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EQUATIONS FOR PERIODIC SOLUTIONS OF A LOGISTIC DIFFERENCE EQUATION

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Abstract

The paper is concerned with periodic solutions of the difference equation $u_{n+1} = 2au_n - bu_n^2$, where a and b are constants, with $a > \frac{1}{2}$ and $b > 0$. A new method is developed for dealing with this problem and, for period lengths up to 6, polynomial equations are given which allow the periodic solutions to be determined in a precise and practical manner. These equations apply whether the periodic solutions are stable or unstable and the elements of the cycle can be determined with an accuracy which is not affected by instability of the cycle.

A simple transformation puts the equation into the form $w_{n+1} = w_n^2 - A$, where $A = a^2 - a$, and the detailed discussion is based on this simpler form. The discussion includes details such as the number of cyclic solutions for a given value of A, the pattern of the cycles and their stability. For practical purposes, it is enough to consider a restricted range of values of A, namely $-\frac{1}{4} < A < 2$, although the equations obtained are valid for $A > 2$.

1. Introduction

Difference equations of the form

$$u_{n+1} = (2a - bu_n)u_n \tag{1.1}$$

are well known in population dynamics. They can be regarded as a finite analogue of the logistic differential equation

$$N'(t) = N(t)\{C - BN(t)\}, \tag{1.2}$$

where C and B are positive constants and $N(t)$ is the population size at time t. If we replace $N'(t)$ by $(1/h)\{N(t + h) - N(t)\}$, with $h > 0$, and write u_n for

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$N(nh)$, then equation (1.1) is obtained with

$$2a = 1 + hC > 1, \qquad b = hB > 0. \tag{1.3}$$

In this case $u_{n+1} - u_n$ has the same sign as N' for a given population size and, in particular, the equilibrium values are the same. Indeed, equation (1.1) is the Cauchy–Euler approximation to (1.2) and the solutions behave in the same way for h small (Hurewicz [4]). However, the solutions of the difference equation show a much richer variety of possible behaviour for larger values of h. In a review article [8], May has drawn attention to these possibilities and suggested that the difference equation provides a more appropriate model than the differential equation in a number of practical applications. In particular, the difference equation can have solutions which oscillate above and below the equilibrium level $N^* = C/B$, whereas solutions of equation (1.2) approach N^* from one side only.

A change of scale, with $v_n = bu_n$, is enough to eliminate b from the difference equation. The solutions then depend on the value of a and the variety of solutions available becomes greater as a increases. Periodic solutions can arise and a good deal of information is available about the existence and stability of these periodic solutions for different values of a (see May [8], Hoppensteadt and Hyman [3]). For longer periods, much of this information comes from computational studies. In the present paper, algebraic equations are given which allow the periodic solutions to be determined for periods up to 6. These equations are valid whether the solutions are stable or unstable, although information about the stability comes out as a by-product. For periods 5 and 6, the solutions are only stable within narrow intervals of the parameter a and outside these intervals iteration of equation (1.1) can give an apparent periodicity when the initial value is not an element of a periodic solution or an apparent drift away from periodicity when the initial value is a good approximation to an element of a periodic solution.

Section 2 introduces a preliminary transformation which simplifies the discussion. Some known results are cited and the general problem is outlined. Section 3 considers solutions of period 5 in detail, to illustrate the approach that is used. Each periodic solution is identified by the sum of its elements, α, and, for a given value of a, α is determined from a polynomial equation of degree 6. The elements of the periodic solution are then found as the zeros of a polynomial of degree 5.

Corresponding equations for solutions with periods 2, 3, 4 and 6 are summarized in Section 4. In the most complicated case, for solutions with period 6, appropriate values of α come from a polynomial equation of degree 9 and the elements of the periodic solution are the roots of an equation of degree 6.

The results for solutions with periods 3 and 4 are discussed in Section 5 and Section 6 gives a similar discussion for solutions of periods 5 and 6. The results are summarized in Table 1.

2. Preliminary transformation and notation

Starting from equation (1.1), with $2a > 1$ and $b > 0$, we can obtain a simpler form by using $w_n = a - bu_n$, which gives

$$w_{n+1} = w_n^2 - A = F(w_n), \tag{2.1}$$

where $A = a^2 - a$. This is essentially the form used by Chaundy and Phillips [1], who considered the convergence of the sequence $\{w_n\}$. They showed that, as $n \to \infty$,

(i) for $|w_0| > a$, $w_n \to \infty$ monotonically,

(ii) for $|w_0| < a$ and $\frac{1}{2} < a \leqslant \frac{3}{2}$, $w_n \to 1 - a$,

(iii) for $|w_0| < a$ and $\frac{3}{2} < a \leqslant 2$, w_n oscillates finitely, with $|w_n| < a$, and

(iv) for $|w_0| < a$ and $a > 2$, $w_n \to \infty$ in general.

The exceptional cases in (iv) are the values of w_0 which lead to the equilibrium values ($w_n = a$ and $w_n = 1 - a$) or to a periodic solution. Note also that $w_0 = \pm a$ gives $w_n = a$ for $n \geqslant 1$. Thus, in looking for periodic solutions of practical interest, we can impose the restriction that $|w_0| < a$ and consider only sequences with $\frac{3}{2} \leqslant a \leqslant 2$, that is $0.75 \leqslant A \leqslant 2$. (Periodic solutions certainly exist for $a > 2$ but result (iv) indicates that they are unstable.)

For $A = 2$, Lorenz [7] gives an exact solution and this is of considerable help in checking results. If we write $w_0 = 2\cos\phi$, then $w_n = 2\cos(2^n\phi)$ and the condition for w_n to equal w_0 becomes

$$(2^n \pm 1)\phi = 2N\pi, \tag{2.2}$$

for any integer N. It should be noted that this condition gives not only the solutions with basic period n but also those with basic period n/p, where p is a factor of n. (When $p = n$ we get the equilibrium values.) These exact solutions will be referred to later as the trigonometric solutions.

For $a > \frac{1}{2}$, the relationship $A = a^2 - a$ can be inverted to give

$$a = \left(\tfrac{1}{2}\right)\{1 + \sqrt{(1 + 4A)}\},$$

which means that we can treat A as the relevant parameter in discussing equation (2.1) and occasionally refer back to a when comparing results with those found by other authors. Also we can write, for $m = 2, 3, \ldots,$

$$w_{n+m} = F_m(w_n) = F\{F_{m-1}(w_n)\}, \tag{2.3}$$

where $F_1 = F$. Thus F_m is the mth iterate of equation (2.1) and $F_m(w_n)$ will be a polynomial of degree 2^m in w_n. Setting $w_{n+m} = w_n$ gives a polynomial equation of degree 2^m, namely

$$G_m(w_n) \equiv F_m(w_n) - w_n = 0. \tag{2.4}$$

Since $F'(w_n) = 2w_n$, we can write

$$F'_m = 2w_{m+n-1}F'_{m-1} = 2^m w_{m+n-1}w_{m+n-2} \cdots w_n. \tag{2.5}$$

The equation $G_m(w_n) = 0$ must hold for a solution with period m although for $m > 1$ the degree of the equation can be reduced slightly. When m is a prime number, the degree of the equation can be reduced by 2, since $G_m(w_n)$ must include factors $w_n - a$ and $w_n - (1 - a)$, which correspond to the equilibrium solutions. The product of these two factors, $w_n^2 - w_n - A$, can be removed from G_m, leaving an equation of degree $2^m - 2$. When m is a composite number, with factor p, then $G_p(w_n)$ must be a factor of $G_m(w_n)$ and the degree of equation (2.4) can be reduced by 2^p, with the possibility of a further reduction from the cofactor of p. Despite these reductions, the dominant effect is that the degree of the equation increases exponentially with the length of the period and a direct solution soon becomes impracticable.

3. Equations for solutions of period 5

For convenience, we shall denote a solution with basic period 5 as a $C5$ solution, with a similar notation for solutions with other basic periods. Also we write b_1, b_2, b_3, b_4 and b_5 for the elements of the solution, with

$$h_5(x) = \prod_{i=1}^{5} (x - b_i) = x^5 - \alpha x^4 + \beta x^3 - \gamma x^2 + \delta x - \varepsilon. \tag{3.1}$$

This equation defines α, β, γ, δ and ε as symmetric functions of the b's, with full permutational symmetry. It turns out that β, γ, δ and ε can be expressed in terms of α, and so a knowledge of α defines $h_5(x)$. (Since it does not matter which element of the $C5$ solution we take as b_1, it is appropriate to identify $h_5(x)$ by a symmetric function of the elements, such as α. The values of b_i can be determined by solving $h_5(x) = 0$ once α is known.)

If we write $G_5(x) = (x^2 - x - A)H_5(x)$, then H_5 is a polynomial in x of degree 30 and h_5 must be a factor of H_5. At most we can expect six $C5$ solutions for a given value of A, corresponding to a factorization of H_5 into fifth-degree polynomials of the form $h_5(x)$. To avoid carrying out this factorization explicitly, we derive a sixth-degree equation for α and look for real roots of this equation. A fair amount of algebra is required to derive the equation for α but solving it

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for a given value of A is straightforward. For each real α we can then calculate β, γ, δ and ε and solve $h_5(x) = 0$.

The condition for local stability in the general case is that $|F_m'(w_n)| < 1$ and for our $C5$ solution this becomes

$$-\frac{1}{32} < \varepsilon < \frac{1}{32}, \tag{3.2}$$

using equation (2.5) and noting that $\varepsilon = b_1 b_2 b_3 b_4 b_5$. Thus the stability of the solution is determined by the value of ε.

The basic equations for the elements b_i are

$$b_{i+1} = b_i^2 - A, \qquad i = 1, 2, 3, 4, 5, \tag{3.3}$$

where $b_6 = b_1$, and we want to use these equations to express β, γ, δ and ε in terms of α and then to form an equation for α. From the theory of equations (Uspensky [10] or other books on the topic) any symmetrical function of the b_i can be expressed in terms of $\alpha, \beta, \gamma, \delta$ and ε and if we can relate two symmetrical functions by means of equations (3.3) we get an equation relating $\alpha, \beta, \gamma, \delta$ and ε. For example, taking summations over $i = 1$ to 5 and using equation (3.3)

$$\sum b_i^2 = \sum (A + b_{i+1}) = 5A + \alpha, \tag{3.4}$$

$$\sum b_i^4 = \sum (A + b_{i+1})^2 = 5A^2 + 2A\alpha + (5A + \alpha). \tag{3.5}$$

From the theory of equations

$$\sum b_i^2 = \alpha^2 - 2\beta \qquad \text{and} \qquad \sum b_i^4 = \alpha^4 - 4\alpha^2\beta + 2\beta^2 + 4\alpha\gamma - 4\delta, \tag{3.6}$$

and we get

$$\alpha^2 - 2\beta = \alpha + 5A$$

and

$$\alpha^4 - 4\alpha^2\beta + 2\beta^2 + 4\alpha\gamma - 4\delta = \alpha + 5A + 2\alpha A + 5A^2. \tag{3.7}$$

These examples typify the procedure, that is b_i^2 is replaced by $b_{i+1} + A$ wherever it occurs and this links a symmetric function of higher degree with a function of lower degree.

One snag which arises is that the function of lower degree may not have full permutational symmetry, since equations (3.3) have cyclic symmetry rather than permutational symmetry. To deal with this, it is convenient to introduce functions with cyclic symmetry in the b's. If we use Σ_0 to denote cyclic summation (over 1 to 5 in this case), we can write

$$\beta_1 = \Sigma_0\, b_1 b_2 = b_1 b_2 + b_2 b_3 + b_3 b_4 + b_4 b_5 + b_5 b_1, \tag{3.8}$$

$$\beta_2 = \Sigma_0\, b_1 b_3 = \Sigma_0\, b_1 b_4 = \beta - \beta_1. \tag{3.9}$$

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These functions arise naturally in equations such as

$$\sum b_i^3 = \Sigma_0\, b_1(A + b_2) = A\alpha + \beta_1, \tag{3.10}$$

$$\sum b_i^5 = \Sigma_0\, b_1(A + b_2)^2 = \Sigma_0\, b_1\left(A^2 + 2Ab_2 + A + b_3\right) = A^2\alpha + 2A\beta_1 + A\alpha + \beta_2. \tag{3.11}$$

Combining these with standard relationships from the theory of equations gives

$$\alpha^3 - 3\alpha\beta + 3\gamma = \Sigma_0\, b_1^3 = \beta_1 + \alpha A, \tag{3.12}$$

$$\alpha^5 - 5\alpha^3\beta + 5\alpha^2\gamma - 5\beta\gamma + 5\alpha\beta^2 - 5\alpha\delta + 5\varepsilon = \Sigma_0\, b_1^5 = \beta_2 + A\alpha + 2A\beta_1 + A^2\alpha. \tag{3.13}$$

Equations (3.7), (3.9), (3.12) and (3.13) suffice to express β, γ, δ and ε in terms of α and β_1. The detailed expressions are

$$2\beta = \alpha^2 - \alpha - 5A, \qquad 6\gamma = \alpha^3 - 3\alpha^2 - 13\alpha A + 2\beta_1, \tag{3.14}$$

$$24\delta = \alpha^4 - 6\alpha^3 + \alpha^2(3 - 22A) + \alpha(18A - 6) + 45A^2 - 30A + 8\alpha\beta_1, \tag{3.15}$$

$$120\varepsilon = \alpha^5 - 10\alpha^4 + \alpha^3(15 - 30A) + \alpha^2(70A - 18) + \alpha(149A^2 - 126A - 12) - 60A + \beta_1(20\alpha^2 - 20\alpha - 24 - 52A). \tag{3.16}$$

At this stage we need two more equations if we are to solve for β_1 and α. From equation (3.3),

$$\Pi_0(b_3 - b_2) = \Pi_0\left(b_2^2 - b_1^2\right),$$

where Π_0 is used as a cyclic product sign, and if we take the b's as distinct this gives

$$1 = \Pi_0(b_1 + b_2). \tag{3.17}$$

Similarly,

$$1 = \Pi_0(b_1 + b_3). \tag{3.18}$$

This looks promising but in fact both these equations lead to the same relationship

$$0 = (5A^2 + 5A - 1) + \alpha(1 + 4A + 2A^2) + \beta(1 + 3A) + \gamma(1 + 2A) + \delta + 2\varepsilon. \tag{3.19}$$

An equivalent form is

$$48\varepsilon = -\alpha^4 + (2 - 8A)\alpha^3 + (10A - 3)\alpha^2 + (56A^2 - 26A - 6)\alpha + (24 - 30A + 15A^2) - 8\beta_1(\alpha + 1 + 2A). \tag{3.20}$$

An additional equation can be obtained in different ways, one being to use the symmetrical function

$$S = \alpha\delta - 5\varepsilon + \left(\alpha^2 - 2\beta\right)\gamma = \sum\left(b_i^3 b_j b_k + 2b_i^2 b_j b_k b_m\right), \tag{3.21}$$

where the summation is over 1 to 5 for i, j, k and m, with $i \neq j \neq k \neq m$. Using equations (3.3),

$$S = \gamma(2 + 7A) + 5\delta + \beta(1 + 4A) + 2\alpha + 5\alpha A + 10A + \beta_1(1 + A), \tag{3.22}$$

and equating the two expressions for S leads to

$$\begin{aligned} 120\varepsilon = \alpha^5 - 7\alpha^4 &+ (13 - 30A)\alpha^3 + (52A - 9)\alpha^2 \\ &+ (149A^2 - 88A - 6)\alpha + 15A^2 - 30A \\ &+ \beta_1(8\alpha^2 - 32\alpha - 40 - 40A). \end{aligned} \tag{3.23}$$

Equations (3.16), (3.20) and (3.23) can be used to eliminate ε and β_1. This gives

$$0 = (2\alpha - 5)K(\alpha, A), \tag{3.24}$$

where

$$\begin{aligned} K(\alpha, A) \equiv \alpha^6 + \alpha^5 + (3 - 11A)\alpha^4 + (11 - 18A)\alpha^3 + (44 - 19A + 19A^2)\alpha^2 \\ + (36 + 24A + 17A^2)\alpha + (32 - 28A + 40A^2 - 9A^3). \end{aligned} \tag{3.25}$$

The solution $\alpha = \frac{5}{2}$ in equation (3.24) arises because equations (3.17) and (3.18) are satisfied when $b_1 = b_2 = b_3 = b_4 = b_5 = \frac{1}{2}$ and these elements also provide an equilibrium solution of equations (3.3) in the limiting case as $a \to \frac{1}{2}$. Thus the effective equation for α is $K(\alpha, A) = 0$. For a given value of A we can look for real roots of this equation and for each real root we evaluate $\beta, \beta_1, \gamma, \delta$ and ε and obtain the elements of the $C5$ solution, if there is one, from $h_5(x) = 0$. In solving for β_1, the equation used in practice was

$$\begin{aligned} \beta_1(40\alpha^2 - 8 - 24A) = -2\alpha^5 + 15\alpha^4 + (20A - 20)\alpha^3 + (21 - 90A)\alpha^2 \\ - (18A^2 - 122A + 6)\alpha + (120 - 30A + 75A^2), \end{aligned} \tag{3.26}$$

obtained by eliminating ε between equations (3.16) and (3.20).

4. Equations for periods 2, 3, 4 and 6

For $C2$ solutions, the results are well known (May [8], Levin and May [7]) and require little comment. In our notation, the basic equations are, for elements b_1 and b_2,

$$b_2 = b_1^2 - A \qquad \text{and} \qquad b_1 = b_2^2 - A. \tag{4.1}$$

Subtracting one from the other gives $\alpha = b_1 + b_2 = -1$, assuming $b_1 \neq b_2$, while adding them gives

$$\beta = b_1 b_2 = \left(\tfrac{1}{2}\right)(\alpha^2 - \alpha - 2A) = 1 - A. \tag{4.2}$$

Thus b_1 and b_2 are solutions of the equation

$$0 = x^2 - \alpha x + \beta = x^2 + x + 1 - A \tag{4.3}$$

and the condition for real, distinct roots is $A > \frac{3}{4}$. From equation (4.2), the solutions are locally stable if $-1 < 4(1 - A) < 1$, that is, for $\frac{3}{4} < A < \frac{5}{4}$. Levin and May [7] note that in this case local stability implies global stability for $|w_0| < a$.

For solutions with period 3, we can write $G_3(x) = (x^2 - x - A)H_3(x)$, where H_3 is a polynomial of degree 6. A typical $C3$ solution will give a factor

$$h_3(x) = (x - b_1)(x - b_2)(x - b_3) = x^3 - \alpha x^2 + \beta x - \gamma \tag{4.4}$$

and at most we can have two factors of this type in H_3. Detailed analysis along the same lines as in Section 3 gives

$$2\beta = \alpha^2 - \alpha - 3A \quad \text{and} \quad 6\gamma = \alpha^3 - 2\alpha^2 - \alpha(1 + 7A) - 3A, \tag{4.5}$$

with α satisfying the quadratic equation

$$\alpha^2 + \alpha + 2 - A = 0. \tag{4.6}$$

For $A < \frac{7}{4}$ this equation does not have real roots but for $A \geqslant \frac{7}{4}$ there are real roots and corresponding $C3$ solutions, obtained by solving $h_3(x) = 0$. The b's satisfy a relationship

$$(b_1 + b_2)(b_2 + b_3)(b_3 + b_1) = 1, \tag{4.7}$$

which corresponds to equations (3.17) and (3.18) of the $C5$ case. The condition for local stability for the $C3$ solutions is $|\gamma| < \frac{1}{8}$.

For solutions with period 4, we can write $G_4(x) = G_2(x)H_4(x)$, where H_4 is a polynomial of degree 12. A typical $C4$ solution gives a factor

$$h_4(x) = (x - b_1)(x - b_2)(x - b_3)(x - b_4) = x^4 - \alpha x^3 + \beta x^2 - \gamma x + \delta \tag{4.8}$$

and H_4 can at most have three factors of this type. Thus we can expect α to satisfy a cubic equation and the analysis confirms this. For our typical $C4$ solution, there are two equations similar to (4.7). They are

$$(b_1 + b_3)(b_2 + b_4) = -1 \tag{4.9}$$

and

$$(b_1 + b_2)(b_2 + b_3)(b_3 + b_4)(b_4 + b_1) = 1. \tag{4.10}$$

Equation (4.9) gives

$$\beta_1 = \Sigma_0\, b_1 b_2 = b_1 b_2 + b_2 b_3 + b_3 b_4 + b_4 b_1 = -1, \tag{4.11}$$

so, although the distinction between cyclic symmetry and permutational symmetry begins to appear at this stage, it causes no difficulty. As in Section 3, we can use the basic equations for the $C4$ solution to obtain

$$2\beta = \alpha^2 - \alpha - 4A, \qquad 6\gamma = \alpha^3 - 3\alpha^2 - 10\alpha A - 2, \tag{4.12}$$

and

$$24\delta = \alpha^4 - 6\alpha^3 + \alpha^2(3 - 16A) + \alpha(12A - 14) + 24(A^2 - A), \tag{4.13}$$

giving β, γ and δ in terms of α. The equation for α is $J(\alpha, A) = 0$, with

$$J(\alpha, A) = \alpha^3 + (3 - 4A)\alpha + 4. \tag{4.14}$$

This equation has at least one real solution for any value of A but, as in the case of $C2$ solutions, the corresponding equation $h_4(x) = 0$ does not always have real solutions. (A simple example of this is $A = 0$, $\alpha = -1$ and $h_4(x) = x^4 + x^3 + x^2 + x + 1$.) In fact, real $C4$ solutions occur only for $A > \frac{5}{4}$. In the border-line case, $A = \frac{5}{4}$, equation (4.14) has a single real root, $\alpha = -2$, and the equation $h_4(x) = 0$ gives a $C2$ solution which is at the limit for local stability, a result which agrees with the folk-lore for the subject. For the $C4$ solutions, the condition for local stability is $|\delta| < \frac{1}{16}$.

For solutions with period 6 the algebra becomes heavier but the procedure follows the same lines. We can write

$$G_6(x) = (x^2 - x - A)(x^2 + x + 1 - A)H_3(x)H_6(x), \tag{4.15}$$

where H_6 is a polynomial of degree 54. The first three factors cover, respectively, the equilibrium solutions, the $C2$ solutions and the $C3$ solutions, which are included as degenerate cases in the equation $w_{n+6} = w_n$. The $C6$ solutions come from the remaining factor, that is from $H_6(x) = 0$. As before, a $C6$ solution with elements b_1 to b_6 will be represented in H_6 by a factor

$$h_6(x) = \prod_{i=1}^{6} (x - b_i) = x^6 - \alpha x^5 + \beta x^4 - \gamma x^3 + \delta x^2 - \varepsilon x + \theta, \tag{4.16}$$

and at most there can be nine factors of this type. In agreement with this it turns out that α satisfies the equation $L(\alpha, A) = 0$, where

$$\begin{aligned} L(\alpha, A) \equiv{}& \alpha^9 - \alpha^8 + (2 - 24A)\alpha^7 + (14 - 8A)\alpha^6 \\ &+ (49 - 16A + 144A^2)\alpha^5 + (175 - 16A + 112A^2)\alpha^4 \\ &+ (140 + 136A + 160A^2 - 256A^3)\alpha^3 \\ &+ (196 - 552A + 480A^2 - 256A^3)\alpha^2 \\ &+ (448 - 416A - 304A^2 + 256A^3)\alpha + (384A - 592A^2 + 256A^3). \end{aligned} \tag{4.17}$$

Also, β, γ, δ, ε and θ can be expressed in terms of α and β_1, where

$$\beta_1 = \Sigma_0\, b_1 b_2 = b_1 b_2 + b_2 b_3 + b_3 b_4 + b_4 b_5 + b_5 b_6 + b_6 b_1. \tag{4.18}$$

The relevant equations are, for $\alpha + 1 \neq 0$,

$$2\beta = \alpha^2 - \alpha - 6A, \qquad 6\gamma = \alpha^3 - 3\alpha^2 - 16A\alpha + 2\beta_1, \tag{4.19}$$

$$24\delta = \alpha^4 - 6\alpha^3 + (3 - 28A)\alpha^2 + (24A - 6)\alpha + 72A^2 - 36A + 8\alpha\beta_1, \tag{4.20}$$

$$\begin{aligned} 120\varepsilon = {} & \alpha^5 - 10\alpha^4 + (15 - 40A)\alpha^3 + (100A - 22)\alpha^2 \\ & + (264A^2 - 156A - 8)\alpha + (8 - 32A) \\ & + (20\alpha^2 - 20\alpha - 24 - 72A)\beta_1 + 16\{(\beta_1 + 2A - 2)/(1 + \alpha)\}, \end{aligned} \tag{4.21}$$

$$\begin{aligned} 720\theta = {} & \alpha^6 - 11\alpha^5 + (35 - 50A)\alpha^4 + (200A - 37)\alpha^3 \\ & + (544A^2 - 326A + 64)\alpha^2 - (264A^2 - 424A + 4)\alpha \\ & - (80 + 136A - 1080A^2 + 720A^3) - 112\{(\beta_1 + 2A - 2)/(1 + \alpha)\} \\ & + \beta_1\{20\alpha^3 - 100\alpha^2 - (104 + 272A)\alpha - 80 - 48A\}. \end{aligned} \tag{4.22}$$

The cyclic summation β_1 can be written as a rational function of α in two ways and the equation for α comes from equating these two expressions for β_1. More precisely, for $\alpha \neq 3$,

$$0 = \beta_1 M_1(\alpha) + M_2(\alpha) \qquad \text{and} \qquad 0 = \beta_1 N_1(\alpha) + N_2(\alpha), \tag{4.23}$$

where

$$M_1 = 5\alpha^3 + 10\alpha^2 + (9 - 8A)\alpha + 8 - 8A, \tag{4.24}$$

$$\begin{aligned} M_2 = {} & \left(\tfrac{1}{4}\right)(\alpha^6 - 4\alpha^5 - 2\alpha^3 + 15\alpha^2 + 150\alpha + 96) \\ & - A(5\alpha^4 + 24\alpha^2 + 52\alpha + 15) + 16A^2(\alpha^2 + \alpha), \end{aligned} \tag{4.25}$$

$$\begin{aligned} N_1 = {} & 5\alpha^5 + (35 - 60A)\alpha^4 + (49 - 148A)\alpha^3 + (87 - 376A + 240A^2)\alpha^2 \\ & + (208 - 548A + 320A^2)\alpha + (80 - 180A + 80A^2), \end{aligned} \tag{4.26}$$

and

$$\begin{aligned} N_2 = {} & \left(\tfrac{1}{4}\right)\alpha^8 - 2\alpha^7 - \alpha^6 + (30A - 3)\alpha^5 \\ & + \{36A^2 + 36A - (127/4)\}\alpha^4 + (312A^2 - 126A - 50)\alpha^3 \\ & - \left\{160A^3 - 856A^2 + 482A + \left(\tfrac{87}{2}\right)\right\}\alpha^2 \\ & + (99 - 1060A + 1560A^2 - 640A^3)\alpha \\ & + (240 - 990A + 1140A^2 - 480A^3). \end{aligned} \tag{4.27}$$

Eliminating β_1 from equations (4.23) gives

$$0 = 4(M_1N_2 - M_2N_1) = 15(\alpha + 1)(4A - 3)L(\alpha, A). \tag{4.28}$$

Thus $A = \frac{3}{4}$, $\alpha = -1$ and $\alpha = 3$ have to be examined separately, but it turns out that the equation $L(\alpha, A) = 0$ covers these special cases.

With the equations above, the $C6$ solutions can be explored systematically. For a given value of A, the real roots of $L(\alpha, A) = 0$ can be found to any desired accuracy and for each real root equations (4.19) to (4.27) can be used to evaluate β_1, β, γ, δ, ε and θ. The elements of the $C6$ solution then come from solving $h_6(x) = 0$. The criterion for stability is that $|\theta| < \frac{1}{64}$.

For the $C6$ solutions, the equations corresponding to (3.17) and (3.18) are

$$(b_1 + b_4)(b_2 + b_5)(b_3 + b_6) = -1, \tag{4.29}$$

$$1 = \Pi_0(b_1 + b_2) \quad \text{and} \quad 1 = \Pi_0(b_1 + b_3), \tag{4.30}$$

where the cyclic product Π_0 is now taken over 1 to 6. Equations (4.23) for β_1 come essentially from equations (4.30).

5. Discussion of $C3$ and $C4$ solutions

We can take the results for the $C2$ solutions as sufficiently well known not to require further discussion and look briefly at the $C3$ and $C4$ solutions, which illustrate some of the ideas that occur in the $C5$ and $C6$ solutions. For the $C3$ solutions, the equation for α, equation (4.6), has no real solutions for $A < \frac{7}{4}$ but, for $A > \frac{7}{4}$, the equation has real, distinct roots, say α_1 and α_2, with $\alpha_1 < -\frac{1}{2} < \alpha_2$. For the critical value $A = \frac{7}{4}$, the equation has a double root $\alpha = -\frac{1}{2}$. It can be shown that there is a $C3$ solution corresponding to each real value of α. Thus we get $C3$ solutions for $A \geqslant A^* = 1.75$ and we can refer to A^* as the critical value for A, with α^* as the corresponding double root value for α. This notation will be used in other cases where a similar situation arises.

For $A = \frac{7}{4}$ and $\alpha = -\frac{1}{2}$, $\gamma = \frac{1}{8}$ and the corresponding $C3$ solution represents a limiting case for local stability. For $A > \frac{7}{4}$, α_1 decreases and α_2 increases as A increases. Now, from equations (4.5) and (4.6), we can write

$$\gamma = -(\alpha^3 + 2\alpha^2 + 3\alpha + 1),$$

which gives $d\gamma/d\alpha = -(3\alpha^2 + 4\alpha + 3) < 0$. For the solution corresponding to the α_1 roots, γ increases with A and these solutions are unstable. On the other hand, γ decreases with A for the α_2 family of solutions and there is a range of A for which $-\frac{1}{8} < \gamma < \frac{1}{8}$ and the solutions are locally stable. It is easy to check that the equation $\gamma = -(\alpha^3 + 2\alpha^2 + 3\alpha + 1) = -\frac{1}{8}$ gives a unique real value of α and a corresponding value for A, say A^{**}, with local stability for $A^* < A < A^{**}$.

This type of behaviour occurs for $C4$, $C5$ and $C6$ solutions also, in the sense that real solutions for α mostly occur in pairs for $A > A^*$, where A^* is a critical value of A. At $A = A^*$ there is a double root, $\alpha = \alpha^*$, and the corresponding solution is on the limit of stability. As A increases from A^*, one family of α values gives unstable solutions while the other family gives local stability for $A^* < A < A^{**}$, where A^{**} is an upper critical value.

For $C4$ solutions, α is obtained from $J(\alpha, A) = 0$, with $J(\alpha, A)$ as defined in equation (4.14). The equation has a negative root for all values of A but these negative roots do not lead to real solutions for b_1, b_2, b_3 and b_4 as long as $A < \frac{5}{4}$. This can be proved algebraically. For $A = \frac{5}{4}$, $\alpha = -2$ and the equation $h_4(x) = 0$ gives $b_1 = b_3 = \frac{1}{2}(-1 + \sqrt{2})$ and $b_2 = b_4 = \frac{1}{2}(-1 - \sqrt{2})$, that is, a $C2$ solution with $b_1 b_2 = -\frac{1}{4}$. For $A > \frac{5}{4}$, the negative root for α gives a real $C4$ solution which is locally stable for a small range of values of A. In Table 1, where the results are summarized, the negative roots are denoted by α_1 and the critical value A^* is given as 1.25, in the sense that the values of α_1 lead to real $C4$ solutions for $A > A^*$.

Positive roots for α appear for $A \geqslant A^* = \frac{3}{4}\{1 + (2)^{2/3}\}$, with a double root for $A = A^*$ and distinct roots α_2 and α_3 for $A > A^*$. It can be shown that each positive root leads to a real $C4$ solution. Taking $\alpha_2 < \alpha_3$, the table indicates that the α_3 values lead to locally stable solutions for $A^* < A < A^{**}$, while the α_2 values lead to unstable solutions.

The last column of Table 1 shows values of the b_i for a typical solution and it will be observed that the solution corresponding to negative values of α is similar to a $C2$ solution in its behaviour. It might be described as a split-level $C2$ solution. The solutions corresponding to positive values of α show a decrease from b_1 to b_4, with a sharp increase from b_4 to b_1 at the beginning of the next cycle. In terms of the population problem, this represents a gradual increase in the population over three seasons (or generations) with a catastrophic drop at the end of that time, a pattern reminiscent of some of the "plague" outbreaks described by Elton [2].

The critical values for the $C3$ solutions agree with those given by May [8, Table 3]. For the $C4$ solutions, May's table could be misleading since he records only the critical values corresponding to the α_3 family of solutions. (It should be noted that May's parameter a corresponds to twice the parameter a as defined in this paper.)

6. Discussion of $C5$ and $C6$ solutions

Table 1 shows similar results for $C5$ and $C6$ solutions. The roots are arranged in order of increasing magnitude, with $\alpha_1 < \alpha_2 < \alpha_3 < \alpha_4 < \alpha_5 < \alpha_6$ when six real, distinct roots occur in the $C5$ case. In particular, all six values of α were determined for $A = 2$ and found to agree with the trigonometric solutions

TABLE 1

Critical values for $C3$, $C4$, $C5$ and $C6$ solutions

	Roots	Critical values A^*	Critical values α^*	Stable sequence	Limit of stability A^{**}	Typical periodic solution $b_1, b_2, \ldots$
$C3$	α_1, α_2	1.75	−0.50	α_2	1.768529	+1.302, −0.055, −1.747
$C4$	α_1	1.25	−2.00	α_1	1.368099	+0.494, −1.132, −0.094, −1.368
	α_2, α_3	1.940551	1.259921	α_3	1.941538	+1.825, +1.388, −0.013, −1.940
$C5$	α_1, α_2	1.624395	−2.501045	α_2	1.628438	+1.016, −0.594, −1.272, −0.006, −1.625
	α_3, α_4	1.860587	−0.914998	α_3	1.861365	+1.600, +0.684, −1.407, +0.105, −1.864
	α_5, α_6	1.985410	+3.217952	α_6	1.985468	+1.956, +1.842, +1.408, −0.003, −1.985
$C6$	α_1, α_2	1.474695	−3.462728	α_1	1.479736	+0.70, −0.99, −0.50, −1.225, +0.025, −1.47
	α_3, α_4	1.907251	−1.200622	α_4	1.907368	+1.73, +1.09, −0.73, −1.38, −0.005, −1.91
	α_5	1.768529	−0.727756	α_5	1.777222	+1.56, +0.43, −1.81, +1.29, −0.34, −1.89
	α_6, α_7	1.966764	+0.932874	α_6	1.966800	+1.90, +1.65, +0.75, −1.40, +0.0024, −1.97
	α_8, α_9	1.996375	+5.211297	α_9	1.996379	+1.99, +1.96, +1.85, +1.41, −0.001, −1.996

mentioned in Section 2. The corresponding periodic solutions also agreed with the trigonometric solutions. A similar check was made for the $C6$ solutions for $A = 2$.

For the $C5$ solutions, a computer programme was used to evaluate $K(\alpha, A)$ for different values of A and it was evident that the real roots occurred in pairs and led to real $C5$ solutions, except in one special case. The special case arose for $A = \frac{4}{3}$ and $\alpha = -1$, which gives a minimum value of $K(\alpha, A)$, with $K = 0$ at the minimum. For neighbouring values of A and α, K is positive and a slight increase in A does not lead to two real, distinct solutions for α. Equation (3.26) for β_1 gives an indeterminate form at $A = \frac{4}{3}$ and $\alpha = -1$ but it can be shown that

$$\beta_1^2 = (2 + 2A)\beta + 4\alpha A + 5A^2 + 2\delta - \beta_1(1 + 2A), \tag{6.1}$$

which leads to a quadratic equation for β_1. In the general case, this provides a useful check on the solution for β, β_1 and δ. For $A = \frac{4}{3}$ and $\alpha = -1$, equation (6.1) gives $\beta_1^2 + \frac{13}{3}\beta_1 + \frac{64}{9} = 0$ and we get complex solutions for β_1, which shows that this special case does not lead to a real $C5$ solution.

In the critical cases listed in Table 1, $\alpha = \alpha^*$ and $A = A^*$ gives a point where $K = 0$ and $\partial K/\partial\alpha = 0$ but $\partial K/\partial A \neq 0$. In practice, it was fairly easy to find approximate solutions for α^* and A^* and then to improve the approximation by iteration, using the equations $K = 0$ and $\partial K/\partial\alpha = 0$. In each case, the (α^*, A^*) solution gave $\varepsilon = 0.03125$, thus corresponding to a limiting value of ε for local stability.

The $C5$ solutions corresponding to α_1, α_2, α_3 and α_4 show similar characteristics, in that $b_1 > b_2 > b_3$ but there is a jump upward from b_3 to b_4 and then a drop to b_5. In a paper by May and Oster [9] the theorem by Li and Yorke [6] about chaotic behaviour is mentioned and there is a statement that the proof of the theorem applies if there exists any cycle with an odd period. This claim is doubtful since the $C5$ pattern above is not covered by the statement of Li and Yorke's theorem.

For solutions of period 6, the equation for α is $L(\alpha, A) = 0$, with $L(\alpha, A)$ given by equation (4.17). Since $L(\alpha, A)$ is a polynomial of odd degree in α, it must have at least one real zero for all values of A. This zero is indicated by α_5 in Table 1 and its value decreases from zero at $A = 0$ to -0.6 at $A = 1$ and to -0.76 at $A = 2$. It corresponds to the α_1 family of roots in the $C4$ case and has similar characteristics. For $A < A^*$, the α_5 values do not lead to real $C6$ solutions. For $A = A^* = 1.768529$, the equation $h_6(x) = 0$ gives a degenerate $C6$ solution, with $b_1 = b_4 = 1.3494$, $b_2 = b_5 = 0.0525$, $b_3 = b_6 = -1.7658$ and $b_1b_2b_3 = -0.125$. In fact, this is the $C3$ solution which is at the limit for local stability, as indicated by the value for A^{**} in the $C3$ entry in Table 1. For $A > A^*$, the α_5 roots lead to real $C6$ solutions whose form is similar to that for

$C3$ solutions and which are locally stable for $1.768529 < A < 1.777222$. As mentioned earlier, the stability criterion is that $|\theta| < \frac{1}{64}$.

The other real roots occur in pairs and lead to real $C6$ solutions except in one special case similar to that which arose when considering the $C5$ solutions. The special case occurs when $\alpha = \alpha_0 = 0.597634$ and $A = A_0 = 1.4097134$. For these values of α and A, the polynomials M_1, M_2, N_1 and N_2, as defined in equations (4.24) to (4.27), are all zero. This means that both equations for β_1 are indeterminate (equation 4.23) and it is easy to verify from equation (4.28) that L, $\partial L/\partial\alpha$ and $\partial L/\partial A$ are all zero at (α_0, A_0). As in the corresponding $C5$ case, L has a minimum at (α_0, A_0) and is positive for neighbouring points, so (α_0, A_0) is not a starting point for families of real roots of $L(\alpha, A) = 0$.

Although equations (4.23) give indeterminate forms for β_1, there are quadratic equations available which can be used to show that β_1, γ and δ are complex when $\alpha = \alpha_0$ and $A = A_0$. Thus the corresponding values for $b_1, b_2, \ldots, b_6$ cannot all be real.

As for the $C5$ solutions, a computer programme was used to evaluate $L(\alpha, A)$ for different values of A and to examine the solutions of $h_6(x) = 0$ corresponding to zeros of $L(\alpha, A)$. It will be seen from Table 1 that $L(\alpha, A) = 0$ has at most three real roots for $A \leqslant 1.9$ but additional pairs of roots appear as A increases from 1.9 to 2.0. For $A = 2$,

$$\begin{aligned} L(\alpha, 2) &= \alpha^9 - \alpha^8 - 46\alpha^7 - 2\alpha^6 + 593\alpha^5 + 591\alpha^4 \\ &\quad -996\alpha^3 - 1036\alpha^2 + 448\alpha + 448 \\ &= (\alpha^2 - 1)(\alpha^3 - 21\alpha - 28)(\alpha^2 + p\alpha - 4)\{\alpha^2 - (p+1)\alpha - 4\}, \end{aligned} \tag{6.2}$$

where p is the positive root of $p^2 + p = 16$. From this, $L(\alpha, 2) = 0$ gives nine real roots which are, in ascending order,

$$\alpha_1 = -4.43, \quad \alpha_2 = -3.65, \quad \alpha_3 = -1.49, \quad \alpha_4 = -1.00, \quad \alpha_5 = -0.76,$$
$$\alpha_6 = 0.90, \quad \alpha_7 = 1.00, \quad \alpha_8 = 5.14, \quad \alpha_9 = 5.29.$$

In terms of the trigonometric solutions in Section 2, the corresponding values of ϕ are, respectively, $22\pi/65$, $22\pi/63$; $10\pi/63$, $10\pi/65$; $14\pi/65$; $6\pi/65$, $6\pi/63$; $2\pi/63$; $2\pi/65$. Thus the ϕ values pair off in the same way as the α values in Table 1, with the larger ϕ value in each pair corresponding to the unstable α sequence. For α_2, α_3, α_7 and α_8, $\theta = +1$ and this checks that these are the α sequences for which θ increases as A increases. For α_1, α_4, α_5, α_6 and α_9, $\theta = -1$, which also agrees with the conclusions about stability. A similar check can be used for the $C3$, $C4$ and $C5$ entries in Table 1.

Typical $C6$ solutions are shown in the last column of Table 1. For α_1 and α_2, the solution oscillates above and below the equilibrium value $w = 1 - a$, in the manner of a $C2$ solution but with three distinct levels on either side of the equilibrium level. For α_3, α_4, α_6 and α_7, the solutions show the same type of behaviour, with a decrease from b_1 to b_4, a partial recovery at b_5 and a sharp drop to b_6. For α_8 and α_9, the typical solution shows a decrease from b_1 to b_6, followed by a sharp increase to start the next cycle. As mentioned before, the solutions corresponding to α_5 are reminiscent of $C3$ solutions.

For the $C5$ and $C6$ solutions, the critical values agree with those given by May [8, Table 3], although May's table does not have an entry corresponding to the "solitary" root, α_5, in his values for cycles with period 6.

7. Conclusion and acknowledgments

This work arose from looking at papers in which it was hard to tell what had been proved and what had been deduced from numerical simulation, so to some extent the aim was to set up equations which could be regarded as exact. The idea of "factorizing" the equation $H_n(x) = 0$ emerged from this and it will be seen that for $n = 4$, 5 and 6 it results in a considerable simplification of the original problem. To continue the method to $C7$ solutions would involve finding an equation of degree 18 for α and an equation of degree 7 for the elements of the corresponding periodic solution. It seems likely that this could be done without any additional technical problems but it would require a fair amount of time and patience.

The original draft of the paper contained some additional algebraic details but at the suggestion of the referees the paper has been reduced in length and corroboration of some less important points has been omitted.

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