THREE PROOFS OF MINKOWSKI'S SECOND INEQUALITY IN THE GEOMETRY OF NUMBERS

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Dedicated to Kurt Mahler on his sixtieth birthday

1. Introduction

Let K be a bounded, open convex set in euclidean n-space R_n, symmetric in the origin 0. Further let L be a lattice in R_n containing 0 and put

$$m_i = \text{infimum } u_i \qquad i = 1, 2, \cdots, n;$$

extended over all positive real numbers u_i for which u_iK contains i linearly independent points of L. Denote the Jordan content of K by $V(K)$ and the determinant of L by $d(L)$. Minkowski's second inequality in the geometry of numbers states that

$$(1) \qquad m_1 m_2 \cdots m_n V(K) \leqq 2^n d(L).$$

Minkowski's original proof has been simplified by Weyl [6] and Cassels [7] and a different proof has been given by Davenport [1].

Professor Mahler, during a seminar at Notre Dame University, suggested to the authors that it would be worthwhile to reexamine these proofs with a view to possible generalisations. Each author then gave a proof, one based on Weyl's paper [6] and two on Davenport's [1]. These three proofs are given here.

In Weyl's proof all considerations are made in the quotient space determined by the lattice L. The aim of the first proof is to show that the existence of the quotient space is needed only to deduce the so-called monotone property, thus suggesting that the theorem is true for some point sets other than lattices. Davenport's proof of (1) depends on certain functions constructed in the course of the argument being continuous. He states without proof that the construction can be made to ensure this. The third proof here shows how this may be done. Earlier Siegel in lectures at New York University gave without proof a method for making this construction. The second proof shows that by working with iterated integrals instead of

Jordan contents Davenport's proof can be made independent of this continuity. Theorem A of this argument, a variant of the Minkowski-Blichfeldt theorem, may have some independent interest.

2. Reduction of the problem

The following normalisation is common to all three proofs. It is well known and easy to prove that there exist n linearly independent points $F_1, F_2, \cdots, F_n$ of L such that F_i lies on the boundary of $m_i K$ for each $i = 1, 2, \cdots, n$. It is then possible to select a basis $P_1, P_2, \cdots, P_n$ of L so that the linear space generated by the points $P_1, P_2, \cdots, P_i$ is the same as that generated by $F_1, F_2, \cdots, F_i$, and the intersection of this linear space with L is the i-dimensional lattice L_i generated by the points P_1, $P_2, \cdots, P_i$ for $i = 1, 2, \cdots, n$. Since the inequality (1) is invariant with respect to the full linear group of transformations of R_n it follows that without loss of generality $L = L_n$ may be assumed to be the integral lattice and $P_1, P_2, \cdots, P_n$ the points given by

$$P_i = (\delta_{i1}, \delta_{i2}, \cdots, \delta_{in}) \qquad i = 1, 2, \cdots, n;$$

where δ_{ij} is the Kronecker delta.

3. First proof

Let C be a bounded Lebesgue measurable point set and S a discrete point set in R_n. For $s \in S$ and a positive integer m the set of points contained in $C+s$ and exactly $m-1$ of the sets $C+t$, $t \in S-s$, simultaneously is measurable with measure M_m say. Let k be the largest integer for which M_k is not zero and put

$$D(C, S, s) = M_1 + \tfrac{1}{2}M_2 + \tfrac{1}{3}M_3 + \cdots + \tfrac{1}{k}M_k.$$

The affine invariance of Lebesgue measure implies that

(A) If $x, y \in R_n$ then $D(C+x, S+y, s+y) = D(C, S, s)$;

(B) If T is a nonsingular linear transformation of R_n with determinant $|T|$ then

$$D(TC, TS, Ts) = ||T||D(C, S, s).$$

For $i < n$ we convene that R_i is the subspace of R_n composed of those points for which the last $n-i$ coordinates vanish. Denote by $C[x_{i+1}, x_{i+2}, \cdots, x_n]$ the section of C consisting of those points of C whose last $n-i$ coordinates have the fixed values $x_{i+1}, x_{i+2}, \cdots, x_n$ respectively. Now if $S \subset R_i$ then for any point $t \in S$ the section of $C+t$ consisting of those points of $C+t$ whose last $n-i$ coordinates have the fixed values $x_{i+1}, x_{i+2}, \cdots, x_n$ respectively is given by $C[x_{i+1}, x_{i+2}, \cdots, x_n]+t$ and therefore by Fubini's theorem

(C) If $S \subset R_i$ then $D(C, S, s) = \int_{R_{n-i}} D(C[x_{i+1}, x_{i+2}, \cdots, x_n], S, s)dx_{i+1} \cdots dx_n$ where $D(C[x_{i+1}, \cdots, x_n], S, s)$ is understood to be the corresponding function in R_i.

By (A) it follows that $D(C, L_i, s)$ is independent of s so in this case we write $D(C, L_i)$ in place of $D(C, L_i, s)$.

Denote by Π_i the set of all points $(x_1, \cdots, x_n) \in R_n$ for which

$$0 \leqq x_j < 1 \qquad j = 1, 2, \cdots, i;$$

Thus Π_i is a fundamental region of R_n modulo L_i. Let C/L_i denote the set of all points of C reduced modulo L_i to Π_i and $m(C/L_i)$ its measure. A fundamental principle of the geometry of numbers — the Minkowski-Blichfeldt theorem — may be stated as

(D) $$D(C, L_i) = m(C/L_i).$$

PROOF. m points of C are congruent modulo L_i and congruent to no other points of C if and only if each of the m points lies in that part of C which is covered by exactly $m-1$ of the sets $C+z$, $z \in L_i-0$. Since this part of C has measure weighted by the factor $1/m$ in $D(C, L_i)$ the result follows.

It follows immediately that

(E) If C_1, C_2 are bounded Lebesgue measurable subsets of R_n such that $C_1 \subset C_2$ then $D(C_1, L_i) \leqq D(C_2, L_i)$.

A subset X of R_n is said to be star with respect to a point $x \in X$ if whenever $y \in X$ so is the complete line segment joining y to x contained in X. The particular property of such sets required here is that if $t \geqq 1$ then

$$X \subset tX + (x - tx)$$

that is to say tX may be translated into a position so as to cover X. We observe that a convex set is star with respect to any point in it. For a given $t \geqq 1$ denote by T_i the linear transformation

$$T_i(x_1, x_2, \cdots, x_n) = (tx_1, tx_2, \cdots, tx_i, x_{i+1}, \cdots, x_n)$$

and by K the convex body of the introduction. Then

$$T_i(K[x_{i+1}, \cdots, x_n]) = (T_iK)[x_{i+1}, \cdots, x_n]$$

and so $(T_iK)[x_{i+1}, \cdots, x_n]$ may be translated into a position in the same section so as to contain $K[x_{i+1}, \cdots, x_n]$. Hence (A) and (E) imply

$$D(K[x_{i+1}, \cdots, x_n], L_i) \leqq D((T_iK)[x_{i+1}, \cdots, x_n], L_i)$$

and therefore also by (C)

(F) $$D(K, L_i) \leqq D(T_iK, L_i).$$

LEMMA 1. $D(tK, L_i) \geqq t^{n-i} D(K, L_i)$.

PROOF. If $T^{(n-i)}$ denotes the transformation

$$T^{(n-i)}(x_1, \cdots, x_n) = (x_1, \cdots, x_i, tx_{i+1}, \cdots, tx_n)$$

then $tK = T^{(n-i)} T_i K$ and, using (B) and (F),

$$D(tK, L_i) = D(T^{(n-i)} T_i K, T^{(n-i)} L_i) = ||T^{(n-i)}|| D(T_i K, L_i) \geqq t^{n-i} D(K, L_i).$$

LEMMA 2. *If K contains no point of $L_n - L_i$ then*

$$D(\tfrac{1}{2}K, L_n) = D(\tfrac{1}{2}K, L_{n-1}) = \cdots = D(\tfrac{1}{2}K, L_i).$$

PROOF. If $x \in L_n - L_i$ then $\frac{1}{2}K + x$ does not intersect $\frac{1}{2}K$ for otherwise the point x is in K. The lemma then follows from the definition of $D(\frac{1}{2}K, L_i)$.

The inequality (1) now follows quickly, for by the definition of m_{i+1} for $i = 0, 1, \cdots, n-1$ no point of $L_n - L_i$ lies in $m_{i+1}K$ whence by lemma 2, $D(\frac{1}{2}m_{i+1}K, L_{i+1}) = D(\frac{1}{2}m_{i+1}K, L_i)$ and therefore also by lemma 1,

$$\begin{aligned} 1 \geqq D(\tfrac{1}{2}m_n K, L_n) &= D(\tfrac{1}{2}m_n K, L_{n-1}) \geqq \frac{m_n}{m_{n-1}} D(\tfrac{1}{2}m_{n-1}K, L_{n-2}) \\ &\geqq \frac{m_n}{m_{n-1}} \left(\frac{m_{n-1}}{m_{n-2}}\right)^2 \cdots \left(\frac{m_2}{m_1}\right)^{n-1} D(\tfrac{1}{2}m_1 K, L_0) \\ &= m_n m_{n-1} \cdots m_1 (\tfrac{1}{2})^n V(K). \end{aligned}$$

4. Second proof

By the definition of m_i, it is clear that if X is a point of L_n in $m_i K$ then $x_i = x_{i+1} = \cdots = x_n = 0$. By convexity and symmetry of K the following three statements are equivalent:

(i) The sets $\frac{1}{2}m_i K$, $\frac{1}{2}m_i K + A$ overlap;
(ii) $\frac{1}{2}m_i K$ contains two points X, Y such that $A = X - Y$;
(iii) A lies in $m_i K$.

In particular if $\frac{1}{2}m_i K$ has two points X, Y such that $A = X - Y$ lies in L_n then $x_i = y_i, \cdots, x_n = y_n$.

In the rest of this proof we will assume that each function that appears in any integral is nonnegative and is nonzero only in some bounded set, not necessarily independent of the function. The symbol

$$\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f(x_n, \cdots, x_1) dx_1 dx_2 \cdots dx_i$$

will stand for the iterated Riemann integral

$$\int_{-\infty}^{\infty} dx_i \int_{-\infty}^{\infty} dx_{i-1} \cdots \int_{-\infty}^{\infty} f(x_n, \cdots, x_1) dx_1.$$

We next prove a few almost obvious results about these integrals.

LEMMA 1. *Suppose that* $I = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f(x_n, \cdots, x_1)\, dx_1 \cdots dx_i$ $1 \leqq i \leqq n$ *exists. Then, given real numbers* $a_1, \cdots, a_i$,

$$I = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f(x_n, \cdots, x_{i+1}, x_i - a_i, \cdots, x_1 - a_1) dx_1 \cdots dx_i.$$

PROOF. For $i = 1$ it follows from the definition and by induction for $i > 1$.

LEMMA 2. *Suppose that* $I = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f(x_n, \cdots, x_1) dx_1 \cdots dx_n$ exists. *Then for given numbers* $a_1 > 0, \cdots, a_i > 0$ *the integral* $I(a) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f(x_n, \cdots, x_{i+1}, (x_i/a_i), \cdots, (x_1/a_1)) dx_1 \cdots dx_i$ exists and $I(a) = a_1 \cdots a_i I$.

PROOF. For $i = 1$ by definition and for $i > 1$ by induction.

LEMMA 3. (i) *If*

$$I_1 = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f(x_n, \cdots, x_1) dx_1 \cdots dx_n$$

and

$$I_2 = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} g(x_n, \cdots, x_1) dx_1 \cdots dx_n$$

both exist then so does

$$I = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} (f+g) dx_1 \cdots dx_n$$

and $I = I_1 + I_2$.
(ii) *If in* (i), $f \leqq g$ *for all points* X *then* $I_1 \leqq I_2$.

PROOF. Clear.

THEOREM A. *Suppose that* S *is a bounded set in* R_n. *Let* $\chi(X) = \chi(x_n, \cdots, x_1)$ *be its characteristic function. Suppose further that*

$$I = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \chi(X) dx_1 \cdots dx_n \text{ exists and } I > 1.$$

Then the sets $S + A$, $A \in L$ *overlap so that there exist points* X, Y *in* S, $X \neq Y$, *such that* $X - Y = A \in L$.

PROOF. Suppose S lies in the box $|x_i| \leqq k$. Suppose further that the sets $S + A$, $A \in L$ do not overlap. For a fixed positive integer N consider the set Σ of points A of L such that $1 \leqq a_i \leqq N$, $i = 1, \cdots, n$. Define

$$F(X) = \sum_{A \in \Sigma} \chi(X - A)$$

Since the sets $S + A$ do not overlap $F(X) \leqq 1$ for all X. Since the sets lie in the box B given by $-k \leqq x_i \leqq N + k$, so $F(X) = 0$ if $X \notin B$. By the lemmas 1 and 3,

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$$\begin{aligned}(N+2k)^n &\geqq \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} F(X)dx_1\cdots dx_n\\ &= \sum_{A\in\Sigma}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\chi(X-A)dx_1\cdots dx_n\\ &= \sum_{A\in\Sigma}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\chi(X)dx_1\cdots dx_n\\ &= (N+1)^k I\end{aligned}$$

so that

$$I \leqq \left(\frac{N+2k}{N+1}\right)^n.$$

By making $N\to\infty$, we get $I\leqq 1$ and the theorem follows.

LEMMA 4. *It is possible to construct sets* $K_1, \cdots, K_n$ *such that*

(1) $K_1 = \frac{1}{2}m_1 K$,

(2) $K_i \subset \frac{1}{2}m_i K$

(3) *If* $X, Y \in K_i$ *for* $i>1$ *and* $x_i = y_i, \cdots, x_n = y_n$ *then there exist points* X', Y' *in* K_{i-1} *such that* $X-Y = X'-Y'$ *and*

(4) *If* $\chi_i(X) = \chi_i(x_n, \cdots, x_1)$ *is the characteristic function of* K_i *then*

$$V_i = \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\chi_i(x_n,\cdots,x_1)dx_1\cdots dx_n$$

exists and

$$V_i = \begin{cases}(\frac{1}{2}m_1)^n V(K) & \text{if } i=1\\ \left(\dfrac{m_i}{m_{i-1}}\right)^{n-i+1} V_{i-1} & \text{if } i>1.\end{cases}$$

PROOF. Take $K_1 = \frac{1}{2}m_1 K$. To prove (4) we observe that K_1 has volume $(\frac{1}{2}m_1)^n V(K)$ and since K_1 is convex all its sections by hyperplanes of various dimensions are convex and hence have volumes in their appropriate dimensions. Therefore

$$\begin{aligned}(\tfrac{1}{2}m_1)^n V(K) = V(K_1) &= \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\chi(X)dx_1\cdots dx_n\\ &= \int_{-\infty}^{\infty}dx_n\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\chi(X)dx_1\cdots dx_{n-1} = \cdots\\ &= \int_{-\infty}^{\infty}dx_n\int_{-\infty}^{\infty}dx_{n-1}\cdots\int_{-\infty}^{\infty}\chi(X)dx_1 = V_1\end{aligned}$$

so that K_1 has the required properties.

Suppose that $K_1, \cdots, K_{i-1}$ $(2\leqq i\leqq n)$ have been constructed to satisfy (1) through (4). For each point $(0, 0, \cdots, 0, x_i, \cdots, x_n)$ of the projection of $\frac{1}{2}m_{i-1}K$ on $x_1 = \cdots = x_{i-1} = 0$ choose a point

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$$(\varphi_1(x_i, \cdots, x_n), \cdots, \varphi_{i-1}(x_i, \cdots, x_n), x_i, \cdots, x_n) \text{ of } \tfrac{1}{2}m_{i-1}K.$$

Define K_i to be the set of points Y where

$$y_1 = x_1 + \left(\frac{m_i}{m_{i-1}} - 1\right) \varphi_1(x_i, \cdots, x_n)$$

$$y_2 = x_2 + \left(\frac{m_i}{m_{i-1}} - 1\right) \varphi_2(x_i, \cdots, x_n)$$

$$\cdots$$

$$y_{i-1} = x_{i-1} + \left(\frac{m_i}{m_{i-1}} - 1\right) \varphi_{i-1}(x_i, \cdots, x_n)$$

$$y_i = \frac{m_i}{m_{i-1}} x_i$$

$$\cdots$$

$$y_n = \frac{m_i}{m_{i-1}} x_n$$

and $X \in K_{i-1}$.

Since $X = (x_1, \cdots, x_n)$ and $\tilde{X} = (\varphi_1, \cdots, \varphi_{i-1}, x_i, \cdots, x_n)$ lie in $\frac{1}{2}m_{i-1}K$ which is convex so does $Z = \{X + (m_{i-1}/m_i - 1)\tilde{X}\}|m_{i-1}/m_i$. Therefore $Y = (m_i/m_{i-1})Z$ lies in $\frac{1}{2}m_iK$. This proves (1) and (2) for K_i while (3) is a direct consequence of the definition.

$$\begin{aligned} V_i &= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \chi_i(y_n, \cdots, y_1) dy_1 \cdots dy_n \\ &= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \chi_{i-1}(a^{-1}y_n, \cdots, a^{-1}y_i, y_{i-1} - (a-1)\varphi_{i-1}(a^{-1}y_i, \cdots, a^{-1}y_n) \\ &\qquad \cdots, y_1 - (a-1)\varphi_1(a^{-1}y_i, \cdots, a^{-1}y_n)) dy_1 \cdots dy_n \end{aligned}$$

where $a = (m_i/m_{i-1})$, exists because if we write

$$\chi(x_n, \cdots, x_i) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \chi_{i-1}(x_n, \cdots, x_i, x_{i-1}, \cdots, x_1) dx_1 \cdots dx_{i-1}$$

then χ exists by the induction hypothesis and equals

$$\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \chi_{i-1}(x_n, \cdots, x_i, x_{i-1} - (a-1)\varphi_{i-1}(a^{-1}x_i, \cdots, a^{-1}x_n), \cdots, \\ x_1 - (a-1)\varphi_1(a^{-1}x_i, \cdots, a^{-1}x_n)) dx_1 \cdots dx_{i-1}$$

by lemma 1, so that by lemma 2,

$$\begin{aligned} V_i &= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \chi_{i-1}(a^{-1}y_n, \cdots, a^{-1}y_i, y_{i-1}, \cdots, y_1) dy_1 \cdots dy_n \\ &= a^{n-i+1} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \chi_{i-1}(x_n, \cdots, x_i, \cdots, x_1) dx_1 \cdots dx_n \\ &= a^{n-i+1} V_{i-1}. \end{aligned}$$

This proves the lemma.

Proof of Minkowski's inequality (1). Suppose that $m_1 \cdots m_n V(K) > 2^n$. Then by lemma 4

$$V_n = m_1 \cdots m_n 2^{-n} V(K) > 1,$$

and by theorem A there exist points X, Y in K_n such that $0 \neq X-Y \in L_n$. Since $K_n \subset \frac{1}{2}m_n K$ this implies that $x_n = y_n$ and by property (1) of the sets K_i, K_{n-1} contains points $X^{(1)}$, $Y^{(1)}$ such that

$$X-Y = X^{(1)}-Y^{(1)}.$$

Since $K_{n-1} \subset \frac{1}{2}m_{n-1}K$ this implies that $x^{(1)}_{n-1} = y^{(1)}_{n-1}$, $x^{(1)}_n = y^{(1)}_n$ and there exist points $X^{(2)}$, $Y^{(2)}$ in K_{n-2} with

$$X-Y = X^{(2)}-Y^{(2)}.$$

Repeating this argument a number of times we obtain points X^*, Y^* in $\frac{1}{2}m_1 K$ such that $X-Y = X^*-Y^*$. But $X^*-Y^* \in L_n$ implies that $X^* = Y^*$ and $X-Y = 0$, which is a contradiction. This proves the theorem.

5. Third proof

In view of the fact that the Jordan content of K and the successive minima $m_1, \cdots, m_n$ are continuous functions of K it is only necessary to prove the inequality for strictly convex bodies K. Thus it is assumed that from now on K is strictly convex.

As with Davenport (1) this proof depends upon the inductive construction of sets $K_1, \cdots, K_n$ so that K_i has the following properties

(1) K_i is a bounded Jordan measurable open star body symmetric in the origin;

(2) $K_i \subset \frac{1}{2}m_i K$;

(3) $K_i \cap R_j \subset \frac{1}{2}m_j K$ for $j = 1, 2, \cdots, i-1$;

(4) $V(K_i) = \left(\dfrac{m_i}{m_{i-1}}\right)^{n-i+1} V(K_{i-1})$ if $i \geqq 2$;

(5) $V(K_1) = (\frac{1}{2})^n m_1^n V(K)$.

(6) No two translates of K_i by points of L_n have a point in common.

Putting $K_1 = \frac{1}{2}m_1 K$ properties (1)—(6) are satisfied for $i = 1$. Assuming that K_i has been constructed to satisfy (1)—(6) define K_{i+1} as follows.

Let $X \in R_n$. Since K is strictly convex the linear manifold $X+R_i$ contains a unique point, $f(X)$ say, that is closest to the origin in the metric determined by K. If the coordinates of X are $x_1, \cdots, x_n$ then the coordinates of $f(X)$ are independent of $x_1, \cdots, x_i$ and continuous functions of $x_{i+1}, \cdots, x_n$. Moreover as K is star and symmetric in the origin so

$$f(tX) = tf(X) \text{ for all real } t.$$

Any point of R_n can now be written in the form

$$A+f(X)$$

where $A \in R_i$ and $A, f(X)$ are uniquely determined. Define K_{i+1} to be the set of all points

$$A + \left(\frac{m_{i+1}}{m_i}\right) f(X)$$

for which $A+f(X) \in K_i$. It remains to verify that K_{i+1} has properties (1)—(6).

Let $P \in K_{i+1}$ so that $P = A+(m_{i+1}/m_i)f(X)$ where $A+f(X) \in K_i$. Since K_i is star and symmetric in the origin it follows that $tA+tf(X) \in K_i$ for $|t| \leqq 1$. Therefore also

$$tA + \left(\frac{m_{i+1}}{m_i}\right) tf(X) = tP \in K_{i+1}$$

and K_{i+1} is star and symmetric the origin. As $f(X)$ is continuous in each of its coordinates so K_{i+1} is open and Jordan measurable and so satisfies (1). Also $A+f(X) \in K_i \subset \frac{1}{2}m_iK$ and therefore by construction $f(X) \in \frac{1}{2}m_iK$. By the convexity of $\frac{1}{2}m_iK$ it follows that

$$\left(\frac{m_i}{m_{i+1}}\right) (A+f(X)) + \left(1- \frac{m_i}{m_{i+1}}\right) f(X) \in \tfrac{1}{2}m_iK$$

i.e.

$$\left(\frac{m_i}{m_{i+1}}\right) A+f(X) \in \tfrac{1}{2}m_iK$$

hence

$$A + \left(\frac{m_{i+1}}{m_i}\right) f(X) \in \tfrac{1}{2}m_{i+1}K$$

and K_i satisfies (2).

By construction $K_{i+1} \cap R_j = K_i \cap R_j$ for $j = 1, 2, \cdots, i$ so that K_{i+1} satisfies (3).

By Fubini's theorem K_{i+1} has the same Jordan content as the point set obtained from K_i by dilating the last $n-i$ coordinates by the factor m_{i+1}/m_i. Hence

$$V(K_{i+1}) = \left(\frac{m_{i+1}}{m_i}\right)^{n-i} V(K_i)$$

and K_{i+1} has property (4).

Property (5) refers only to the case $i = 1$ so it remains to verify (6). If two translates of K_{i+1} by points of L_n have a point in common then there exist a pair of points X, Y say in K_{i+1} such that $X-Y \in K_{i+1}$. Since $K_{i+1} \subset \frac{1}{2}m_{i+1}K$ it follows that $X-Y \in L_i \subset R_i$. Now

$$X = A + \left(\frac{m_{i+1}}{m_i}\right) f(Z), \quad Y = B + \left(\frac{m_{i+1}}{m_i}\right) f(Z')$$

where $A, B \in R_i$ and

$$A+f(Z) \in K_i, \quad B+f(Z') \in K_i.$$

As $X-Y \in R_i$ so $f(Z) = f(Z')$ and

$X-Y = A-B = (A+f(Z))-(B+f(Z'))$. Thus there are two distinct points of K_i whose difference is in L_n and therefore two translates of K_i by L_n have a point in common which is impossible. K_{i+1} therefore has properties (1)—(6).

The sets $K_1, \cdots, K_n$ having been constructed property (6) implies that

$$V(K_n) \leqq 1$$

or $m_1 m_2 \cdots m_n V(K) \leqq 2^n$ which completes the proof.

The above construction of K_n raises the question whether a convex centrally symmetric body exists with properties similar to (1)—(6). Further if C_i denotes the set of points in R_i that are a distance m_i from the origin in the metric determined by K and if D denotes the set of all points that are closer to the origin than to any point of $\bigcup_{i=1}^{n} C_i$ then it can be shown that D is an open star body symmetric in the origin the translates of which by points of L_n do not intersect. This raises the question of whether the content of D is large enough, i.e. at least as large as $V(K_n)$, to produce another proof of the inequality.

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