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Nuij Type Pencils of Hyperbolic Polynomials

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Abstract. Nuij's theorem states that if a polynomial $p \in \mathbb{R}[z]$ is hyperbolic (*i.e.*, has only real roots), then $p + sp'$ is also hyperbolic for any $s \in \mathbb{R}$. We study other perturbations of hyperbolic polynomials of the form $p_a(z,s) := p(z) + \sum_{k=1}^{d} a_k s^k p^{(k)}(z)$. We give a full characterization of those $a = (a_1, \dots, a_d) \in \mathbb{R}^d$ for which $p_a(z,s)$ is a pencil of hyperbolic polynomials. We also give a full characterization of those $a = (a_1, \dots, a_d) \in \mathbb{R}^d$ for which the associated families $p_a(z,s)$ admit universal determinantal representations. In fact, we show that all these sequences come from special symmetric Toeplitz matrices.

1 Introduction

Hyperbolic polynomials emerged from PDE's (*cf.* Gårding [2]), and they now appear in various branches of mathematics; see for instance an excellent survey of Pemantle [8] for applications in combinatorics. In real algebraic geometry many activities concern hyperbolic polynomials and their determinantal representations. Vinnikov's survey [11] is a good source on recent developments in this subject. The goal of this paper is a study of 1-parameter families of hyperbolic polynomials and their universal determinantal representations. Recall that a polynomial $p \in \mathbb{R}[z]$ is called *hyperbolic* if all its roots are real. Clearly any monic hyperbolic polynomial of degree d is a characteristic polynomial of a symmetric $d \times d$ matrix. First, we recall the following theorem proved by W. Nuij [7].

Theorem 1.1 *Let $p \in \mathbb{R}[z]$ be a hyperbolic polynomial; then $p + sp'$ is hyperbolic for any $s \in \mathbb{R}$.*

We give below a proof of this result, based on the existence of determinantal representation of the family of the polynomials $p + sp'$, $s \in \mathbb{R}$. In fact, we state and prove a generalization of Nuij's result. To this end, we propose the following definition.

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Definition 1.2 We say that $a = (a_1, \dots, a_d) \in \mathbb{R}^d$ is a *Nuij sequence* if for any hyperbolic polynomial p of degree d, the polynomial

$$p_a(z,s) := p(z) + \sum_{k=1}^{d} a_k s^k p^{(k)}(z) \in \mathbb{R}[z], \tag{1.1}$$

is hyperbolic for any $s \in \mathbb{R}$. We denote by $\mathcal{N}_d$ the set of all Nuij sequences in $\mathbb{R}^d$.

Note that by Theorem 1.1, $a = (1, 0, \dots, 0)$ is a Nuij sequence for any $d \in \mathbb{N}$, $d \geq 1$. On the other hand, repeated application of Theorem 1.1 also produces Nuij sequences; for instance, we have

$$p + sp' + s(p + sp')' = p + 2sp' + s^2 p''.$$

Hence, $(2, 1, 0, \dots, 0)$ is a Nuij sequence for any $d \in \mathbb{N}$, $d \geq 2$. In Section 3 we shall see, however, that there is an essential difference between those two families, with respect to their determinantal representations.

Surprisingly, the set $\mathcal{N}_d$ has a nice explicit description.

Theorem A *A sequence $a = (a_1, \dots, a_d) \in \mathbb{R}^d$ is a Nuij sequence if and only if the polynomial*

$$q_a(z) := z^d + \sum_{k=1}^{d} a_k (z^d)^{(k)} = z^d + \sum_{k=1}^{d} a_k \frac{d!}{(d-k)!} z^{d-k} \tag{1.2}$$

is hyperbolic.

In other words, the theorem states that to check that a given $a = (a_1, \dots, a_d) \in \mathbb{R}^d$ is a Nuij sequence, it is enough to check hyperbolicity of $p_a(z,s)$ only for $p(z) = z^d$. The proof is given in Section 2; it is based on a deep result of Borcea and Brändén [1] which gives a characterization of linear maps (on the space of polynomials) preserving hyperbolic polynomials. A nice exposition of the results of Borcea and Brändén is given in Wagner's paper [12].

The second part, developed in Section 3, concerns universal determinantal representation of some Nuij sequences.

Definition 1.3 We say that $a = (a_1, \dots, a_d) \in \mathcal{N}_d \subset \mathbb{R}^d$ admits a *universal determinantal representation* if there exists a symmetric matrix A_a such that for any hyperbolic polynomial p of degree d, we have $p_a(z,s) = \det(zI + D + sA_a)$, where D is a diagonal matrix whose characteristic polynomial is equal to $p = p_a(z, 0)$. The matrix A_a will be referred to as a *matrix associated with the sequence* $a = (a_1, \dots, a_d)$. We denote by $\mathcal{UN}_d$ the set of all Nuij sequences in $\mathbb{R}^d$ that admit universal determinantal representations.

Recall that a square matrix is *Toeplitz* if all parallels to the principal diagonal are constant. We say that a symmetric Toeplitz matrix is *special* if all entries outside the principal diagonal are equal to some $\beta \in \mathbb{R}$, and of course, all entries on the principal diagonal are equal to some $\alpha \in \mathbb{R}$. In the sequel, we will denote such a $d \times d$ matrix by $T_{\alpha,\beta}(d)$ and its determinant by $t_{\alpha,\beta}(d) := \det T_{\alpha,\beta}(d) = (\alpha - \beta)^{d-1}(\alpha + (d-1)\beta)$.

We obtain the following characterization of all Nuij sequences that admit universal determinantal representations.

Theorem B *A sequence $a = (a_1, \dots, a_d) \in \mathbb{R}^d$ is a Nuij sequence with a universal determinantal representation if and only if there exit $\alpha, \beta \in \mathbb{R}$ such that*

$$a_i = \frac{1}{i!} t_{\alpha,\beta}(i), i = 1, \dots, d.$$

2 Hyperbolic Polynomials and Nuij Sequences

First, we recall some facts about the space $\mathcal{H}_1^d$ of hyperbolic (monic) polynomials of some fixed degree d. For $x = (x_1, \dots, x_d) \in \mathbb{R}^d$, we have the k-th elementary symmetric polynomial

$$c_k(x) = \sum_{i_1 < \cdots < i_k} x_{i_1} \cdots x_{i_k},$$

for $k = 1, \dots, d$. We will identify any $b = (b_1, \dots, b_d) \in \mathbb{R}^d$ with a monic polynomial $h_b := z^d + \sum_{k=1}^d b_k z^{d-k}$. Thus, we can write $\mathcal{H}_1^d = c(\mathbb{R}^n)$, where $c = (c_1, \dots, c_d) : \mathbb{R}^d \to \mathbb{R}^d$ is the Viète map; hence, by the Tarski–Seidenberg theorem, it follows that $\mathcal{H}_1^d$ is semialgebraic. Moreover, the Viète map $c = (c_1, \dots, c_d) : \mathbb{R}^d \to \mathbb{R}^d$ is generically a submersion; hence, $\mathcal{H}_1^d = c(\mathbb{R}^n)$ has nonempty interior. In fact, $\mathcal{H}_1^d$ is a basic semi-algebraic set which can be described using generalized discriminants or Bezoutians (see a nice exposition in [9] or a more detailed one in [10]). Recent developments on hyperbolic univariate polynomials are given by Kostov in his survey [4].

For the proof of Theorem A we need to recall several definitions and results from [1].

Definition 2.1 ([1, Definition 1]) We say that a polynomial

$$f(z_1, \dots, z_n) \in \mathbb{C}[z_1, \dots, z_n]$$

is *stable* if $f(z_1, \dots, z_n) \neq 0$ for all n-tuples $(z_1, \dots, z_n) \in \mathbb{C}^n$ with $im(z_j) > 0$, for $j = 1, \dots, n$. If in addition f has real coefficients, it will be referred to as *real stable*. The set of stable and real stable polynomials in n variables will be denoted by $\mathcal{H}_n(\mathbb{C})$ and $\mathcal{H}_n(\mathbb{R})$, respectively. Note that for $n = 1$, a polynomial f is real stable, which precisely means that f is hyperbolic.

Let $T : \mathbb{C}_d[z] \to \mathbb{C}_d[z]$ be a linear map, where $\mathbb{C}_d[z]$ stands for the vector space (over $\mathbb{C}$) of complex polynomials of degree at most d. We extend it to a linear map $T : \mathbb{C}_d[z, w] \to \mathbb{C}_d[z, w]$, by setting $T(z^k w^l) := T(z^k) w^l$ for all $k = 1, \dots, d$ and $l \in \mathbb{N}$. We now state the result that is crucial for the proof of Theorem A.

Theorem 2.2 ([1, Theorem 4]) *Let $T : \mathbb{C}_d[z] \to \mathbb{C}_d[z]$ be a linear map. Then T preserves stability if an only if either*

(i) *T has range of dimension at most one and is of the form $T(f) = \alpha(f) P$, where $\alpha : \mathbb{C}_d[z] \to \mathbb{C}$ is a linear functional and $P \in \mathcal{H}_1(\mathbb{C})$; or*
(ii) $T((z + w)^d) \in \mathcal{H}_2(\mathbb{C})$.

Proof of Theorem A Assume that $a = (a_1, \dots, a_d) \in \mathbb{R}^d$ is a Nuij sequence. Hence, by Definition 1.2 applied to $p(z) = z^d$ with $s = 1$, we obtain that the polynomial p_a defined by (1.1) is hyperbolic.

To prove the converse, let us fix some $a = (a_1, \dots, a_d) \in \mathbb{R}^d$ and assume that the polynomial q_a defined by (1.2) is hyperbolic. We associate with the sequence $a = (a_1, \dots, a_d)$ a linear operator $T_a: \mathbb{C}_d[z] \to \mathbb{C}_d[z]$ defined by

$$T_a(p)(z) := p(z) + \sum_{k=1}^{d} a_k p^{(k)}(z) \in \mathbb{R}[z]. \tag{2.1}$$

Lemma 2.3 $T_a((z+w)^d) = q_a(z+w)$.

Proof We first expand the right-hand side of (2.1):

$$T_a\Big((z+w)^d\Big) = T\Big(\sum_{i=0}^{d}\binom{d}{i} z^i w^{d-i}\Big) = \sum_{i=0}^{d}\binom{d}{i} w^{d-i} T(z^i).$$

Note that

$$T_a(z^i) = \sum_{j=0}^{i} a_j (z^i)^{(j)} = \sum_{j=0}^{i} a_j (z^i)^{(j)} = \sum_{j=0}^{i} a_j \frac{i!}{(i-j)!} z^{i-j},$$

so

$$\sum_{i=0}^{d}\binom{d}{i} w^{d-i} T(z^i) = \sum_{i=0}^{d}\binom{d}{i} w^{d-i}\Big(\sum_{j=0}^{i} a_j \frac{i!}{(i-j)!} z^{i-j}\Big),$$

hence

$$T_a\Big((z+w)^d\Big) = \sum_{i=0}^{d} \frac{d!}{(d-i)!i!} z^{i-j} w^{d-i}\Big(\sum_{j=0}^{i} a_j \frac{i!}{(i-j)!} z^{i-j}\Big). \tag{2.2}$$

On the other hand,

$$q_a(z+w) = \sum_{i=0}^{d} \frac{d!}{(d-i)!} a_i (z+w)^{d-i}. \tag{2.3}$$

- The coefficient in (2.2) that comes with a_j, $j = 0, 1, \dots, d$ is equal to

$$\sum_{i=j}^{d} \frac{d!}{(d-i)!i!} \frac{i!}{(i-j)!} z^{i-j} w^{d-i} = \sum_{i-j=k=0}^{d} \frac{d!}{(d-k-j)!k!} z^k w^{d-j-k}.$$

- The coefficient in (2.3) that comes with a_j, $j = 0, 1, \dots, d$ is equal to

$$\frac{d!}{(d-j)!}(z+w)^{d-j} = \frac{d!}{(d-j)!} \sum_{k=0}^{d} \binom{d-j}{k} z^k w^{d-j-k} = \sum_{i-j=k=0}^{d} \frac{d!}{(d-k-j)!} z^k w^{d-j-k}.$$

Hence, these coefficients are equal, which proves the lemma. ∎

By the assumption, q_a has only real roots. Hence, $q_a(z+w)$ is a stable polynomial in variables (z, w). Indeed, if $im(z) > 0$ and $im(w) > 0$, then $im(z+w) > 0$, so $q_a(z+w) \neq 0$. By Lemma 2.3, we have $T_a((z+w)^d) = q_a(z+w)$. Applying

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Theorem 2.2 we conclude that the operator T_a preserves stability, hence T_a restricted to $\mathbb{R}_d[z]$ preserves hyperbolicity. Thus, we have proved that

$$p_a(z,1) = p(z) + \sum_{k=1}^{d} a_k p^{(k)}(z)$$

is hyperbolic whenever $p \in \mathbb{R}_d[z]$ is hyperbolic. Let us take $s \in \mathbb{R}^*$ and denote $a(s) := (sa_1, \dots, s^k a_k, \dots, s^d a_d)$. Then the polynomial

$$q_{a(s)}(z) := z^d + \sum_{k=1}^{d} s^k a_k (z^d)^{(k)} = z^d + \sum_{k=1}^{d} s^k a_k \frac{n!}{(n-k)!} z^{d-k}$$

is again hyperbolic, since $q_{a(s)}(z) = s^{-d} q_a(sz)$. Thus, by applying the above argument to the sequence $a(s)$, we conclude that

$$p_a(z,s) := p(z) + \sum_{k=1}^{d} a_k s^k p^{(k)}(z)$$

is hyperbolic for all $s \in \mathbb{R}$ and any $p \in \mathbb{R}_d[z]$ hyperbolic. This ends the proof of Theorem A. ∎

Corollary 2.4 *If $(a_1, a_2, \dots, a_d)$ is a Nuij sequence for hyperbolic polynomials of degree d, then $(a_1, a_2, \dots, a_{d-i})$ is also a Nuij sequence for hyperbolic polynomials of degree $d - i$, $i = 1, \dots, d-1$. Moreover $(a_1, a_2, \dots, a_d)$ is a Nuij sequence for hyperbolic polynomials of arbitrary degrees if and only if it is Viète, the iteration of the standard Nuij sequence.*

Proof The first assertion is easily deduced by differentiation of (1.2).

The second affirmation is a consequence of the fact that $(a_1, a_2, \dots, a_d, 0, \dots, 0)$ is a Nuij sequence for hyperbolic polynomials of degree $k = d + i$, $i = 1, 2, \dots$ and satisfies (1.2) for all $k = d + i$, $i \geq 1$.

Simplifying each obtained equation by the corresponding z^i, we can obtain a sequence of hyperbolic polynomials of degree d convergent to $z^d + a_1 z^{d-1} + a_2 z^{d-2} + \cdots + a_d$, and this implies the claim. Namely, we have

$$\big(ka_1, k(k-1)a_2, \dots, k(k-1)\cdots(k-d+1)a_d\big) = \sigma\big(x_1(k), \dots, x_d(k)\big), \quad \forall k \geq d,$$

for some $x(k) = (x_1(k), \dots, x_d(k)) \in \mathbb{R}^d$. Now we can see that $\sigma(x(k)/k)$ tends to $(a_1, a_2, \dots, a_d)$ as $k \to \infty$. ∎

2.1 Iterations of Nuij's Sequences

Let $a = (a_1, \dots, a_d) \in \mathbb{R}^d$ and $b = (b_1, \dots, b_d) \in \mathbb{R}^d$ be two Nuij sequences, we define their composition $b \circ a := c = (c_1, \dots, c_d)$ in the following way. For any polynomial $p(z) \in \mathbb{R}[z]$,

$$p_c(z,s) = (p_a)_b(z,s) = p_a(z,s) + \sum_{k=1}^{d} b_k s^k \frac{\partial^k p_a}{\partial z^k} = p + \sum_{k=1}^{d} c_k s^k p^{(k)}.$$

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Note that with the convention $a_0 = b_0 = 1$, we have

$$c_k = \sum_{i=0}^{k} a_i b_{k-i}.$$

Let $a^1, \dots, a^r \in \mathbb{R}^d$. We define by induction the composition of r copies of sequences:

$$I_1(a^1) = a^1, \quad I_r(a^1, \dots, a^r) := I_{r-1}(a^1, \dots, a^{r-1}) \circ a^r.$$

Explicitly, if $I_r(a^1, \dots, a^r) = c = (c_1, \dots, c_d)$, then

$$c_k = \sum_{i_1 < \cdots < i_r, i_1 + \cdots + i_r = k} a^1_{i_1} \cdots a^r_{i_r}.$$

Let us consider the original Nuij sequences of the form

$$a^i = (x_i, 0, \dots, 0) \in \mathbb{R}^d,$$

where $x_i \in \mathbb{R}$, $i = 1, \dots, d$. Then $I_d(a^1, \dots, a^d) = c = (c_1, \dots, c_d)$ is the Nuij sequence obtained by the iteration of a^i and

$$c_k = \sum_{i_1 < \cdots < i_k} x_{i_1} \cdots x_{i_k},$$

for $k = 1, \dots, d$. Thus, $c_k = c_k(x_1, \dots, x_d)$ is in fact the k-th elementary symmetric polynomial of $x_1, \dots, x_d$. Denote by $c = (c_1, \dots, c_d): \mathbb{R}^d \to \mathbb{R}^d$ the Viète map and recall that $\mathcal{H}_1^d = c(\mathbb{R}^n)$. Thus, we obtain that $\mathcal{H}_1^d \subset \mathcal{N}_d$. For $d \in \mathbb{N}$, let us denote by $b_d: \mathbb{R}^d \to \mathbb{R}^d$ the following linear map:

$$b_d(a_1, \dots, a_k, \dots, a_d) := \Big(d a_1, \dots, \frac{d!}{(d-k)!} a_k, \dots, d! a_d \Big).$$

Theorem A and the above discussion can be summarized as follows.

Corollary 2.5 *For any $d \in \mathbb{N}$, we have $\mathcal{H}_1^d \subset \mathcal{N}_d = b_d^{-1}(\mathcal{H}_1^d)$.*

Example 2.6 For $d = 2$, we have $\mathcal{H}_1^2 = \{a_1^2 - 4a_2 \geq 0\} \subset \mathcal{N}_2 = \{a_1^2 - 2a_2 \geq 0\}$.

3 Universal Determinantal Representations

We will consider 1-parameter families of hyperbolic polynomials. A polynomial

$$p(z, s) = z^d + a_1(s) z^{d-1} + \cdots + a_d(s)$$

will be called a *pencil of hyperbolic polynomials* if and only if:

- for each $s \in \mathbb{R}$ the polynomial $z \mapsto p(s, z)$ is hyperbolic,
- each coefficient $a_i(s) \in \mathbb{R}[s]$ is of degree at most i.

For any $d \geq 1$, we shall denote by $\mathcal{PH}_d$ the space of such pencils of hyperbolic polynomials.

We say that a polynomial $p(z, s)$ admits a determinantal representation if there are real symmetric matrices A_0, A_1 such that

$$p(z, s) = \det(zI + A_0 + sA_1),$$

and clearly in this case $p(z, s)$ is a pencil of hyperbolic polynomials.

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The following is an easy reformulation of a remarkable theorem of Helton and Vinnikov [3].

Theorem 3.1 *Any polynomial $p(z,s) \in \mathcal{PH}_d$ admits a determinantal representation.*

Indeed, let us set $z = x^{-1}$ and $s = x^{-1}y$ and finally

$$f(x,y) := x^d p(z,s) = x^d p(x^{-1}, x^{-1}y).$$

Then f is a real zero polynomial in the sense of Helton–Vinnikov, so it has a determinantal representation according to [3, Theorem 2.2]. In fact, as noticed by Lewis, Parrilo, and Ramana [6], Theorem 3.1 is a positive answer to the nonhomogeneous version of the Lax conjecture [5].

We want to characterize all Nuij sequences $a = (a_1, \dots, a_d) \in \mathbb{R}^d$ such that for any $p \in \mathbb{R}[z]$, hyperbolic polynomial of degree d, the associated pencil of hyperbolic polynomials

$$p_a(z,s) := p + \sum_{k=1}^{d} a_k s^k p^{(k)} \in \mathbb{R}[z]$$

admits a *universal determinantal representation*; by this we mean that there exists a symmetric matrix A_a such that $p_a(z,s) = \det(zI + D + sA_a)$, where D is a diagonal matrix. In other words, $-D$ has on the diagonal all the roots of p written in an arbitrary order. The matrix A_a will be referred as a *matrix associated with the sequence* $a = (a_1, \dots, a_d)$. We denote by $\mathcal{UN}_d$ the set of all Nuij sequences in $\mathbb{R}^d$ that admit universal determinantal representations.

3.1 Special Toeplitz Matrices

Recall that a square matrix is called a *Toeplitz matrix* if all parallels to the principal diagonal are constant. We say that a symmetric Toeplitz matrix is *special* if all entries outside the principal diagonal are equal to some $\beta \in \mathbb{R}$, and of course all entries on the principal diagonal are equal to some $\alpha \in \mathbb{R}$. We will denote such a matrix by $T_{\alpha,\beta}$.

In the next proposition we will show that special Toeplitz matrices give all Nuij sequences which admit universal determinantal representations.

Proposition 3.2 *Let $a = (a_1, \dots, a_d) \in \mathcal{UN}_d$. Then there exists a special Toeplitz matrix $T_{\alpha,\beta}$ that is associated with the sequence a. The constant α is unique. For $d = 2$, we have two choices β or $-\beta$. If $d \geq 3$, then β is uniquely determined.*

Proof Let us fix a sequence $a = (a_1, \dots, a_d) \in \mathcal{UN}_d$, and let A_a be a symmetric matrix associated to a. It means that for any hyperbolic polynomial $p \in \mathbb{R}[z]$ we have

$$p_a(z,s) = \det(zI + D + sA_a), \tag{3.1}$$

where D is a diagonal matrix with characteristic polynomial equal to p. We will find a special Toeplitz matrix $T_{\alpha,\beta}$ such that

$$p_a(z,s) = \det(zI + D + sT_{\alpha,\beta}).$$

Following convention, we recall that a $j \times j$ minor of A_a is *principal* if it is the determinant of a matrix obtained from A_a by deleting rows and columns containing

$d - j$ elements from the principal diagonal. With the assumption of Proposition 3.2, we have the following lemma.

Lemma 3.3 *For any $j = 1, \dots, d$, all $j \times j$ principal minors of A_a are equal.*

Let $-\lambda_1, \dots, -\lambda_d$ be the roots of p. Since p can be chosen arbitrarily, we can consider both sides of the identity (3.1) as polynomials with real coefficients in variables $w_i := z + \lambda_i$, $i = 1, \dots, d$. Since $\mathbb{R}$ is a field of characteristic 0, the coefficients corresponding to the monomials in $w_{i_1} \cdots w_{i_j}$, where $i_1 < \cdots < i_j$, on right and left-hand sides are equal. It is enough to expand both sides to check the statement of the lemma. In particular the 1×1 minors, which are actually the entries on the principal diagonal, are all equal to some $\alpha \in \mathbb{R}$.

Lemma 3.4 *Let $A_a = (a_{ij})$. Then there exists $\beta \in \mathbb{R}$ such that for any distinct i, j we have $a_{ij}^2 = \beta^2$.*

Indeed, with each entry a_{ij}, $i \neq j$ we can associate the 2×2 principal minor

$$\det \begin{pmatrix} \alpha & a_{ij} \\ a_{ij} & \alpha \end{pmatrix} = \alpha^2 - a_{ij}^2.$$

Hence, by Lemma 3.3 all a_{ij}^2 are equal for $i \neq j$. We put $\beta^2 = a_{ij}^2$. Clearly the statement of Proposition 3.2 is trivial for $\beta = 0$, so in the sequel we assume that $\beta \neq 0$.

Before analyzing the case of $j \times j$ principal minors, where $j \geq 3$, we need an explicit formula for the determinant of a special Toeplitz matrix $T_{\alpha,\beta}$.

Lemma 3.5 *If $T_{\alpha,\beta}$ is a special Toeplitz matrix of size $d \times d$, then*

$$t_{\alpha,\beta}(d) := \det T_{\alpha,\beta} = (\alpha - \beta)^{d-1}\big(\alpha + (d-1)\beta\big).$$

Next we consider the 3×3 principal minors of the matrix A_a. We know by Lemma 3.4 that for any $i \neq j$ we have $a_{ij} = \epsilon_{ij}|\beta|$, where $\epsilon_{ij} \in \{-1, 1\}$. We will show that the sign of ϵ_{ij} can be uniformly chosen, which means that either $\epsilon_{ij} = 1$ for all $i \neq j$, or $\epsilon_{ij} = -1$ for all $i \neq j$. Let us write this minor in the form

$$\det \begin{pmatrix} \alpha & \epsilon_{ij}|\beta| & \epsilon_{ik}|\beta| \\ \epsilon_{ij}|\beta| & \alpha & \epsilon_{jk}|\beta| \\ \epsilon_{ik}|\beta| & \epsilon_{jk}|\beta| & \alpha \end{pmatrix} = \alpha^3 + 2\epsilon_{ij}\epsilon_{ik}\epsilon_{jk}\beta^2|\beta| - 3\alpha\beta^2.$$

By Lemma 3.3 all these minors are equal, so there exists $\xi \in \{-1, 1\}$ such that for all choices $1 \leq i < j < k \leq d$ we have

$$\epsilon_{ij}\epsilon_{ik}\epsilon_{jk} = \xi. \tag{3.2}$$

This shows that we can chose $\epsilon_{ij} = \xi$ for all $i \neq j$.

Assume now that $d \geq 4$. We have to show that if we put $\epsilon_{ij} = \xi$ for any $i \neq j$, then actually all principal minors $j \times j$, $j \geq 4$ are equal to the value of a principal minor $j \times j$, $j \geq 4$ for the original matrix A_a, so in fact they are determined just by ξ. Note that it

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is enough to consider the case $\alpha = 0$ and $\beta = 1$. First, we consider the case $d = 4$, so

$$A_a = \begin{pmatrix} 0 & \epsilon_{12} & \epsilon_{13} & \epsilon_{14} \\ \epsilon_{12} & 0 & \epsilon_{23} & \epsilon_{24} \\ \epsilon_{13} & \epsilon_{23} & 0 & \epsilon_{34} \\ \epsilon_{14} & \epsilon_{24} & \epsilon_{34} & 0 \end{pmatrix}.$$

For each $i \geq 2$, we multiply the i-th row of A_a by ϵ_{1i} and use relation (3.2). Thus we obtain the matrix

$$B_a := \begin{pmatrix} 0 & \epsilon_{12} & \epsilon_{13} & \epsilon_{14} \\ 1 & 0 & \xi\epsilon_{13} & \xi\epsilon_{14} \\ 1 & \xi\epsilon_{12} & 0 & \xi\epsilon_{14} \\ 1 & \xi\epsilon_{12} & \xi\epsilon_{13} & 0 \end{pmatrix}.$$

For each $j \geq 2$, we multiply the j-th column of B_a by ϵ_{1j} and use the fact that $\epsilon_{1i}^2 = 1$. So we obtain the matrix

$$C_a := \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & \xi & \xi \\ 1 & \xi & 0 & \xi \\ 1 & \xi & \xi & 0 \end{pmatrix}.$$

Multiplying the first row and the first column of C_a by ξ, we can see that

$$\det C_a = \xi^2 \det T_{0,1} = t_{0,1}(4) = -3.$$

But on the other hand, $\det C_a = (\epsilon_{12}\epsilon_{13}\epsilon_{14})^2 \det A_a = \det A_a$. Accordingly, we can assume that $A_a = T_{0,\xi}$. The same argument applies for any $d > 4$. Hence the existence in Proposition 3.2 follows.

To proof the uniqueness, note that α and β^2 are uniquely determined. Clearly the equation $a_3 = \frac{1}{3!}(\alpha - \beta)^2(\alpha + 2\beta)$ uniquely determines β. ■

As a consequence we obtain the following characterization of Nuij sequences that admit universal determinantal representations.

Theorem B *A sequence $a = (a_1, \dots, a_d) \in \mathbb{R}^d$ is a Nuij sequence with a universal determinantal representation if and only if there exits $\alpha, \beta \in \mathbb{R}$ such that*

$$a_i = \frac{1}{i!} t_{\alpha,\beta}(i),\ i = 1, \dots, d.$$

Proof If $T_{\alpha,\beta}$ is a special Toeplitz matrix, then for any hyperbolic polynomial $p(z) = (z + \lambda_1)\dots(z + \lambda_d)$, we have a pencil of polynomials

$$p_a(z,s) := p + \sum_{k=1}^{d} a_k s^k p^{(k)}(z) = \det(zI + D + sT_{\alpha,\beta}),$$

where $a_i = \frac{1}{i!} t_{\alpha,\beta}(i)$, and D is a diagonal matrix with entries $\lambda_1, \dots, \lambda_d$. So the sequence $a = (a_1, \dots, a_d)$ is a Nuij sequence with a universal determinantal representation. Conversely, if $a = (a_1, \dots, a_d) \in \mathbb{R}^d$ is a Nuij sequence with a universal determinantal representation, then by Proposition 3.2 the associated matrix can be chosen as a special Toeplitz matrix $T_{\alpha,\beta}$. Hence, $a_i = \frac{1}{i!} t_{\alpha,\beta}(i)$. ■

Example 3.6 Note that the original Nuij sequence $a = (1, 0, \dots, 0)$ has a universal determinantal representation. Indeed, $T_{1,1}$, which has all entries equal to 1, is the matrix associated with this sequence. Note that this also proves Nuij's Theorem 1.1.

Remark 3.7 A composition of the original Nuij sequence $a = (1, 0, \dots, 0)$ with itself gives a Nuij sequence $b = (2, 1, 0, \dots, 0)$ that has no universal determinantal representation for $d \geq 3$. Indeed, if there exist $\alpha, \beta \in \mathbb{R}$ such that $b_i = \frac{1}{i!} t_{\alpha,\beta}(i)$, $i = 1, 2, 3$, then $\alpha = 2$ and $\alpha^2 - \beta^2 = 2$. Hence, $\beta = \pm\sqrt{2}$. But, then $6b_3 = \alpha^3 + 2\beta^3 - 3\alpha\beta^2 \neq 0$, so $b_3 \neq 0$, which is a contradiction.

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