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SOME PROPERTIES OF BOUNDED HOLOMORPHIC MAPPINGS DEFINED ON BOUNDED HOMOGENEOUS DOMAINS

BY

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ABSTRACT. Let F be a bounded holomorphic mapping defined on a bounded homogeneous domain in $\mathbb{C}^N$. We study the relation between the Jacobian $J_F(z)$ and the radius $d_F(z)$ of univalence of F.

1. **Introduction**. $\mathbb{C}^N$ will denote the complex vector space with the ordinary inner product $\langle z, w\rangle = \Sigma_{j=1}^{N} z_j\bar{w}_j$ and the associated norm $\|z\|^2 = \langle z, z\rangle$. Let D be a domain in $\mathbb{C}^N$ and let F be a holomorphic mapping of D into $\mathbb{C}^N$. We define a continuous function by

$$d_F(z) = \sup\{r > 0\text{: there exists a domain } \Omega,\ z \in \Omega \subset D, \text{ such that } F \text{ maps } \Omega \text{ univalently into } B(F(z), r)\}$$

if $J_F(z) \neq 0$, and $d_F(z) = 0$ if $J_F(z) = 0$. Here J_F denotes the Jacobian of F and $B(w, r)$ denotes the ball $\{z \in \mathbb{C}^N : \|z - w\| < r\}$.

The purpose of this paper is to study the relation between J_F and d_F in the case that D is a bounded homogeneous domain and F is a bounded mapping. Various results of Seidel and Walsh [8] have been extended to several variables by Hahn [1], [3]. Our results are also higher dimensional generalizations of some of results in [8].

2. **Holomorphic mappings defined on balls in** $\mathbb{C}^N$. The following higher dimensional analogue of Landau's theorem plays an essential role in our paper. It has been proved by Hahn [2]. For the sake of completeness we include another proof which relies on the generalized Rouché's theorem [4].

THEOREM A. *Let F be a holomorphic mapping of RB_N into MB_N with $F(0) = 0$, where B_N denotes the unit ball $\{z : \|z\| < 1\}$ and $rB_N = \{rz : z \in B_N\}$. Let $\lambda_F(z)$ be the positive square root of the smallest characteristic value of $A_F(z)A_F(z)^*$, where $A_F(z)$ is the Jacobian matrix of F at z. If $\lambda_F(0) \neq 0$, then for any λ with $0 < \lambda \leqq \lambda_F(0)$ there exists a domain Δ, $0 \in \Delta \subset c_1R^2M^{-1}\lambda B_N$, such that F maps Δ univalently onto $c_2R^2M^{-1}\lambda^2B_N$, where c_1 and c_2 are absolute constants independent of F, λ, M, R and N with $0 < c_i < 1$, $i = 1, 2$.*

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PROOF. Since there are unitary matrices U and V such that

$$UA_F(0)V = \begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \lambda_N \end{bmatrix}, \quad \lambda_j\text{: real},$$

we may assume that

$$A_F(0) = \begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \lambda_N \end{bmatrix},$$

where $\lambda_1^2, \dots, \lambda_N^2$ are the characteristic values of $A_F(0)A_F(0)^*$. Furthermore we may assume that $A_F(0) = \lambda_F(0)I$, where I is the identity matrix (we may consider $F^* = (\lambda_F(0)\,\lambda_1^{-1}f_1, \dots, \lambda_F(0)\lambda_N^{-1}f_N)$ instead of $F = (f_1, \dots, f_N)$).

Take a point t in ∂B_N and consider the mapping

$$G(\zeta) = F(\zeta t) - \lambda_F(0)\zeta t.$$

Then $\|\zeta^{-2}G(\zeta)\|^2$ is subharmonic in $|\zeta| < R$. Since

$$\limsup_{|\zeta| \to R} \|\zeta^{-2}G(\zeta)\| \leqq \frac{M + \lambda_F(0)R}{R^2}$$

and since $\lambda_F(0) \leqq MR^{-1}$ (see [5]), we have

$$\|G(\zeta)\| \leqq \frac{2M}{R^2}|\zeta|^2 \qquad (|\zeta| < R).$$

Thus we have the inequality

$$\|F(z) - \lambda_F(0)z\| \leqq \frac{2M}{R^2}\|z\|^2 \qquad (z \in RB_N).$$

Hence, for $w \in 8^{-1}R^2M^{-1}\lambda^2B_N$ and $z \in \partial(4^{-1}R^2M^{-1}\lambda B_N)$, we have

$$\|F(z)\| \geqq \|\lambda_F(0)z\| - \|F(z) - \lambda_F(0)z\| \geqq \frac{R^2\lambda^2}{8M} > \|w\| \geqq \|(F(z) - w) - F(z)\|,$$

and so

$$\|\lambda_F(0)z\| > \|F(z) - \lambda_F(0)z\|.$$

Now it follows from the generalized Rouche's theorem ([4], Theorem 3) that for each w in $8^{-1}R^2M^{-1}\lambda^2B_N$, there is only one point z in $4^{-1}R^2M^{-1}\lambda B_N$ such that $F(z) = w$. Therefore we can deduce the theorem.

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COROLLARY B. *Let F be a holomorphic mapping of RB_N into MB_N with $F(0) = 0$. If $J_F(0) \neq 0$, then for any η with $0 < \eta \leqq J_F(0)$, there exists a domain Δ, $0 \in \Delta \subset c_1^* R^{N+1} M^{-N} \eta B_N$, such that F maps Δ univalently onto $c_2^* R^{2N} M^{-2N+1} \eta^2 B_N$, where c_1^* and c_2^* are absolute constants independent of F, η, M, R and N with $0 < c_i^* < 1$, $i = 1, 2$.*

PROOF. Let $\lambda_1^2, \ldots, \lambda_N^2$ be the characteristic values of $A_F(0)A_F(0)^*$. Since $\lambda_j^2 \leqq (M/R)^2 (j = 1, \ldots, N)$, we have

$$|J_F(0)|^2 = \lambda_1^2 \ldots \lambda_N^2 \leqq \left(\frac{M}{R}\right)^{2N-2} \lambda_F(0)^2.$$

Hence Corollary B follows from Theorem A.

3. **Holomorphic mappings defined on bounded homogeneous domains**. Now we study properties of bounded holomorphic mappings defined on bounded homogeneous domains. Let D be a bounded homogeneous domain in $\mathbb{C}^N$ containing the origin 0. Let K_D denote the Bergman kernel function of D, and let $T_D = (T_{p\bar{q}})$, $T_{p\bar{q}} = \partial^2 \log K_D / \partial z_p \partial \bar{z}_q$. Then, as is well known, if φ is an automorphism of D, we have

$$K_D(z, \bar{z}) = K_D(\varphi(z), \overline{\varphi(z)}) |J_\varphi(z)|^2 \tag{1}$$

and

$$T_D(z, \bar{z}) = A_\varphi(z)^* T_D(\varphi(z), \overline{\varphi(z)}) A_\varphi(z). \tag{2}$$

The following theorem is a higher dimensional generalization of the fact: If f is holomorphic and $|f| < 1$ in $|z| < 1$, then

$$d_f(z) \leqq |f'(z)|(1 - |z|^2) < C d_f(z)^{\frac{1}{2}} \qquad (|z| < 1),$$

where C is a positive absolute constant. (See [8], p. 152).

THEOREM 1. *If D is a bounded homogeneous domain in $\mathbb{C}^N$, then there exist positive constants C_1, C_2 and C_3 depending only on D such that, for all holomorphic mappings F of D into B_N,*

$$C_1 d_F(z)^N \leqq |J_F(z)| K_D(z, \bar{z})^{-\frac{1}{2}} \leqq C_2 d_F(z)^{\frac{1}{2}} \qquad (z \in D)$$

and

$$\lambda_F(z) \Lambda_D(z)^{-1} \leqq C_3 d_F(z)^{\frac{1}{2}} \qquad (z \in D),$$

where Λ_D is the positive square root of the largest characteristic value of T_D.

Moreover, if D contains the origin 0, C_1, C_2 and C_3 can be so chosen that

$$C_1 = R_D^{-N} K_D(0,0)^{-\frac{1}{2}},$$
$$C_2 = 2^{N-\frac{1}{2}} c_2^{*-\frac{1}{2}} r_D^{-N} K_D(0,0)^{-\frac{1}{2}},$$
$$C_3 = 2^{\frac{1}{2}} c_2^{-\frac{1}{2}} r_D^{-1} \lambda_D(0)^{-1}$$

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where

$$R_D = \inf\{R : RB_N \supset D\}, \qquad r_D = \sup\{r : rB_N \subset D\}$$

and $\lambda_D(0)$ *is the positive square root of the smallest characteristic value of* $T_D(0, 0)$.

PROOF. We may suppose that D contains the origin 0, without loss of generality. Take any $z \in D$. We may assume that $d_F(z) \neq 0$, since otherwise Theorem 1 is trivial. Set $G = F \circ \varphi - F(z)$, where φ is an automorphism of D with $\varphi(0) = z$. Since $d_G(0) = d_F(z)$, there exists a domain Ω, $0 \in \Omega \subset D$, such that G maps Ω univalently onto $d_F(z)B_N$. Let H denote the inverse of the restriction of G to Ω. Then H maps $d_F(z)B_N$ into R_DB_N and $H(0) = 0$. Hence

$$|J_H(0)| \leqq \left(\frac{R_D}{d_F(z)}\right)^N.$$

Since $J_H(0) = J_F(z)^{-1}J_\varphi(0)^{-1}$, together with (1) this shows that

$$R_D^{-N}K_D(0,0)^{-\frac{1}{2}}d_F(z)^N \leqq |J_F(z)|K_D(z,\bar{z})^{-\frac{1}{2}} \qquad (z \in D).$$

Next, since $G = F \circ \varphi - F(z)$ maps r_DB_N into $2B_N$ and $G(0) = 0$, we have, by Corollary B and Theorem A,

$$d_F(z) = d_G(0) \geqq 2^{-2N+1}c_2^* r_D^{2N}|J_G(0)|^2 \tag{3}$$

and

$$d_F(z) = d_G(0) \geqq 2^{-1}c_2 r_D^2 \lambda_G(0)^2. \tag{4}$$

From (1) and (3) we have

$$|J_F(z)|K_D(z,\bar{z})^{-\frac{1}{2}} \leqq 2^{N-\frac{1}{2}}c_2^{*-\frac{1}{2}}r_D^{-N}K_D(0,0)^{-\frac{1}{2}}d_F(z)^{\frac{1}{2}}.$$

From $A_G(0) = A_F(z)A_\varphi(0)$ and $T_D(0,0) = A_\varphi(0)^*T_D(z,\bar{z})A_\varphi(0)$, we have

$$\lambda_G(0) \geqq \lambda_F(z)\lambda_\varphi(0) \tag{5}$$

and

$$\Lambda_D(z)\lambda_\varphi(0) \geqq \lambda_D(0). \tag{6}$$

Indeed, we may suppose that

$$T_D(z,\bar{z}) = \begin{bmatrix} \mu_1 & 0 & \dots & 0 \\ 0 & \mu_2 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \mu_N \end{bmatrix}, \qquad 0 \leqq \mu_1 \leqq \dots \leqq \mu_N = \Lambda_D(z)^2,$$

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hence

$$\lambda_D(0)^2 = \inf_{\|x\|=1} \langle T_D(0,0)x, x\rangle = \inf_{\|x\|=1} \langle A_\varphi(0)^* T_D(z,\bar{z}) A_\varphi(0)x, x\rangle$$
$$= \inf_{\|x\|=1} \sum_{k=1}^{N} \mu_k \left| \sum_{j=1}^{N} a_{kj} x_j \right|^2 \leqq \Lambda_D(z)^2 \inf_{\|x\|=1} \langle A_\varphi(0)^* A_\varphi(0)x, x\rangle$$
$$= \Lambda_D(z)^2 \lambda_\varphi(0)^2,$$

where $A_\varphi(0) = (a_{ij})$. Therefore we obtain from (4), (5) and (6) that

$$\lambda_F(z)\Lambda_D(z)^{-1} \leqq 2^{\frac{1}{2}} c_2^{-\frac{1}{2}} r_D^{-1} \lambda_D(0)^{-1} d_F(z)^{\frac{1}{2}} \qquad (z \in D).$$

COROLLARY 1. *Let D be a bounded homogeneous domain in $\mathbb{C}^N$ and let F be a holomorphic mapping of D into B_N. Let $\{z^{(n)}\}$ be a sequence of points in D. Then,* $\lim_{n\to\infty} d_F(z^{(n)}) = 0$ *if and only if*

$$\lim_{n\to\infty} |J_F(z^{(n)})| K_D(z^{(n)}, \overline{z^{(n)}})^{-\frac{1}{2}} = 0.$$

COROLLARY 2. *Let D be a bounded homogeneous domain in $\mathbb{C}^N$. Then there exist positive constants C_1^*, C_2^* and C_3^* depending only on D such that*

$$C_1^*\{\text{dist }(z, \partial D)\}^{-1} \leqq K_D(z, \bar{z}) \leqq C_2^*\{\text{dist }(z, \partial D)\}^{-2N}$$

and

$$C_3^*\{\text{dist }(z, \partial D)\}^{-\frac{1}{2}} \leqq \Lambda_D(z),$$

where dist *denotes the euclidean distance in $\mathbb{C}^N$.*

Consequently, for every $\zeta \in \partial D$

$$\lim_{n\to\zeta} K_D(z, \bar{z}) = \infty$$

and

$$\lim_{z\to\zeta} \Lambda_D(z) = \infty.$$

PROOF. Applying Theorem 1 to, for example, the mapping $F(z) = z/M$, $M = \inf\{r : rB_N \supset D\}$, we have this corollary.

Next we consider mappings F which admit sequences $\{z^{(n)}\}$ of points in D such that $z^{(n)} \to \partial D$ as $n \to \infty$ and $\inf |J_F(z^{(n)})| K_D(z^{(n)}, \overline{z^{(n)}})^{-\frac{1}{2}} > 0$, and derive some geometric properties of F.

THEOREM 2. *Let D be a bounded homogeneous domain in $\mathbb{C}^N$ and let F be a holomorphic mapping of D into B_N. A necessary and sufficient condition that F admits a sequence $\{z^{(n)}\}$ of points in D tending to the boundary of D with*

$$(7) \qquad |J_F(z^{(n)})| K_D(z^{(n)}, \overline{z^{(n)}})^{-\frac{1}{2}} \geqq \delta > 0 \qquad (n = 1, 2, \ldots)$$

is that there exists a sequence $\{\Omega_m\}$ of subdomains of D such that Ω_m, $m = 1, 2, \ldots,$ are mutually disjoint and all Ω_m are mapped by F univalently onto the same ball $B(w, r)$.

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PROOF. We may assume that D contains the origin 0. Suppose that there exists a sequence $\{z^{(n)}\}$ of points in D tending to the boundary of D with (7). We may assume that $F(z^{(n)}) \to w$, $w \in \bar{B}_N$, as $n \to \infty$. Set $G_n = F \circ \varphi_n - F(z^{(n)})$, where φ_n is an automorphism of D with $\varphi_n(0) = z^{(n)}$. Then G_n maps $r_D B_N$ into $2B_N$, and $G_n(0) = 0$, further

$$|J_{G_n}(0)| = |J_F(z^{(n)})| K_D(z^{(n)}, \overline{z^{(n)}})^{-\frac{1}{2}} K_D(0,0)^{\frac{1}{2}} \geqq \delta K_D(0,0)^{\frac{1}{2}} = \delta' > 0.$$

Hence it follows from Corollary B that there exists a domain Δ_n, $0 \in \Delta_n \subset \rho B_N$, $\rho = 2^{-N} c_1^* r_D^{N+1} \delta'$, such that F maps $\varphi_n(\Delta_n)$ univalently onto $B_N(F(z^{(n)}), 2r)$, $r = 2^{-2N} c_2^* r_D^{2N} \delta'^2$. Since $F(z^{(n)}) \to w$ as $n \to \infty$, we conclude that there exist domains Ω_n, $n > n_0$, such that F maps Ω_n univalently onto $B_N(w, r)$ and $\Omega_n \subset \varphi_n(\rho B_N)$.

Let K be any compact subset of D. Suppose that $\varphi_n(\rho B_N) \cap K \neq \emptyset$ for $n > n^*$. Take points $\zeta^{(n)}$, $n > n^*$, such that $\zeta^{(n)} \in \rho B_N$ and $\varphi_n(\zeta^{(n)}) \in K$. We may assume that $\zeta^{(n)} \to \zeta^* \in \rho \bar{B}_N$ and $\varphi_n(\zeta^{(n)}) \to w^* \in K$ as $n \to \infty$. Since the family $\{\varphi_n\}$ is normal, we may also assume that $\{\varphi_n\}$ converges to a holomorphic mapping $\varphi : D \to \mathbb{C}^N$ uniformly on every compact subset of D. By H. Cartan's theorem (see [6], p. 78) it follows that either φ is an automorphism of D or $\varphi(D) \subset \partial D$. Since $\varphi(\zeta^*) = \lim_{n\to\infty} \varphi_n(\zeta^{(n)}) = w^*$, φ is an automorphism of D, but $\varphi(0) = \lim_{n\to\infty} \varphi_n(0) = \lim_{n\to\infty} z^{(n)} \in \partial D$. This is a contradiction. Therefore we can choose a subsequence $\{\Omega_m\}$ or $\{\Omega_n\}$ such that Ω_m, $m = 1, 2, \ldots,$ are mutually disjoint.

Conversely, suppose that there exists such a sequence $\{\Omega_m\}$ of subdomains of D. Let $z^{(m)}$ be the point in Ω_m with $F(z^{(m)}) = w$. Then $d_F(z^{(m)}) \geqq r$, and so, by Theorem 1,

$$|J_F(z^{(m)})| K_D(z^{(m)}, \overline{z^{(m)}})^{-\frac{1}{2}} \geqq C_1 r^N = \delta > 0 \qquad (m = 1, 2, \ldots).$$

If $z^{(m_\upsilon)} \to z^* \in D$ for some subsequence $\{z^{(m_\upsilon)}\}$ of $\{z^{(m)}\}$, then $|J_F(z^*)| > \delta K_D(z^*, \overline{z^*})^{\frac{1}{2}} > 0$. Hence F is univalent in a neighborhood of z^*, in contradiction to the fact that $z^{(m_\upsilon)} \to z^*$ as $\upsilon \to \infty$ and $F(z^{(m_\upsilon)}) = w$ for $\upsilon = 1, 2, \ldots$. Thus the sequence $\{z^{(m)}\}$ tends to the boundary of D.

COROLLARY 3. *Let D be a bounded homogeneous domain in $\mathbb{C}^N$ and let F be a holomorphic mapping of D into B_N. If $\zeta \in \partial D$ is a local peak point for $A(D)$ and if there exists a sequence $\{z^{(n)}\}$ of points of D such that $z^{(n)} \to \zeta$ as $n \to \infty$ and $|J_F(z^{(n)})| K_D(z^{(n)}, \overline{z^{(n)}})^{-\frac{1}{2}} \geqq \delta > 0$ for $n = 1, 2, \ldots,$ then the set $\cap_{r>0} F(B_N(\zeta, r) \cap D)$ contains an open set.*

PROOF. It follows from Theorem 2 that there exist domains Ω_n with $F(\Omega_n) = B_N(w, \eta)$. As was shown in the proof of Theorem 2 we may assume that $\Omega_n \subset \varphi_n(\rho B_N)$, where φ_n is an automorphism of D with $\varphi_n(0) = z^{(n)}$. Since $\varphi_n(0) \to \zeta$ as $n \to \infty$ and since ζ is a local peak point for $A(D)$, $\varphi_n(z) \to \zeta$ uniformly on every compact subset of D. (See [7], p. 306). Hence $\cap_{r>0} F(B_N(\zeta, r) \cap D)$ contains $B_N(w, \eta)$.

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The following example shows that we cannot delete the hypothesis that ζ is a local peak point for $A(D)$ from Corollary 3. Define F by

$$F(z) = N^{-\frac{1}{2}}(f(z_1), z_2, \ldots, z_N), \qquad f(\xi) = \exp\left(\frac{\xi - 1}{\xi + 1}\right).$$

Then F is a holomorphic mapping of the unit polydisc U^N into B_N.

There are a sequence $\{\xi^{(n)}\}$ of points in $|\xi| < 1$ and a sequence $\{\Delta_n\}$ of subdomains of $|\xi| < 1$, such that $f(\xi^{(n)}) = e^{-1}$, $\xi^{(n)} \in \Delta_n$, $\xi^{(n)} \to -1$ as $n \to \infty$ and f maps Δ_n univalently onto $\{\xi : |\xi - e^{-1}| < \eta\}$. Set

$$\Omega_n' = \Delta_n \times \{z_2 : |z_2| < \eta\} \times \cdots \times \{z_N : |z_N| < \eta\}$$

and $z_n = (\xi^{(n)}, 0, \ldots, 0)$. Then F maps Ω_n' univalently onto

$$\{w_1 : |w_1 - e^*| < \rho\} \times \{w_2 : |w_2| < \rho\} \times \cdots \times \{w_N : |w_N| < \rho\},$$

where $e^* = N^{-\frac{1}{2}}e^{-1}$ and $\rho = N^{-\frac{1}{2}}\eta$, hence there are infinitely many domains Ω_n such that F maps Ω_n univalently onto $B_N(w^*, \rho)$, $w^* = (e^*, 0, \ldots, 0)$ and $z^{(n)} \in \Omega_n$. The sequence $\{z^{(n)}\}$ tends to the boundary point $z^* = (-1, 0, \ldots, 0)$ of U^N. Since $F(z^{(n)}) = w^*$, by Theorem 1,

$$|J_F(z^{(n)})| K_D(z^{(n)}, \overline{z^{(n)}})^{-\frac{1}{2}} \geqq C_1 \rho^N > 0 \qquad (n = 1, 2, \ldots).$$

However, the set $\cap_{r>0} F(B_N(z^*, r) \cap U^N)$ contains no open sets.

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