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SECOND FOX SUBGROUPS OF ARBITRARY GROUPS

MARIO CURZIO AND C. KANTA GUPTA

ABSTRACT. We give a complete description of the second Fox subgroup $G \cap (1 + \Delta^2(G)\Delta(H))$ relative to a given normal subgroup H of an arbitrary finitely generated group G.

Introduction. Let H be a normal subgroup of a finitely generated group G and let $\Delta(G) = \mathbf{Z}G(G-1)$, $\Delta(H) = \mathbf{Z}G(H-1)$ denote the augmentation ideals of the integral group ring $\mathbf{Z}G$. The n-th Fox subgroup of G relative to H is defined to be $G \cap \big(1 + \Delta^n(G)\Delta(H)\big)$. It is a normal subgroup of G consisting of all elements $g \in G$ with $g - 1 \in \Delta^n(G)\Delta(H)$. Identification of the subgroup $G \cap \big(1 + \Delta^n(G)\Delta(H)\big)$ is the so-called *general Fox problem*. The identification $G \cap \big(1 + \Delta(G)\Delta(H)\big) = [H,H]$ follows easily from the corresponding well-known Magnus-Schumann-Fox theorem when G is assumed to be free ([8; page 6], *cf.* [1], [3]). Identification of the n-th Fox subgroup when G is a free group is now completely known: Enright [2], Hurley [9] and Gupta [4] for $n = 2$; Gupta and Gupta [5] for $H = G'$; N. Gupta [6] for G/H finite; N. Gupta [7], Yunus [13] and Hurley [10] for arbitrary H. We refer the reader to Chapter III of N. Gupta [8] for details. In the general case, when $n = 2$ and G is a split extension of H, a solution can be found in Khambadkone [11]. When G is an arbitrary finitely generated group, the identification of the general n-th Fox subgroup for $n \geq 2$ remains a long-standing open problem. In this paper we resolve the case $n = 2$ by proving that: $G \cap \big(1 + \Delta^2(G)\Delta(H)\big) = [H,H,H][H \cap G', H \cap G']K_G(H)$, where $K_G(H)$ is a certain specifically defined subgroup contained in H' (Theorem B).

Preliminaries. We use notation and terminology from Chapter III of [8]. Let F be a free group of finite rank, and let T, R be normal subgroups of F with $T \leq R$. Denote by $\mathfrak{f} = \mathbf{Z}F(F-1)$, $\mathfrak{r} = \mathbf{Z}F(R-1)$, $\mathfrak{t} = \mathbf{Z}F(T-1)$, the ideals of the free integral group ring $\mathbf{Z}F$ of F. With $G = F/T$ and $H = R/T$, in the language of free group rings, the n-th general Fox subgroup problem translates to the identification of the normal subgroup $F \cap (1 + \mathfrak{f}^n\mathfrak{r} + \mathfrak{t})$ of F. In what follows we shall restrict to the case $n = 2$.

We may assume that $F = \langle x_1, x_2, \ldots, x_m\rangle$ is free of finite rank $m \geq 2$ and that F/R admits a pre-abelian presentation where R is the normal closure

$$(1) \qquad R = \langle x_1^{e_1}\xi_1, x_2^{e_2}\xi_2, \ldots, x_m^{e_m}\xi_m, \xi_{m+1}, \xi_{m+2}, \ldots\rangle^F$$

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with $e_m \mid e_{m-1} \mid \cdots \mid e_1 \geq 0$, $\xi_i \in F' = [F,F]$, $i = 1,2,\dots$ (see, for instance, [12, Section 3.3]), $T \leq R$. Being a subgroup of the free group F, R is itself a free group and we may assume that

$$(2) \qquad R = \mathrm{sgp}\{r_1, r_2, \dots, r_m, r_{m+1}, r_{m+2}, \dots\},$$

where $r_j \in F'$ for $j \geq m+1$ and $r_i = x_i^{e_i}\xi_i$ for $1 \leq i \leq m$ and $T \leq R$. Modulo $[R \cap F', R \cap F'][R,R,R]$, every element $w \in R'$ can be written as

$$(3) \qquad w = \prod_{1 \leq i < j \leq m} [r_i, r_j]^{a_{ij}} \prod_{\substack{1 \leq k \leq m \\ q \geq m+1}} [r_k, r_q]^{b_{kq}},$$

where $a_{ij}, b_{kq} \in Z$.

For $w \in R'$ as in (3), define

$$(4) \qquad y_k(w) = \prod_{i<k} r_i^{-a_{ik}} \prod_{k<j} r_j^{a_{kj}} \in R,$$

$$(5) \qquad z_k(w) = \prod_{q \geq m+1} r_q^{b_{kq}} \in R \cap F'.$$

THE SECOND FOX SUBGROUPS. Let F be a free group of finite rank and T, R be normal subgroups of F with $T \leq R$. Define

$$(6) \qquad W = \mathrm{sgp}\{w \in R' \mid \big(y_k(w)z_k(w)\big)^{e_k} \in R'T \text{ for all } 1 \leq k \leq m\},$$

where $y_k(w)$, $z_k(w)$ are as defined in (4), (5).

We state and prove our main result as:

THEOREM A. *Let R, T, W be subgroups of the free group $F = \langle x_1, x_2, \dots, x_m\rangle$ as defined by (1), (2) and (6). Then $F \cap (1 + \mathfrak{f}^2\mathfrak{r} + \mathfrak{t}) = W[R \cap F', R \cap F'][R,R,R]T$.*

PROOF. Since $[R \cap F', R \cap F'] - 1 \subseteq (\mathfrak{r} \cap \mathfrak{f}^2)(\mathfrak{r} \cap \mathfrak{f}^2) \subseteq \mathfrak{f}^2\mathfrak{r}$, $[R,R,R] - 1 \subseteq \mathfrak{r}\mathfrak{r}\mathfrak{r} \subseteq \mathfrak{f}^2\mathfrak{r}$ and $T - 1 \subseteq \mathfrak{t}$, it follows that each of the factors $[R \cap F', R \cap F']$, $[R,R,R]$ and T is contained in $F \cap (1 + \mathfrak{f}^2\mathfrak{r} + \mathfrak{t})$. To see that W is also contained in $F \cap (1 + \mathfrak{f}^2\mathfrak{r} + \mathfrak{t})$, let $w \in R'$ be an arbitrary generating element of W as defined by (6). Then, by (3),

$$w = \prod_{1 \leq i < j \leq m} [r_i, r_j]^{a_{ij}} \prod_{\substack{1 \leq k \leq m \\ q \geq m+1}} [r_k, r_q]^{b_{kq}}, \quad a_{ij},\ b_{kq} \in Z,$$

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and expansion of $w-1$ modulo $\mathfrak{f}^2\mathfrak{r}$ gives

$$\begin{aligned} w-1 &\equiv \sum_{1\le i<j\le m}\{a_{ij}(r_i-1)(r_j-1)-a_{ij}(r_j-1)(r_i-1)\} \\ &\quad + \sum_{1\le k\le m}(r_k-1)\Big(\prod_{q\ge m+1} r_q^{b_{kq}}-1\Big) \\ &\equiv \sum_{1\le i<j\le m}\{(r_i-1)(r_j^{a_{ij}}-1)+(r_j-1)(r_i^{-a_{ij}}-1)\} \\ &\quad + \sum_{1\le k\le m}(r_k-1)\big(z_k(w)-1\big) \quad \text{(by (5))} \\ &\equiv \sum_{1\le k\le m}(r_k-1)\Big(\prod_{i<k} r_i^{-a_{ik}}\prod_{k<j} r_j^{a_{kj}}-1\Big)+\sum_{1\le k\le m}(r_k-1)\big(z_k(w)-1\big) \\ &\equiv \sum_{1\le k\le m}(r_k-1)\big(y_k(w)-1\big)+\sum_{1\le k\le m}(r_k-1)\big(z_k(w)-1\big) \quad \text{(by (4))} \\ &\equiv \sum_{1\le k\le m}(x_k^{e_k}-1)\big(y_k(w)z_k(w)-1\big) \\ &\equiv \sum_{1\le k\le m} e_k(x_k-1)\big(y_k(w)z_k(w)-1\big) \\ &\equiv \sum_{1\le k\le m}(x_k-1)\Big(\big(y_k(w)z_k(w)\big)^{e_k}-1\Big). \end{aligned}$$

Thus, by (6), $w-1\subseteq \mathfrak{f}(\mathfrak{r}^2+\mathfrak{t})\subseteq \mathfrak{f}^2\mathfrak{r}+\mathfrak{t}$ and consequently,

$$W[R\cap F', R\cap F'][R,R,R]T \le F\cap(1+\mathfrak{f}^2\mathfrak{r}+\mathfrak{t}).$$

For the reverse inequality, we set

$$X = W[R\cap F', R\cap F'][R,R,R]T$$

and assume by way of contradiction that

$$f\in F\cap(1+\mathfrak{f}^2\mathfrak{r}+\mathfrak{t}) \quad \text{and}\ f\notin X.$$

Then, for all $x\in X$,

$$fx\in F\cap(1+\mathfrak{f}^2\mathfrak{r}+\mathfrak{t}) \quad \text{and}\ fx\notin X.$$

It follows that for each x there exists $t_x\in T$ such that $fx-1\equiv t_x-1 \pmod{\mathfrak{f}^2\mathfrak{r}+\mathfrak{f}\mathfrak{t}}$. Equivalently, $fxt_x^{-1}\in F\cap(1+\mathfrak{f}^2\mathfrak{r}+\mathfrak{f}\mathfrak{t})$. Replacing x by xt_x^{-1}, if necessary, we may assume that, for all $x\in X$,

$$fx\in F\cap(1+\mathfrak{f}^2\mathfrak{r}+\mathfrak{f}\mathfrak{t}) \quad \text{and}\ fx\notin X. \tag{7}$$

Since $F\cap(1+\mathfrak{f}^2\mathfrak{r}+\mathfrak{f}\mathfrak{t})\le F\cap(1+\mathfrak{f}\mathfrak{r})=R'$, by (3) we may write

$$fx\equiv \prod_{1\le i<j\le m}[r_i,r_j]^{a_{ij}}\prod_{\substack{1\le k\le m\\ q\ge m+1}}[r_k,r_q]^{b_{kq}} \pmod{[R\cap F', R\cap F'][R,R,R]},$$

where $a_{ij}, b_{kq}\in Z$.

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Expansion of $fx - 1$ modulo $\mathfrak{f}^2\mathfrak{r}$ gives, as before,

$$fx - 1 \equiv \sum_{1\le k\le m} (x_k - 1)\Big(\big(y_k(fx)\, z_k(fx)\big)^{e_k} - 1\Big), \tag{8}$$

where $y_k(fx)$, $z_k(fx)$ are defined by (4) and (5).

Since, by hypothesis, $fx - 1 \in \mathfrak{f}^2\mathfrak{r} + \mathfrak{f}\mathfrak{t} = \mathfrak{f}(\mathfrak{f}\mathfrak{r} + \mathfrak{t})$, it follows from (8) that

$$\sum_{1\le k\le m} (x_k - 1)\Big(\big(y_k(fx)\, z_k(fx)\big)^{e_k} - 1\Big) \in \mathfrak{f}(\mathfrak{f}\mathfrak{r} + \mathfrak{t}). \tag{9}$$

Now, since $\mathfrak{f}$ is a free right ZF-module with basis $\{x_k - 1\,;\, 1 \le k \le m\}$ (see [3] or [8]), (9) yields

$$\Big(\big(y_k(fx)\, z_k(fx)\big)^{e_k} - 1\Big) \in \mathfrak{f}\mathfrak{r} + \mathfrak{t} \quad \text{for all } k = 1, \ldots, m.$$

Since $F \cap (1 + \mathfrak{f}\mathfrak{r} + \mathfrak{t}) = R'T$ (see [8]), it follows that $\big(y_k(fx)\, z_k(fx)\big)^{e_k} \in R'T$ for each k. By (6), this yields $fx \in W$ which, in turn, implies $f \in WX = X$, contrary to the choice of f (by (7)). This completes the proof of the theorem.

Let H be a normal subgroup of a finitely generated group G. We may choose a set $\{g_1, \ldots, g_m\}$ of elements of G so that

(i) G/G' is generated by $\{g_1G', \ldots, g_mG'\}$;

(ii) HG'/G' is generated by $\{h_1G', \ldots, h_mG'\}$ with $h_i = g_i^{e_i}$, $e_i \ge 0$ for each i.

For each $g \in H'$ of the form

$$g \equiv \prod_{1\le i<j\le m} [h_i, h_j]^{a_{ij}} \pmod{[H, H, H]},$$

put

$$y_k(g) = \Big(\prod_{i<k} h_i^{-a_{ik}} \prod_{k<j} h_j^{a_{kj}}\Big), \qquad 1 \le k \le m.$$

Define

$$K_G(H) = \mathrm{sgp}\Big\{g \equiv \prod_{1\le i<j\le m} [h_i, h_j]^{a_{ij}}\ ;\ y_k(g)^{e_k} \in H'(H \cap G')^{e_k}, 1 \le k \le m\Big\}.$$

Then, with $G = F/T$ and $H = R/T$, we have the natural isomorphisms $ZG \cong ZF/\mathfrak{t}$ and $ZH \cong \mathfrak{r}/\mathfrak{t}$ which translate the subgroup W of F given by (6) to the subgroup $K_G(H)$ defined above. Thus, we may state Theorem A as,

THEOREM B. $G \cap \big(1 + \Delta^2(G)\Delta(H)\big) = K_G(H)[H, H, H][H \cap G', H \cap G']$.

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Department of Mathematics
University of Napoli
via Cinitia
Napoli 80126
Italy

Department of Mathematics
University of Manitoba
Winnipeg, Manitoba
R3T 2N2