ell(w')$, then there is no $x \in W_0$ such that $xwx^{-1} = w'$. So if $w' \in {}^S\mathrm{Adm}(\mu)^\circ_{\mathrm{cyc}}$, $w' \leq_S w$ and $\ell(w') = \ell(w)$, then $w = w'$. Thus, by the induction hypothesis, we have $S_{A^\lambda,\varphi} \subseteq \pi(X_w(b)^0)$. By [2, Propositions 2.11(5) and 3.4], the closure of $S_{A^\lambda,\varphi}$ is contained in a union of semi-module strata T_λ such that $\dim(T_\lambda \backslash S_{A^\lambda,\varphi}) < \dim S_{A^\lambda,\varphi}$. Thus, by the induction hypothesis and Lemma 5.3, we have $\pi(X_w(b)^0) \subseteq S_{A^\lambda,\varphi}$. Therefore, it follows that $\pi(X_w(b)^0) = S_{A^\lambda,\varphi}$, which completes the proof. ∎

6.3 The Ekedahl–Oort stratification for $\omega_1 + \omega_{n-3}$

Throughout this subsection, we set $\mu = \omega_1 + \omega_{n-3}$. Also we assume that $n \geq 7$. Note that the unique dominant cocharacter μ' with $\mu' \prec \mu$ is $\mu' = \omega_{n-2}$.

Proposition 6.7 *There exist at least $2(n-4)$ elements of length $3n-11$ in ${}^S\mathrm{Adm}(\mu)^\circ_{\mathrm{cyc}} := {}^S\mathrm{Adm}(\mu)_{\mathrm{cyc}} \backslash {}^S\mathrm{Adm}(\omega_{n-2})_{\mathrm{cyc}}$. There also exists an element w of length $3n-14$ in ${}^S\mathrm{Adm}(\mu)$ such that $p(w)$ is not a n-cycle and $X_w(b) \neq \varnothing$. Moreover, the semi-module stratification of $X_\mu(b)$ is not a refinement of the Ekedahl–Oort stratification.*

Proof For any $1 \leq j \leq n-4$, set $c_j = s_{n-3}s_{n-2}s_{n-1}s_{n-4}\cdots s_{j+2}s_{j+1}s_1\cdots s_{j-1}s_j$. For $j = n-3$, set $c_{n-3} = s_1 s_2 \cdots s_{n-1}$. Then we have $\varpi^\mu c_j \in {}^S\mathrm{Adm}(\mu)^\circ_{\mathrm{cyc}}$ and $\ell(\varpi^\mu c_j) = 3n-9$ for any $1 \leq j \leq n-3$. If $1 \leq j \leq n-5$, then $c_j s_{n-3}s_{n-2}$ and $c_j s_{n-3}s_{n-4}$ are n-cycles of length $3n-11$ satisfying $\varpi^\mu c_j s_{n-3}s_{n-2}, \varpi^\mu c_j s_{n-3}s_{n-4} \in {}^S\mathrm{Adm}(\mu)^\circ_{\mathrm{cyc}}$. Further $c_{n-4}s_{n-3}s_{n-2}$ and $c_{n-3}s_{n-4}s_{n-3}$ are also n-cycles of length $3n-11$ satisfying $\varpi^\mu c_{n-4}s_{n-3}s_{n-2}, \varpi^\mu c_{n-3}s_{n-4}s_{n-3} \in {}^S\mathrm{Adm}(\mu)^\circ_{\mathrm{cyc}}$. Thus, we have found $2(n-4)$ distinct elements of length $3n-11$ in ${}^S\mathrm{Adm}(\mu)^\circ_{\mathrm{cyc}}$.

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Set $y = c_{n-5}s_{n-3}s_{n-2}s_{n-4}s_{n-6}s_{n-5} = (1\,2\,\cdots\,n-6\;n-2\;n\;n-3)(n-4\;n-5\;n-1)$. Then $\varpi^\mu y \in {}^S\mathrm{Adm}(\mu)$ and $\chi_{1,n-1}, \chi_{n-5,n} \in \Phi_+ \backslash \Phi_{\varpi^\mu y}$. By Theorem 2.9, $X_{\varpi^\mu y}(b) \neq \emptyset$. This shows the second assertion. We can easily check the last assertion using Lemma 5.4, similarly as the proof of Proposition 6.4. ■

6.4 The Ekedahl–Oort stratification for $\omega_1 + \omega_2, \omega_4 + \omega_{n-1}$

Note that the unique dominant cocharacter μ' with $\mu' < \omega_1 + \omega_2$ is ω_3. By an explicit calculation, it is easy to verify the following statements (cf. Proposition 6.5).

Proposition 6.8 *Assume that $n = 5$. Set $\mu = \omega_1 + \omega_2$. For any $1 \leq j \leq 3 (= \dim X_\mu(b))$, set ${}^S\mathrm{Adm}(\mu)^\circ_{\mathrm{cyc}} := {}^S\mathrm{Adm}(\mu)_{\mathrm{cyc}} \backslash {}^S\mathrm{Adm}(\omega_3)_{\mathrm{cyc}}$. Then we have*

$$ {}^S\mathrm{Adm}(\mu)^\circ_{\mathrm{cyc}} = \{s_0s_4s_3s_2s_1s_0\tau^3, s_0s_1s_4s_3s_0s_4\tau^3, s_0s_4s_3s_2\tau^3, s_0s_1s_4s_3\tau^3\}. $$

Let $\varpi^\mu y \in {}^S\mathrm{Adm}(\mu)^\circ_{\mathrm{cyc}}$. Then there exists $v \in \mathrm{LP}(\varpi^\mu y)$ such that $v^{-1}yv$ is a Coxeter element. Moreover, $X_w(b) = \emptyset$ for any $w \in {}^S\mathrm{Adm}(\mu) \backslash {}^S\mathrm{Adm}(\mu)_{\mathrm{cyc}}$, and the semi-module stratification of $X_\mu(b)$ is a refinement of the Ekedahl–Oort stratification.

Lemma 6.9 *Assume that $n = 7$ or 8. Let μ be $\omega_1 + \omega_2$ (resp. $\omega_4 + \omega_{n-1}$). Set $c = s_1s_2\cdots s_{n-1}$. Then $\varpi^\mu cs_1s_2s_3 \in {}^S\mathrm{Adm}(\mu)$ and $X_{\varpi^\mu cs_1s_2s_3}(b) \neq \emptyset$ (resp. $\varpi^\mu c^{-1}s_5s_4s_3 \in {}^S\mathrm{Adm}(\mu)$ and $X_{\varpi^\mu c^{-1}s_5s_4s_3}(b) \neq \emptyset$). Further $cs_1s_2s_3$ (resp. $c^{-1}s_5s_4s_3$) is not n-cycle.*

6.5 The Ekedahl–Oort stratification for $\omega_2 + \omega_{n-3}$

We set $\mu = \omega_2 + \omega_{n-3}$. Also we assume that $n \geq 5$.

Lemma 6.10 *If n is odd (resp. even), set $y = s_2s_3\cdots s_{n-3}s_1s_2\cdots s_{n-3}$ (resp. $y = s_2s_3\cdots s_{n-3}s_1s_2\cdots s_{n-2}$). Then $\varpi^\mu y \in {}^S\mathrm{Adm}(\mu), X_{\varpi^\mu y}(b) \neq \emptyset$ and y is not a n-cycle.*

Proof If n is odd (resp. even), then $y = (1\;3\;\cdots\;n-2)(2\;4\;\cdots\;n-1\;n)$ (resp. $(1\;3\;\cdots\;n-1)(2\;4\;\cdots\;n)$) and $\varpi^\mu y \in {}^S\mathrm{Adm}(\mu)$. Note that $\chi_{1,n}, \chi_{2,n-1} \in \Phi_+ \backslash \Phi_{\varpi^\mu y}$. So by Lemma 2.9, $X_{\varpi^\mu y}(b) \neq \emptyset$. The proof is finished. ■

7 Comparison of two stratifications

Keep the notations and assumptions in Section 3 .

7.1 Known cases

The following results are known in (the proof of) [39, Corollary 5.5 and Theorem 5.9].

Proposition 7.1 *Let $\cong$ denote a universal homeomorphism.*

(i) *Assume that $n \geq 3$. Set $\mu = 2\omega_1, w = \varpi^\mu s_1s_2\cdots s_{n-1}$ and*

$$ \lambda = \begin{cases} \chi^\vee_{2,n-1} + \chi^\vee_{4,n-3} + \cdots + \chi^\vee_{\frac{n-1}{2},\frac{n+3}{2}} & (\frac{n-1}{2} \text{ even}) \\ \chi^\vee_{1,n} + \chi^\vee_{3,n-2} + \cdots + \chi^\vee_{\frac{n-1}{2},\frac{n+3}{2}} & (\frac{n-1}{2} \text{ odd}). \end{cases} $$

Then we have $X_\mu(b)^0 = X^\lambda_\mu(b) = \pi(X_w(b)^0) \cong \mathbb{A}^{\frac{n-1}{2}}$.

(ii) *Assume that* $n \geq 3$. *Set* $\mu = 2\omega_1 + \omega_{n-1}$, $w_j = \varpi^{\mu} s_{n-1}s_{n-2}\cdots s_{n-j+1}s_1s_2\cdots s_{n-j}$ *and*

$$\lambda_j = \begin{cases} \chi^{\vee}_{1,2j} + \chi^{\vee}_{2,2j-1} + \cdots + \chi^{\vee}_{j,j+1} & (j \leq \frac{n}{2}) \\ \chi^{\vee}_{2j+1-n,n} + \chi^{\vee}_{2j+2-n,n-1} + \cdots + \chi^{\vee}_{j,j+1} & (j \geq \frac{n}{2}). \end{cases}$$

for $j = 1, 2, \ldots, n-1$. *Then we have* $X_\mu(b)^0 = \bigsqcup_{1\leq j\leq n-1} X^{\lambda_j}_{\mu}(b)$ *and* $X^{\lambda_j}_{\mu}(b) = \pi(X_{w_j}(b)^0) \cong \mathbb{A}^{n-1}$ *for each j.*

(iii) *Assume that* $n = 5$. *Set* $\mu = 3\omega_1$, $w = \varpi^{\mu}s_1s_2s_3s_4$ *and* $\lambda = \chi^{\vee}_{1,2} + \chi^{\vee}_{3,4}$. *Then we have* $X_\mu(b)^0 = X^{\lambda}_{\mu}(b) = \pi(X_w(b)^0) \cong \mathbb{A}^4$.

(iv) *Assume that* $n = 4$. *Set* $\mu = 3\omega_1$, $w = \varpi^{\mu}s_1s_2s_3$ *and* $\lambda = \chi^{\vee}_{3,2}$. *Then we have* $X_\mu(b)^0 = X^{\lambda}_{\mu}(b) = \pi(X_w(b)^0) \cong \mathbb{A}^3$.

(v) *Assume that* $n = 3$. *Set* $\mu = 4\omega_1$, $w = \varpi^{\mu}s_1s_2$ *and* $\lambda = \chi^{\vee}_{3,1}$. *Then we have* $X_\mu(b)^0 = X^{\lambda}_{\mu}(b) = \pi(X_w(b)^0) \cong \mathbb{A}^3$.

(vi) *Assume that* $n = 3$. *Set* $\mu = 3\omega_1 + \omega_2$, $w_1 = \varpi^{\mu}s_1s_2$, $w_2 = \varpi^{\mu}s_2s_1$, $\lambda_1 = \chi^{\vee}_{2,3}$ *and* $\lambda_2 = \chi^{\vee}_{3,2}$. *Then we have* $X_\mu(b)^0 = X^{\lambda_1}_{\mu}(b) \sqcup X^{\lambda_2}_{\mu}(b)$ *and* $X^{\lambda_j}_{\mu}(b) = \pi(X_{w_j}(b)^0) \cong \mathbb{A}^3$ *for each j.*

(vii) *Assume that* $n = 2$. *Set* $\mu = m\omega_1$ *with* $m \geq 1$, $w = \varpi^{\mu}s_1$ *and*

$$\lambda = \begin{cases} \frac{m-1}{2}\chi^{\vee}_{1,2} & (\frac{m-1}{2} \text{ odd}) \\ \frac{m-1}{2}\chi^{\vee}_{2,1} & (\frac{m-1}{2} \text{ even}). \end{cases}$$

Then we have $X_\mu(b)^0 = X^{\lambda}_{\mu}(b) = \pi(X_w(b)^0) \cong \mathbb{A}^{\frac{m-1}{2}}$.

7.2 Proof of the main theorem

Theorem 7.2 *Let* $\mu \in X_*(T)_+$. *The following assertions are equivalent.*

(i) *The semi-module stratification of* $X_{\leq\mu}(b)$ *gives a refinement of the Ekedahl–Oort stratification.*

(ii) *For any* $w \in {}^S\mathrm{Adm}(\mu)$ *with* $X_w(b) \neq \varnothing$, *there exists* $v \in \mathrm{LP}(w)$ *such that* $v^{-1}p(w)v$ *is a Coxeter element.*

(iii) *The cocharacter* μ *has one of the following forms:*

$$\begin{array}{lr} \omega_1, \quad \omega_{n-1}, & (n \geq 1), \\ \omega_2, \quad 2\omega_1, \quad \omega_{n-2}, \quad 2\omega_{n-1}, & (\text{odd } n \geq 3), \\ \omega_2 + \omega_{n-1}, \quad 2\omega_1 + \omega_{n-1} \quad \omega_1 + \omega_{n-2}, \quad \omega_1 + 2\omega_{n-1}, & (n \geq 3), \\ \omega_3, \quad \omega_{n-3}, & (n = 7, 8), \\ 3\omega_1, \quad 3\omega_{n-1}, & (n = 4, 5), \\ \omega_1 + \omega_2, \quad \omega_3 + \omega_4, & (n = 5), \\ 4\omega_1, \quad \omega_1 + 3\omega_2, \quad 4\omega_2, \quad 3\omega_1 + \omega_2, & (n = 3), \\ m\omega_1 \text{ with } m \text{ odd}, & (n = 2). \end{array}$$

If one of the above conditions holds, then for any $w \in {}^S\mathrm{Adm}(\mu)_{\mathrm{cyc}}$, *there exist* $\mu' \in X_*(T)_+$ *with* $\mu' \leq \mu$ *and a cyclic extended semi-module* (A^{λ}, φ) *for* μ' *such that* $\pi(X_w(b)^0) = X^{\lambda}_{\leq\mu}(b) = S_{A^{\lambda},\varphi}$. *Moreover,* $\pi(X_w(b)^0) \cong \mathbb{A}^{\mathcal{V}(A^{\lambda},\varphi)}$.

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Proof For any $w = \varpi^{\mu} y \in {}^S\tilde{W}$ with μ dominant, set $w^* = \varpi^{(\mu(1),\dots,\mu(1))}\varsigma(w)$ (cf. Section 2.5 and Section 3.2). Then $w^* \in {}^S\tilde{W}$ and $p(w^*) = w_{\max} y w_{\max}^{-1}$ (cf. Section 2.5 and Section 3.2). Note that the arguments and results in Section 5 and Section 6 for (μ, w, b) also hold for (μ^*, w^*, b^*). Thus, in this proof, it suffices to treat the case for either μ or μ^*.

First assume that $n \geq 6$. Let $1 \leq m_0 < n$ be the residue of m modulo n. If $4 \leq m_0 \leq n-4$, then $\omega_{m_0} + \lfloor \frac{m}{n} \rfloor \omega_n \leq \mu$. So by Lemma 6.1 and Proposition 6.4, μ satisfies neither (i) nor (ii). If $n \geq 10$ and $m_0 = 3$, then by Lemma 6.2, μ satisfies neither (i) nor (ii). If $n = 7, 8$ and $m_0 = 3$, then by Proposition 6.5, $\mu = \omega_3$ satisfies (i) and (ii). If, moreover, $\mu \neq \omega_3$, then $\omega_1 + \omega_2 + \lfloor \frac{m}{n} \rfloor \omega_n \leq \mu$ or $\omega_4 + \omega_{n-1} + (\lfloor \frac{m}{n} \rfloor - 1)\omega_n \leq \mu$. So by Lemma 5.6 and Lemma 6.9, μ satisfies neither (i) nor (ii). If $m_0 = n-2$, then $\omega_1 + \omega_{n-3} + \lfloor \frac{m}{n} \rfloor \omega_n \leq \mu$ unless $\mu = \omega_{n-2}$ or $2\omega_{n-1}$. If $m_0 = n-1$, then $\omega_2 + \omega_{n-3} + \lfloor \frac{m}{n} \rfloor \omega_n \leq \mu$ unless $\mu = \omega_{n-1}, \omega_1 + \omega_{n-2}$ or $\omega_1 + 2\omega_{n-1}$. Thus, the equivalence of (i), (ii) and (iii) for $m_0 = n-2, n-1$ follows from Theorem 4.17, Proposition 6.5, Proposition 6.7, Proposition 6.10 and Proposition 7.1.

Assume that $n = 5$. If $m_0 = 3$, then $\omega_1 + \omega_3 + \omega_4 + \lfloor \frac{m}{n} \rfloor \omega_n \leq \mu$ unless $\mu = \omega_3, 2\omega_4, \omega_1 + \omega_2$ or $3\omega_1$. If $m_0 = 4$, then $2\omega_2 + \lfloor \frac{m}{n} \rfloor \omega_n \leq \mu$ unless $\mu = \omega_4, \omega_1 + \omega_3$ or $\omega_1 + 2\omega_4$. Set $y_5 = (1\,5\,3)(2\,4)$. Then it is easy to check that $\varpi^{\omega_1+\omega_3+\omega_4} y_5 \in {}^S\mathrm{Adm}(\omega_1 + \omega_3 + \omega_4)$ and $X_{\varpi^{\omega_1+\omega_3+\omega_4} y_5}(\tau^8) \neq \emptyset$. Assume that $n = 4$. If $m_0 = 3$, then $2\omega_2 + \omega_3 + \lfloor \frac{m}{n} \rfloor \omega_n \leq \mu$ unless $\mu = \omega_3, \omega_1 + \omega_2, \omega_1 + 2\omega_3$ or $3\omega_1$. Set $y_4 = (1\,3)(2\,4)$. Then it is easy to check that $\varpi^{2\omega_2+\omega_3} y_4 \in {}^S\mathrm{Adm}(2\omega_2 + \omega_3)$ and $X_{\varpi^{2\omega_2+\omega_3} y_4}(\tau^7) \neq \emptyset$. Assume that $n = 3$. If $m_0 = 2$, then $2\omega_1 + 3\omega_2 + \lfloor \frac{m}{n} \rfloor \omega_n \leq \mu$ unless $\mu = \omega_2, 2\omega_1, \omega_1 + 2\omega_2, 3\omega_1 + \omega_2$ or $4\omega_2$. Set $y_3 = (1\,3)$. Then it is easy to check that $\varpi^{2\omega_1+3\omega_2} y_3 \in {}^S\mathrm{Adm}(2\omega_1 + 3\omega_2)$ and $X_{\varpi^{2\omega_1+3\omega_2} y_3}(\tau^8) \neq \emptyset$. Thus, the equivalence of (i), (ii) and (iii) for $n = 2, 3, 4, 5$ also follows from Theorem 4.17, Proposition 6.5, Proposition 6.10 and Proposition 7.1. The case for $n = 1$ is trivially true.

Assume that μ satisfies one of the conditions in the theorem, which is equivalent to each other as we have just proved. Except the cases where μ or μ^* is $\omega_1 + \omega_{n-2}$ $(n \geq 4)$ or $\omega_1 + \omega_2$ $(n = 5)$, it follows from [43, Theorem 5.3] and Proposition 7.1 that each $X_\mu^\lambda(b) (\neq \emptyset)$ is universally homeomorphic to an affine space. Here we will treat the case $\mu = \omega_1 + \omega_{n-2}$. The proof for $\mu = \omega_1 + \omega_2$ is similar.

Set $\mu = \omega_1 + \omega_{n-2}$ and $\mu_\bullet = (\mu_1, \mu_2) = (\omega_1, \omega_{n-2})$. By [32, Theorem 1.5] and the Cartesian square right after it, pr induces a bijection between $\mathrm{pr}^{-1}(X_\mu(b)) (\subseteq X_{\mu_\bullet}(b_\bullet))$ and $X_\mu(b)$ (cf. [40, Lemma 3.11]). Since pr is proper, it induces a universally homeomorphism onto its image. Thus by Theorem 3.12, it suffices to show that for any fixed $1 \leq j \leq n-2$ and $[\lambda] \in \mathbb{A}_\mu^j$, there exists a unique $\lambda_\bullet = (\lambda_1, \lambda_2) \in \mathcal{A}_{\mu_\bullet}^j$ such that $\lambda_1 = \lambda$. If $\lambda_\bullet \in \mathcal{A}_{\mu_\bullet}^j$, then by [32, Proposition 2.9], we have

$$\lambda_2 - \lambda_1 \in W_0 \omega_1, \qquad b\lambda_1 - \lambda_2 \in W_0 \omega_{n-2}.$$

By Lemma 5.3, we may assume that $[\lambda] \in \mathbb{A}_\mu^j$ has one of the following forms:

(1) $\lambda = (1, \dots, 1, 0, \dots, 0, -1, \dots, -1, 0, \dots, 0)$,
(2) $\lambda = (0, \dots, 0, 1, \dots, 1, 0, \dots, 0, -1, \dots, -1)$,
(3) $\lambda = (1, \dots, 1, 0, \dots, 0, -1, \dots, -1)$.

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Here the numbers of 1 and −1 are equal. In the case (1) (resp. (2)), let $i = \max\{i' \mid \lambda(i') = -1\}$ (resp. $\min\{i' \mid \lambda(i') = 1\}$). Then $(\lambda_2 - \lambda)(i) = \lambda_2(i) + 1$ and $(b\lambda - \lambda_2)(i) = 1 - \lambda_2(i)$ (resp. $(\lambda_2 - \lambda)(i-1) = \lambda_2(i-1)$ and $(b\lambda - \lambda_2)(i-1) = 2 - \lambda_2(i-1)$). So if $\lambda_2 - \lambda \in W_0\omega_1$ and $b\lambda - \lambda_2 \in W_0\omega_{n-2}$, then $\lambda_2(i) = 0$ (resp. $\lambda_2(i-1) = 1$). Hence, the i-th (resp. $(i-1)$-th) entry of $\lambda_2 - \lambda$ is equal to 1, and other entries are equal to 0. So λ_2 is uniquely determined by λ. In the case (3), we have $(\lambda_2 - \lambda)(n) = \lambda_2(n) + 1$ and $(b\lambda - \lambda_2)(n) = 1 - \lambda_2(n)$. So if $\lambda_2 - \lambda \in W_0\omega_1$ and $b\lambda - \lambda_2 \in W_0\omega_{n-2}$, then $\lambda_2(n) = 0$. So λ_2 is also uniquely determined by λ.

Other statements follow from the results (and proofs) in Section 5 and Section 6. ∎

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