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INVARIANT KÄHLER METRICS AND PROJECTIVE EMBEDDINGS OF THE FLAG MANIFOLD

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We determine explicitly the space of invariant Hermitian and Kähler metrics on the flag manifold. In particular, we show that a Killing metric is not Kähler. The Chern forms are also computed in terms of the Maurer–Cartan form, and this calculation is used to prove that the flag manifold is projective algebraic. An explicit projective embedding of the flag manifold is also given.

1. INTRODUCTION

Let G be a simple compact connected Lie group and also let T be a maximal torus in G. The coset space G/T is called a flag manifold. For example, taking $G = SU(n)$ and $T = S(U(1)^n)$ we obtain

$$G/T = F_{1,2,\ldots,n}(\mathbb{C}^n),$$

which is the space of all flags in $\mathbb{C}^n$. Flag manifolds are important in that they are the basic building blocks of all compact homogeneous complex spaces [3]. Moreover, a flag manifold is nonsymmetric and Kähler–Einstein of nonconstant holomorphic sectional curvature; hence, exhibits interesting differential geometric properties not encountered in, say, the complex projective space.

In this paper we take the space $F_{1,2,3}(\mathbb{C}^3) = SU(3)/S\left(U(1)^3\right)$ and determine explicitly the space of invariant Hermitian as well as Kähler metrics. In particular, we show that a Killing metric is not Kähler. We also compute the Chern forms of $F_{1,2,3}(\mathbb{C}^3)$ and show that the first Chern form is positive, thus establishing via the Kodaira embedding theorem that the flag manifold is projective algebraic. A noteworthy feature of our exposition is that the calculations are made quite explicitly in terms of the Maurer–Cartan form of $SU(3)$. Moreover, it will be made clear that a similar analysis applies to any flag manifold.

The second section of our paper contains a description of an embedding of $F_{1,2,3}(\mathbb{C}^3)$ into the complex Grassmannian $\mathrm{Gr}(3, 8)$, and is based upon the work [2]. (In general, an arbitrary flag manifold of complex dimension n can be embedded in a

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similar way into the complex Grassmannian $\mathrm{Gr}(n, N)$, where N is the real dimension of G). Via the Plücker embedding $\mathrm{Gr}(3, 8) \hookrightarrow \mathbb{P}^{55}$ we can thus realise $F_{1,2,3}(\mathbb{C}^3)$ as a smooth projective variety. We should mention that the flag manifolds are essentially the only known Kähler–Einstein smooth projective varieties that are not of constant holomorphic sectional curvature.

1. Invariant Metrics

We consider the complex flag manifold

$$F(\mathbb{C}^3) = F_{1,2,3}(\mathbb{C}^3) = SU(3)/S\big(U(1)^3\big).$$

The group $S\big(U(1)^3\big)$, diagonally included in $SU(3)$, is the isotropy subgroup at the reference flag

$$[\varepsilon_1] \subset [\varepsilon_1 \wedge \varepsilon_2] \subset \mathbb{C}^3,$$

where (ε_i) denotes the canonical basis for $\mathbb{C}^3$.

Let ε_{ij} denote the 3 by 3 matrix with $+1$ at the (i, j)-th entry and zeros elsewhere, and put

$$E_{ij} = \varepsilon_{ij} - \varepsilon_{ji}, \quad F_{ij} = \varepsilon_{ij} + \varepsilon_{ji}.$$

Then the Lie algebra of $SU(3)$ decomposes as

$$\mathfrak{su}(3) = \mathfrak{t} \oplus \sum_{i<j} V_{ij},$$

where $\mathfrak{t}$ denotes the Lie algebra of the maximal torus $T = S\big(U(1)^3\big)$, and

$$V_{ij} = \mathbb{R}E_{ij} \oplus \sqrt{-1}\mathbb{R}F_{ij}.$$

The spaces (V_{ij}) are the root spaces of $SU(3)$ with respect to T.

The vector subspace

$$\mathfrak{m} = \oplus \sum_{i<j} V_{ij} \subset \mathfrak{su}(3)$$

is an $\mathrm{Ad}(T)$-invariant complement to $\mathfrak{t}$, and it will be identified with the tangent space to $F(\mathbb{C}^3)$ at the identity coset via π_{*e}, where

$$\pi\colon SU(3) \to F(\mathbb{C}^3), \quad g \mapsto gT.$$

The $\mathfrak{su}(3)$-valued Maurer-Cartan form $\Omega = (\Omega^i_j)_{1 \leqslant i, j \leqslant 3}$ decomposes into

$$\Omega = \Omega_{\mathfrak{t}} \oplus \sum_{i<j} \Omega_{V_{ij}},$$

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where $\Omega_{V_{ij}}$ denotes the V_{ij}-component of Ω. So

$$\Omega_{V_{ij}} = \operatorname{Re}\Omega^i_j \otimes E_{ij} + \operatorname{Im}\Omega^i_j \otimes F_{ij}.$$

The standard complex structure of $F(\mathbb{C}^3)$ is given by letting the pullbacks of the following complex-valued 1-forms to be of type $(1,0)$:

$$\Omega^1_2,\ \Omega^1_3,\ \Omega^2_3.$$

Note that the real and imaginary parts of these forms constitute the m-component of the Maurer–Cartan form. By way of notation we put

$$\omega^i_j = s^*\Omega^i_j,$$

where s is a local section of the principal fibration $SU(3) \to F(\mathbb{C}^3)$.

THEOREM. *Any invariant Hermitian metric on $F(\mathbb{C}^3)$ is given by the $\operatorname{Ad}(T)$-invariant tensor product*

$$ds^2_{(a,b,c)} = a^2\omega^1_2 \otimes \overline{\omega}^1_2 + b^2\omega^1_3 \otimes \overline{\omega}^1_3 + c^2\omega^2_3 \otimes \overline{\omega}^2_3,$$

where a, b, c are positive constants. Thus the totality of invariant Hermitian metrics on $F(\mathbb{C}^3)$ is naturally parameterised by $(\mathbb{R}^+)^3$.

The above result is a straightforward consequence of the following rather general consideration. Let G be a simple compact connected Lie group, and consider the root space decomposition with respect to a maximal torus T

$$\mathfrak{g} = \mathfrak{t} \oplus \sum_{i=1}^{n} V_i,$$

Recall that any two bi-invariant metrics on G are constant multiples of each other; hence, the root spaces of a simple Lie group are defined canonically. The subspace $\mathfrak{m} = \oplus \sum_{i=1}^{n} V_i$ is identified with the tangent space $T_T(G/T)$; an invariant metric on G/T corresponds, via restriction, to an $\operatorname{Ad}(T)$-invariant inner product in $\mathfrak{m}$. Now $\operatorname{Ad}(T)$ restricted to each V_i is irredicible, and consequently each V_i possesses only a one-dimensional family of $\operatorname{Ad}(T)$-invariant inner products. Thus the totality of $\operatorname{Ad}(T)$-invariant inner products in $\mathfrak{m}$ is given by

$$\left\{\sum c_i \cdot \kappa\,|_{V_i},\ c_i < 0\right\} \cong (\mathbb{R}^+)^n,$$

where κ denotes the Killing form of G.

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Theorem. *The metric $ds^2_{(a,b,c)}$ on $F(\mathbb{C}^3)$ is Kähler if and only if*

$$(a,\, b,\, c) = \lambda\left(1,\, \sqrt{2},\, 1\right)$$

for some $\lambda \in \mathbb{R}^+$.

Proof: A unitary coframe for the metric $ds^2_{(a,b,c)}$ is given by

$$\theta^1 = a\omega^1_2, \quad \theta^2 = b\omega^1_3, \quad \theta^3 = c\omega^2_3.$$

We then have

$$d\theta^i = -\psi^i_j \wedge \theta^j + \tau^i,$$

where $\left(\psi^i_j\right)$ is the $\mathfrak{u}(3)$-valued connection form and $\left(\tau^i\right)$ are the torsion forms. It is well-known that the metric $ds^2_{(a,b,c)}$ is Kähler if and only if the torsion forms vanish identically. Using the Maurer–Cartan structure equations of $SU(3)$ we calculate that

$$d\begin{bmatrix}\theta^1\\ \theta^2\\ \theta^3\end{bmatrix} = \begin{bmatrix}\omega^1_1-\omega^2_2 & \frac{a}{bc}\overline{\theta}^3 & 0\\ \frac{-b}{2ac}\theta^3 & \omega^1_1-\omega^3_3 & \frac{b}{2ac}\theta^1\\ 0 & \frac{-c}{ab}\overline{\theta}^1 & \omega^2_2-\omega^3_3\end{bmatrix} \wedge \begin{bmatrix}\theta^1\\ \theta^2\\ \theta^3\end{bmatrix}.$$

It follows that $\left(\tau^i\right) \equiv 0$ if and only if

$$\frac{a}{bc} = \frac{b}{2ac} \quad \text{and} \quad \frac{b}{2ac} = \frac{c}{ab}.$$

And this is so if and only if $(a,\, b,\, c) = \lambda\left(1,\, \sqrt{2},\, 1\right)$ for some $\lambda > 0$. ∎

Definition: The metric $ds^2_{(1,\sqrt{2},1)}$ will be called the normal Kähler metric.

Remark. In general, on a flag manifold G/T of complex dimension n the space of invariant Hermitian metrics is parameterised by $(\mathbb{R}^+)^n$. And amongst these exactly $\mathbb{R}^+$ many of them are Kähler. More precisely, any two invariant Kähler metrics on G/T are homothetically equivalent to each other.

A Killing metric on $F\left(\mathbb{C}^3\right)$ is, by definition, a Hermitian metric coming from a negative multiple of the Killing form restricted to $\mathfrak{m}$. Since for any X, Y in the Lie algebra $\mathfrak{su}(n)$

$$\operatorname{trace}\left(\operatorname{ad}_X \circ \operatorname{ad}_Y\right) = c \cdot \left(\operatorname{trace}\left(X \cdot Y\right)\right)$$

for some dimensional constant c, we see that a Killing metric is given by

$$ds^2_{\lambda(1,1,1)}, \quad \lambda \in \mathbb{R}^+.$$

This observation combined with the above theorem yields the following somewhat surprising corollary.

Corollary. *A Killing metric on $F(\mathbb{C}^3)$ is not Kahler.*

We now compute the Chern forms of $F(\mathbb{C}^3)$ using the normal Kähler metric $ds^2_{(1,\sqrt{2},1)}$. From the computation above we see that the connection matrix of this metric with respect to the unitary coframe

$$\theta^1 = \omega_2^1, \quad \theta^2 = \sqrt{2}\omega_3^1, \quad \theta^3 = \omega_3^2$$

is given by

$$(\psi_j^i) = \begin{bmatrix} \omega_1^1 - \omega_2^2 & (1/\sqrt{2})\overline{\theta}^3 & 0 \\ (-1/\sqrt{2})\theta^3 & \omega_1^1 - \omega_3^3 & (1/\sqrt{2})\theta^1 \\ 0 & (-1/\sqrt{2})\overline{\theta}^1 & \omega_2^2 - \omega_3^3 \end{bmatrix}.$$

The curvature forms $\chi = (\chi_j^i)$ are computed from the formulae

$$\chi_j^i = d\psi_j^i + \psi_k^i \wedge \psi_j^k.$$

We calculate that

$$\chi_1^1 = 2\theta^1 \wedge \overline{\theta}^1 + \frac{1}{2}\theta^2 \wedge \overline{\theta}^2 - \frac{1}{2}\theta^3 \wedge \overline{\theta}^3,$$

$$\chi_2^2 = \frac{1}{2}\theta^1 \wedge \overline{\theta}^1 + \theta^2 \wedge \overline{\theta}^2 + \frac{1}{2}\theta^3 \wedge \overline{\theta}^3,$$

$$\chi_3^3 = -\frac{1}{2}\theta^1 \wedge \overline{\theta}^1 + \frac{1}{2}\theta^2 \wedge \overline{\theta}^2 + 2\theta^3 \wedge \overline{\theta}^3,$$

$$\chi_2^1 = \frac{1}{2}\theta^1 \wedge \overline{\theta}^2, \quad \chi_3^1 = -\frac{1}{2}\theta^1 \wedge \overline{\theta}^3, \quad \chi_3^2 = \frac{1}{2}\theta^2 \wedge \overline{\theta}^3,$$

$$\chi_j^i = -\overline{\chi}_i^j.$$

Let $c_k(\chi)$, $1 \leqslant k \leqslant 3$, denote the k-th Chern form of $F(\mathbb{C}^3)$ constructed using χ so that

$$c_1(\chi) = \frac{i}{2\pi} \operatorname{trace} \chi,$$

$$c_2(\chi) = \left(\frac{i}{2\pi}\right)^2 \sum_{i<j} \left(\chi_i^i \wedge \chi_j^j - \chi_j^i \wedge \chi_i^j\right),$$

$$c_3(\chi) = \left(\frac{i}{2\pi}\right)^3 \det \chi.$$

We find that

$$c_1(\chi) = \frac{i}{\pi} \sum \theta^i \wedge \overline{\theta}^i, \tag{1}$$

$$c_2(\chi) = (-3/4\pi^2)\left(\theta^1 \wedge \overline{\theta}^1 \wedge \theta^2 \wedge \overline{\theta}^2 + 2\theta^1 \wedge \overline{\theta}^1 \wedge \theta^3 \wedge \overline{\theta}^3 + \theta^2 \wedge \overline{\theta}^2 \wedge \theta^3 \wedge \overline{\theta}^3\right), \tag{2}$$

$$c_3(\chi) = (-3i/4\pi^3)\left(\theta^1 \wedge \overline{\theta}^1 \wedge \theta^2 \wedge \overline{\theta}^2 \wedge \theta^3 \wedge \overline{\theta}^3\right). \tag{3}$$

THEOREM. *The flag manifold $F(\mathbb{C}^3)$ equipped with the normal Kähler metric is Kähler-Einstein with constant scalar curvature 24.*

PROOF: The Kähler form of $\left(F(\mathbb{C}^3),\, ds^2_{(1,\sqrt{2},1)}\right)$ is given by

$$\Lambda = \frac{i}{2}\sum \theta^i \wedge \overline{\theta}^i.$$

Then from (1) we see that

$$c_1(\chi) = \frac{2}{\pi}\Lambda,$$

showing that $F(\mathbb{C}^3)$ is Kähler–Einstein. Now the scalar curvature s satisfies

$$c_1(\chi) = (s/12\pi)\Lambda,$$

and $s = 24$. ∎

Incidentally, the Kähler manifold $\left(F(\mathbb{C}^3),\, ds^2_{(1,\sqrt{2},1)}\right)$ is not of constant holomorphic sectional curvature. To see this recall that the curvature forms $\left(\chi^\alpha_\beta\right)$, written relative to a unitary coframe (θ^α), of a Kähler manifold with constant holomorphic sectional curvature c are given by

$$\chi^\alpha_\beta = \frac{c}{4}\left(\theta^\alpha \wedge \overline{\theta}^\beta + \delta^\alpha_\beta \sum \theta^\gamma \wedge \overline{\theta}^\gamma\right).$$

THEOREM. *The flag manifold $F(\mathbb{C}^3)$ is projective algebraic.*

PROOF: The formula (1) shows that the first Chern class of $F(\mathbb{C}^3)$ is positive. Then the anticanonical line bundle $K^* \to F(\mathbb{C}^3)$ must be ample since

$$c_1(F(\mathbb{C}^3)) = c_1(\Lambda^3(F(\mathbb{C}^3))) = c_1(K^*).$$

Thus by the Kodaira embedding theorem a suitable pluri-anticanonical linear system (that is, the linear system of divisors associated with a large enough positive integral power of K^*) gives rise to a projective embedding of $F(\mathbb{C}^3)$. ∎

REMARK. A similar consideration shows that any flag manifold is projective algebraic. In fact, [2] shows that a flag manifold is a rational variety.

2. The flag manifold as a subvariety of the complex Grassmannian

Let G be a semisimple simply connected and connected compact Lie group, and fix a maximal torus $T \subset G$. Then there is a unique holomorphic Lie group G^C with Lie algebra $\mathfrak{g}^C$ (the complexification of $\mathfrak{g}$) containing G.

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REMARK. We are making the simple connectivity assumption here merely to avoid certain technical complications. After all, if $\widetilde{G}$ is the quotient of G by any finite invariant subgroup and $\widetilde{T}$ is the image of T under the projection $G \to \widetilde{G}$, then the spaces G/T and $\widetilde{G}/\widetilde{T}$ are well-known to be diffeomorphic.

A root of the holomorphic Lie group G^C is an element α of $\left(\mathfrak{t}^C\right)^*$ such that the root space

$$\mathfrak{g}_\alpha = \{v \in \mathfrak{g}^C : \mathrm{ad}_h(v) = [h, v] = \alpha(h)(v),\ h \in \mathfrak{t}^C\}$$

is nontrivial. The set of all roots of G^C will be denoted by $\Delta \subset \left(\mathfrak{t}^C\right)^*$. We then have the root space decomposition

$$\mathfrak{g}^C = \mathfrak{t}^C \oplus \sum_{\alpha\in\Delta} \mathfrak{g}_\alpha.$$

We fix a system of positive roots in Δ, and write

$$\Delta = \Delta_+ \cup \Delta_-.$$

We then put

$$\mathfrak{b} = \mathfrak{t}^C \oplus \sum_{\alpha\in\Delta^+} \mathfrak{g}_\alpha, \quad \mathfrak{n} = \oplus \sum_{\alpha\in\Delta^-} \mathfrak{g}_\alpha.$$

The algebra $\mathfrak{b}$ is a Borel subalgebra and $\mathfrak{n}$ is nilpotent. We let B (respectively, N) denote the analytic subgroup of $\mathfrak{b}$ (respectively, $\mathfrak{n}$) in G^C. It can then be verified that

$$G \cap B = T, \quad G \cap N = \{e\},$$

implying that the map

$$G/T \to G^C/B, \quad gT \mapsto gB$$

is a diffeomorphism.

We are interested in the case

$$G = SU(3), \quad T = S\left(U(1)^3\right),$$

$$G^C = SL(3, \mathbb{C}), \quad B = \{\text{upper triangular matrices}\}.$$

We shall identify the complex Lie algebra $\mathfrak{sl}(3, \mathbb{C})$ with $\mathbb{C}^8$ via the map

$$e_a \mapsto \varepsilon_a, \quad 1 \leqslant a \leqslant 8,$$

where

$$e_1 = \varepsilon_{13}, \quad e_2 = \varepsilon_{12}, \quad e_3 = \varepsilon_{23},$$

$$e_4 = \varepsilon_{11} - \varepsilon_{22}, \quad e_5 = \varepsilon_{22} - \varepsilon_{33},$$

$$e_6 = \varepsilon_{32}, \quad \varepsilon_7 = \varepsilon_{21}, \quad \varepsilon_8 = \varepsilon_{13}.$$

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Note that

$$\mathfrak{t}^C = \operatorname{span}\{e_4, e_5\}.$$

In addition, the roots of $SL(3, \mathbb{C})$ corresponding to the root vectors e_1, e_2 and e_3 form a system of positive roots Δ_+ so that

$$\mathfrak{b} = \mathfrak{t}^C \oplus \mathbb{C}e_1 \oplus \mathbb{C}e_2 \oplus \mathbb{C}e_3,$$
$$\mathfrak{n} = \mathbb{C}e_6 \oplus \mathbb{C}e_7 \oplus \mathbb{C}_8.$$

From the formula

$$[\varepsilon_{ij}, \varepsilon_{kl}] = \delta_{jk}\varepsilon_{il} - \delta_{li}\varepsilon_{kj}$$

we compute the image of the adjoint map ad: $\mathfrak{sl}(3, \mathbb{C}) \to \mathfrak{gl}(k, \mathbb{C})$, where $\mathfrak{gl}(8, \mathbb{C})$ is the set of all 8 by 8 complex matrices. Write

$$\operatorname{ad}(\mathfrak{sl}(3, \mathbb{C})) = \left\{X = \sum_{a=1}^{8} x_a \operatorname{ad}(e_a) : x_a \in \mathbb{C}\right\}.$$

Calculations show that X is given by

(*)

$$\begin{bmatrix} x_4 + x_5 & -x_3 & x_2 & -x_1 & -x_1 & 0 & 0 & 0 \\ -x_6 & 2x_4 - x_5 & 0 & -2x_2 & x_2 & x_1 & 0 & 0 \\ x_7 & 0 & -x_4 + 2x_5 & x_3 & -2x_3 & 0 & -x_1 & 0 \\ -x_8 & -x_7 & 0 & 0 & 0 & 0 & x_2 & x_1 \\ -x_8 & 0 & -x_6 & 0 & 0 & x_3 & 0 & x_1 \\ 0 & x_8 & 0 & x_6 & 2x_6 & x_4 - 2x_5 & 0 & -x_2 \\ 0 & 0 & -x_8 & 2x_7 & -x_7 & 0 & -2x_4 + x_5 & x_3 \\ 0 & 0 & 0 & x_8 & x_8 & -x_7 & x_6 & -x_4 - x_5 \end{bmatrix}.$$

And $\operatorname{ad}(\mathfrak{b})$ consists of those matrices with $x_6 = x_7 = x_8 = 0$.

Let $GL(8, \mathbb{C})$ act on the complex Grassmannian $\operatorname{Gr}(3, 8)$ in the usual manner, and also let K denote the isotropy subgroup at the 3-plane $[\varepsilon_1 \wedge \varepsilon_2 \wedge \varepsilon_3]$. Thus

$$K = \left\{\begin{bmatrix} A & * \\ 0 & * \end{bmatrix} \in GL(8, \mathbb{C}) : A \in GL(3, \mathbb{C})\right\}.$$

The Lie algebra of K is given by

$$\mathfrak{f} = \left\{\begin{bmatrix} X & * \\ 0 & * \end{bmatrix} \in \mathfrak{gl}(8, \mathbb{C}) : X \in \mathfrak{gl}(3, \mathbb{C})\right\}.$$

From (*) we then observe that

(†) $$\operatorname{ad}(\mathfrak{b}) = \mathfrak{f} \cap \operatorname{ad}(\mathfrak{sl}(3, \mathbb{C})).$$

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Let $G_1 \subset GL(8, \mathbb{C})$ denote the group generated by $\mathrm{ad}(\mathfrak{sl}(3, \mathbb{C}))$, and let $B_1 \subset GL(8, \mathbb{C})$ denote the group generated by $\mathrm{ad}(\mathfrak{b})$. Then G_1 is locally isomorphic to G^C and B_1 is locally isomorphic to B; in such a case it is well-known (see [**1**], for example) that the spaces G_1/B_1 and G^C/B are biholomorphically identified with each other. Moreover, (†) shows that the map

$$\Phi \colon G_1/B_1 \to GL(8)/K, \quad gB_1 \mapsto gK$$

is a well-defined monomorphism. We have thus arrived at the following theorem.

Theorem. *The flag manifold* $F(\mathbb{C}^3) = G_1/B_1$ *is a smooth subvariety of the complex Grassmannian* $\mathrm{Gr}(3, 8)$ *via the map* Φ.

It would be quite interesting to relate the projective embedding

$$G_1/B_1 \subset \mathrm{Gr}(3, 8) \subset \mathbb{P}^{55}$$

to a pluri-anticanonical projective embedding of $F(\mathbb{C}^3)$, whose existence we established earlier.

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