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LEADING COEFFICIENTS AND CELLULAR BASES OF HECKE ALGEBRAS

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Abstract Let $\boldsymbol{H}$ be the generic Iwahori–Hecke algebra associated with a finite Coxeter group W. Recently, we have shown that $\boldsymbol{H}$ admits a natural cellular basis in the sense of Graham and Lehrer, provided that W is a Weyl group and all parameters of $\boldsymbol{H}$ are equal. The construction involves some data arising from the Kazhdan–Lusztig basis $\{\boldsymbol{C}_w\}$ of $\boldsymbol{H}$ and Lusztig's asymptotic ring $\boldsymbol{J}$. We attempt to study $\boldsymbol{J}$ and its representation theory from a new point of view. We show that $\boldsymbol{J}$ can be obtained in an entirely different fashion from the generic representations of $\boldsymbol{H}$, without any reference to $\{\boldsymbol{C}_w\}$. We then extend the construction of the cellular basis to the case where W is not crystallographic. Furthermore, if $\boldsymbol{H}$ is a multi-parameter algebra, we see that there always exists at least one cellular structure on $\boldsymbol{H}$. Finally, the new construction of $\boldsymbol{J}$ may be extended to Hecke algebras associated with complex reflection groups.

Keywords: Hecke algebras; cellular bases; asymptotic ring

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1. Introduction

Let $\boldsymbol{H}$ be a generic one-parameter Iwahori–Hecke algebra associated with a finite Weyl group W, defined over a suitable ring of Laurent polynomials. (More precise definitions will be given below.) By definition, $\boldsymbol{H}$ has a standard basis usually denoted by $\{T_w \mid w \in W\}$. Using properties of the 'new' basis $\{\boldsymbol{C}_w \mid w \in W\}$ introduced in [**14**], Lusztig has defined a ring $\boldsymbol{J}$ which has a $\mathbb{Z}$-basis $\{t_w \mid w \in W\}$ and integral structure constants, and which can be viewed as an 'asymptotic' version of $\boldsymbol{H}$. All the ingredients in the construction of $\boldsymbol{J}$ can be defined in an elementary way, but the proof that we indeed obtain an associative ring with identity requires a deep geometric interpretation of the basis $\{\boldsymbol{C}_w\}$ (see [**19**, **20**]).

It turns out that $\boldsymbol{J}_{\mathbb{Q}} = \mathbb{Q} \bigotimes_{\mathbb{Z}} \boldsymbol{J}$ is a split semisimple algebra isomorphic to the group algebra of W. Using properties of the irreducible representations of $\boldsymbol{J}_{\mathbb{Q}}$, we have recently proved in [**9**] that $\boldsymbol{H}$ has a natural 'cellular' structure in the sense of Graham and Lehrer [**13**]. The elements of the corresponding 'cellular' basis of $\boldsymbol{H}$ are certain $\mathbb{Z}$-linear combinations of the basis $\{\boldsymbol{C}_w\}$, where the coefficients involve data arising from the action of the basis elements t_w in the irreducible representations of $\boldsymbol{J}_{\mathbb{Q}}$. Note that, although there is an isomorphism between $\boldsymbol{J}_{\mathbb{Q}}$ and the group algebra of W, it does not seem to be possible to see the data that we need easily through this isomorphism. (For example, the image of t_w in the group algebra of W is, in general, a rather complicated sum of group elements.)

Now Lusztig [**17**, **20**] has shown that the construction of $\boldsymbol{J}$ also makes sense—under the assumption that the conjectures (P1)–(P15) in [**20**, § 14.2] hold—when we consider an Iwahori–Hecke algebra $\boldsymbol{H}$ with possibly unequal parameters. The results in [**9**] also extend to this case, assuming that (P1)–(P15) hold.

One of the purposes of this paper is to show that the data required to define a 'cellular' basis of $\boldsymbol{H}$ can be obtained in an alternative way, using the generic irreducible representations of $\boldsymbol{H}$ and the leading matrix coefficients introduced in [**8**]. These coefficients even allow us to construct a ring $\tilde{\boldsymbol{J}}$ with rational structure constants, and show that it is associative with identity, without any reference to the Kazhdan–Lusztig basis $\{\boldsymbol{C}_w\}$ at all. We expect that we have $\boldsymbol{J} = \tilde{\boldsymbol{J}}$ in general but, at present, we can only prove this equality by assuming that Lusztig's conjectures (P1)–(P15) hold.

As an application, we extend the construction of a 'cellular' basis to Iwahori–Hecke algebras associated with non-crystallographic finite Coxeter groups, as announced in [**9**, Remark 3.3]. Using the results in [**10**], we can also show that an Iwahori–Hecke algebra with possibly unequal parameters always admits at least one 'cellular' structure.

Another aspect of our construction of the ring $\tilde{\boldsymbol{J}}$ is that it may actually be applied to other types of algebras, like the cyclotomic Hecke algebras of Broué and Malle [**4**] associated with complex reflection groups. We hope to discuss this in more detail elsewhere.

This paper is organized as follows. In § 2, we briefly recall the main facts about the Kazhdan–Lusztig basis and the $\boldsymbol{a}$-invariants of the irreducible representations of W. Here, we work in the general case of possibly unequal parameters. In Proposition 2.5, we recall a result from [**10**] which shows that the structure constants of Lusztig's ring $\boldsymbol{J}$ can be expressed in terms of the 'leading matrix coefficients' of [**8**]. This is the starting point for our construction of a new ring $\tilde{\boldsymbol{J}}$ (see § 3). For this purpose, we use a definition of the leading matrix coefficients which is somewhat more general than that in [**8**]; this generalization is necessary to obtain the strongest possible statements in our applications. The new definition involves the concept of 'balanced' representations, which will be studied in more detail in § 4. In particular, we establish an efficient criterion for checking if a given representation is balanced or not (see Proposition 4.3). We will show that the analogue of [**9**, Proposition 2.6] (which describes the data required to define a cellular basis) holds for all types of W and all choices of the parameters. In § 5, we formulate the hypothesis $(\widetilde{\text{P15}})$, which is a variant of Lusztig's (P15) in [**20**, § 14.2]. This hypothesis alone allows us to construct a cellular basis of $\boldsymbol{H}$; the statement in Theorem 5.5 is actually slightly stronger than the main result of [**9**]. In the process of doing this, we give a simplified treatment of Lusztig's homomorphism from $\boldsymbol{H}$ into $\boldsymbol{J}$ (see Theorem 5.2).

Let us now introduce some basic notation that will be used throughout this paper. Let (W, S) be a Coxeter system and let $l : W \to \mathbb{Z}_{\geqslant 0}$ be the usual length function. In this paper, we will only consider the case where W is a finite group. Let Γ be an abelian group (written additively). Following [**20**], a function $L : W \to \Gamma$ is called a *weight function* if $L(ww') = L(w) + L(w')$ whenever $w, w' \in W$ are such that $l(ww') = l(w) + l(w')$. Note that L is uniquely determined by the values $\{L(s) \mid s \in S\}$. Furthermore, if $\{c_s \mid s \in S\}$ is a collection of elements in Γ such that $c_s = c_t$ whenever $s, t \in S$ are conjugate in W, then there is a (unique) weight function $L : W \to \Gamma$ such that $L(s) = c_s$ for all $s \in S$.

Let $R \subseteq \mathbb{C}$ be a subring and let $A = R[\Gamma]$ be the free R-module with basis $\{\varepsilon^g \mid g \in \Gamma\}$. There is a well-defined ring structure on A such that $\varepsilon^g \varepsilon^{g'} = \varepsilon^{g+g'}$ for all $g, g' \in \Gamma$. We write $1 = \varepsilon^0 \in A$. Given $a \in A$, we denote by a_g the coefficient of ε^g, so that $a = \sum_{g \in \Gamma} a_g \varepsilon^g$. Let $\boldsymbol{H} = \boldsymbol{H}_A(W, S, L)$ be the *generic Iwahori–Hecke algebra* over A with parameters $\{v_s \mid s \in S\}$, where $v_s := \varepsilon^{L(s)}$ for $s \in S$. This is an associative algebra, which is free as an A-module, with basis $\{T_w \mid w \in W\}$. The multiplication is given by the rule

$$T_s T_w = \begin{cases} T_{sw} & \text{if } l(sw) > l(w), \\ T_{sw} + (v_s - v_s^{-1}) T_w & \text{if } l(sw) < l(w), \end{cases}$$

where $s \in S$ and $w \in W$. The element T_1 is the identity element.

Example 1.1. Assume that $\Gamma = \mathbb{Z}$. Then A is nothing but the ring of Laurent polynomials over R in an indeterminate ε; we will usually set $v = \varepsilon$. Then $\boldsymbol{H}$ is an associative algebra over $A = R[v, v^{-1}]$ with relations

$$T_s T_w = \begin{cases} T_{sw} & \text{if } l(sw) > l(w), \\ T_{sw} + (v^{c_s} - v^{-c_s}) T_w & \text{if } l(sw) < l(w), \end{cases}$$

where $s \in S$ and $w \in W$. This is the setting of Lusztig [**20**].

Example 1.2.

(a) Assume that $\Gamma = \mathbb{Z}$ and L is constant on S; this case will be referred to as the *equal parameter case*. Note that we are automatically in this case when W is of type A_{n-1}, D_n, $I_2(m)$, where m is odd, H_3, H_4, E_6, E_7 or E_8 (since all generators in S are conjugate in W).

(b) Assume that W is irreducible. Then unequal parameters can only arise in types B_n, F_4 and $I_2(m)$, where m is even.

Example 1.3. A 'universal' weight function is given as follows. Let Γ_0 be the group of all tuples $(n_s)_{s \in S}$, where $n_s \in \mathbb{Z}$ for all $s \in S$ and $n_s = n_t$ whenever $s, t \in S$ are conjugate in W. (The addition is defined componentwise). Let $L_0 : W \to \Gamma_0$ be the weight function given by sending $s \in S$ to the tuple $(n_t)_{t \in S}$, where $n_t = 1$ if t is conjugate to s, and $n_t = 0$ otherwise. Let $A_0 = R[\Gamma_0]$ and $\boldsymbol{H}_0 = \boldsymbol{H}_{A_0}(W, S, L_0)$ be the associated Iwahori–Hecke algebra, with parameters $\{v_s \mid s \in S\}$. Then $A_0 = R[\Gamma_0]$ is nothing but the ring of Laurent polynomials in indeterminates v_s $(s \in S)$ with coefficients in R, where $v_s = v_t$ whenever $s, t \in S$ are conjugate in W. Furthermore, if $S' \subseteq S$ is a set of representatives for the classes of S under conjugation, then $\{v_s \mid s \in S'\}$ are algebraically independent.

2. The Kazhdan–Lusztig basis and leading matrix coefficients

We now introduce two concepts whose interplay is the main subject of this paper: the Kazhdan–Lusztig basis and leading matrix coefficients. Both of these essentially rely on the choice of a total ordering $\leqslant$ on Γ which is compatible with the group structure, that is, whenever $g, g', h \in \Gamma$ are such that $g \leqslant g'$, then $g + h \leqslant g' + h$. Such an order on Γ will be called a *monomial order*.

We will assume that such an ordering exists on Γ. One may readily check that this implies that $A = R[\Gamma]$ is an integral domain; we usually reserve the letter K to denote its field of fractions. If we are in the equal parameter case (Example 1.2), the group $\Gamma = \mathbb{Z}$ has a natural monomial order. On the other hand, in the setting of Example 1.3 (assuming that not all elements of S are conjugate), there are infinitely many monomial orders on Γ.

Throughout this paper, we fix a choice of a monomial order, and we assume that

$$L(s) > 0 \quad \text{for all } s \in S.$$

We define $\Gamma_{\geqslant 0} = \{g \in \Gamma \mid g \geqslant 0\}$ and denote by $\mathbb{Z}[\Gamma_{\geqslant 0}]$ the set of all integral linear combinations of terms ε^g, where $g \geqslant 0$. The notations $\mathbb{Z}[\Gamma_{>0}]$, $\mathbb{Z}[\Gamma_{\leqslant 0}]$ and $\mathbb{Z}[\Gamma_{<0}]$ have a similar meaning.

2.1. The *a*-invariants

We set $\mathbb{Z}_W := \mathbb{Z}[2\cos(2\pi/m_{st}) \mid s, t \in S]$ (where m_{st} denotes the order of st in W). Note that $\mathbb{Z}_W = \mathbb{Z}$ if W is a finite Weyl group (or of crystallographic type), that is, if $m_{st} \in \{2, 3, 4, 6\}$ for all $s, t \in S$. Recall that R is a subring of $\mathbb{C}$. We shall always assume that

$$\mathbb{Z}_W \subseteq R \quad \text{and} \quad F \text{ is the field of fractions of } R.$$

Then it is known that F is a splitting field for W (see [**12**, Theorem 6.3.8]). The set of irreducible representations of W (up to isomorphism) will be denoted by

$$\mathrm{Irr}(W) = \{E^\lambda \mid \lambda \in \Lambda\},$$

where Λ is some finite indexing set and E^λ is an F-vector space with a given $F[W]$-module structure. We shall also write

$$d_\lambda = \dim E^\lambda \quad \text{for all } \lambda \in \Lambda.$$

Let K be the field of fractions of A. By extension of scalars, we obtain a K-algebra $\boldsymbol{H}_K = K \bigotimes_A \boldsymbol{H}$. This algebra is known to be split semisimple (see [**12**, 9.3.5]). Furthermore, by Tits's Deformation Theorem, the irreducible representations of $\boldsymbol{H}_K$ (up to isomorphism) are in bijection with the irreducible representations of W (see [**12**, 8.1.7]). Thus, we can write

$$\mathrm{Irr}(\boldsymbol{H}_K) = \{E_\varepsilon^\lambda \mid \lambda \in \Lambda\}.$$

For $\lambda \in \Lambda$, denote by χ^λ the character afforded by E_ε^λ. Thus, we have $\chi^\lambda(T_w) = \mathrm{Tr}(T_w, E_\varepsilon^\lambda)$ for $w \in W$. Then the correspondence $E^\lambda \leftrightarrow E_\varepsilon^\lambda$ is uniquely determined by the following condition:

$$\theta_1(\chi^\lambda(T_w)) = \mathrm{Tr}(w, E^\lambda) \quad \text{for all } w \in W,$$

where $\theta_1 : A \to R$ is the unique R-algebra homomorphism such that $\theta_1(\varepsilon^g) = 1$ for all $g \in \Gamma$. Note also that $\chi^\lambda(T_w) \in A$ for all $w \in W$.

The algebra $\boldsymbol{H}$ is *symmetric*, with trace from $\tau : \boldsymbol{H} \to A$ given by $\tau(T_1) = 1$ and $\tau(T_w) = 0$ for $1 \neq w \in W$. The sets $\{T_w \mid w \in W\}$ and $\{T_{w^{-1}} \mid w \in W\}$ form a pair of dual bases. Hence, we have the following orthogonality relations:

$$\sum_{w \in W} \chi^\lambda(T_w)\chi^\mu(T_{w^{-1}}) = \begin{cases} d_\lambda \boldsymbol{c}_\lambda & \text{if } \lambda = \mu, \\ 0 & \text{if } \lambda \neq \mu \end{cases}$$

(see [**12**, 8.1.8]). Here, $0 \neq \boldsymbol{c}_\lambda \in A$ and, following Lusztig, we can write

$$\boldsymbol{c}_\lambda = f_\lambda \varepsilon^{-2\boldsymbol{a}_\lambda} + \text{combination of terms } \varepsilon^g, \quad \text{where } g > -2\boldsymbol{a}_\lambda;$$

here $\boldsymbol{a}_\lambda \in \Gamma_{\geqslant 0}$ and f_λ is a strictly positive real number (see [**8**, § 3.3]).

Remark 2.1. The invariants $\boldsymbol{a}_\lambda$ and f_λ are explicitly known for all types of W [**20**, Chapter 22]. The elements $\boldsymbol{c}_\lambda \in A$ and the coefficients f_λ are independent of the monomial order $\leqslant$, but $\boldsymbol{a}_\lambda$ heavily depends on it. Note that the statement concerning the independence of f_λ is of interest only in the unequal parameter case; see [**10**, Proposition 5.1 and Table 1] for types F_4 and $I_2(m)$ and [**20**, Proposition 22.14] for type B_n.

The invariants $\boldsymbol{a}_\lambda$ play a fundamental role in Lusztig's study [**18**] of the characters of reductive groups over finite fields. In [**9**], we use these invariants to define an ordering of Λ, which is an essential ingredient in the construction of a 'cellular' basis of $\boldsymbol{H}$.

2.2. Balanced representations

We can now introduce the notion of 'balanced' representations, which is slightly more general than the related concept of 'orthogonal' representations introduced in [**8**]. For this purpose, following [**8**], we consider a certain valuation ring $\mathcal{O}$ in K. Let us write

$$F[\Gamma_{\geqslant 0}] = \text{set of } F\text{-linear combinations of terms } \varepsilon^g, \quad \text{where } g \geqslant 0,$$
$$F[\Gamma_{>0}] = \text{set of } F\text{-linear combinations of terms } \varepsilon^g, \quad \text{where } g > 0.$$

Note that $1 + F[\Gamma_{>0}]$ is multiplicatively closed. Furthermore, every element $x \in K$ can be written in the form

$$x = r_x \varepsilon^{g_x} \frac{1+p}{1+q}, \quad \text{where } r_x \in F,\ g_x \in \Gamma \text{ and } p, q \in F[\Gamma_{>0}].$$

Note that if $x \neq 0$, then r_x and g_x indeed are *uniquely determined* by x; if $x = 0$, we have $r_0 = 0$ and we set $g_0 := +\infty$ by convention. We set

$$\mathcal{O} := \{x \in K \mid g_x \geqslant 0\} \quad \text{and} \quad \mathfrak{p} := \{x \in K \mid g_x > 0\}.$$

Then it is easily verified that $\mathcal{O}$ is a valuation ring in K, with maximal ideal $\mathfrak{p}$. Note that we have

$$\mathcal{O} \cap F[\Gamma] = F[\Gamma_{\geqslant 0}] \quad \text{and} \quad \mathfrak{p} \cap F[\Gamma] = F[\Gamma_{>0}].$$

We have a well-defined F-linear ring homomorphism $\mathcal{O} \to F$ with kernel $\mathfrak{p}$. The image of $x \in \mathcal{O}$ in F is called the *constant term* of x. Thus, the constant term of x is 0 if $x \in \mathfrak{p}$; the constant term equals r_x if $x \in \mathcal{O}^\times$.

Definition 2.2. Choosing a basis of E_ε^λ, we obtain a matrix representation $\rho^\lambda : \boldsymbol{H}_K \to M_{d_\lambda}(K)$. Given $h \in \boldsymbol{H}_K$ and $1 \leqslant i, j \leqslant d_\lambda$, we denote by $\rho_{ij}^\lambda(h)$ the (i,j)-entry of the matrix $\rho^\lambda(h)$. We say that ρ^λ is *balanced* if

$$\varepsilon^{\boldsymbol{a}_\lambda} \rho_{ij}^\lambda(T_w) \in \mathcal{O} \quad \text{for all } w \in W \text{ and all } i, j \in \{1, \dots, d_\lambda\}.$$

If ρ^λ is balanced, we define the *leading matrix coefficient* $c_{w,\lambda}^{ij} \in F$ to be the constant term of $(-1)^{l(w)} \varepsilon^{\boldsymbol{a}_\lambda} \rho_{ij}^\lambda(T_w)$.

Proposition 2.3 (Geck [8, § 4]). *For each $\lambda \in \Lambda$, there exists a balanced representation ρ^λ afforded by E_ε^λ; moreover, ρ^λ can be chosen such that*

$$\Delta^\lambda \rho^\lambda(T_{w^{-1}}) = \rho^\lambda(T_w)^{\mathrm{tr}} \Delta^\lambda \quad \textit{for all } w \in W,$$

where $\Delta^\lambda \in M_{d_\lambda}(\mathcal{O})$ is a diagonal matrix with diagonal coefficients having positive real numbers as constant terms. In particular, $\det(\Delta^\lambda) \in \mathcal{O}^\times$.

Proof. We may assume without loss of generality that $F \subseteq \mathbb{R}$. Let $(\cdot\,,\cdot)$ be any symmetric bilinear form on E_ε^λ which admits an orthonormal basis. We define a new bilinear form $\langle\cdot\,,\cdot\rangle$ by the formula

$$\langle e, e' \rangle := \sum_{w \in W} (T_w.e, T_w.e') \quad \text{for any } e, e' \in E_\varepsilon^\lambda.$$

As in the proof of [**16**, 1.7], it is easily checked that $\langle T_s \cdot e, e' \rangle = \langle e, T_s \cdot e' \rangle$ for all $s \in S$ and, hence, $\langle T_w.e, e' \rangle = \langle e, T_{w^{-1}} \cdot e' \rangle$ for all $w \in W$. Arguing as in step 1 of the proof of [**8**, Proposition 4.3], we see that the following holds:

$$\text{for any } 0 \neq e \in E_\varepsilon^\lambda \text{ we have } \varepsilon^{2g} \langle e, e \rangle \in b + \mathfrak{p}, \tag{$*$}$$

where $g \in \Gamma$ and $b \in F$ is such that $b > 0$. (Recall that $F \subseteq \mathbb{R}$.) Since we are working over a field of characteristic 0, there exists an orthogonal basis, $\{e_1, \dots, e_{d_\lambda}\}$ say, with respect to $\langle\cdot\,,\cdot\rangle$. Now $(*)$ implies that by multiplying the basis vectors e_i by ε^{-g_i} for suitable $g_i \in \Gamma$ we can assume that

$$\langle e_i, e_i \rangle \in b_i + \mathfrak{p}, \quad \text{where } b_i \in F,\ b_i > 0.$$

Let ρ^λ be the matrix representation afforded by E_ε^λ with respect to the basis $\{e_1, \dots, e_{d_\lambda}\}$ and let Δ^λ be the Gram matrix of $\langle\cdot\,,\cdot\rangle$ with respect to that basis. Let D^λ be the diagonal matrix with $b_1, \dots, b_{d_\lambda}$ on the diagonal. Then we have

$$\Delta^\lambda \equiv D^\lambda \bmod \mathfrak{p} \qquad \text{and} \qquad \Delta^\lambda \rho^\lambda(T_{w^{-1}}) = \rho^\lambda(T_w)^{\mathrm{tr}} \Delta^\lambda \quad \text{for all } w \in W.$$

We can now argue as in the proof of [**8**, Theorem 4.4] to show that ρ^λ is balanced. Indeed, let $\gamma \in \Gamma$ be minimal such that $\varepsilon^\gamma \rho_{ij}^\lambda(T_w) \in \mathcal{O}$ for all $w \in W$ and all $1 \leqslant i, j \leqslant d_\lambda$. Let $\hat{c}_{w,\lambda}^{ij} \in F$ be the constant term of $\varepsilon^\gamma \rho_{ij}^\lambda(T_w)$. Choose $i, j \in \{1, \dots, d_\lambda\}$ such that $\hat{c}_{y,\lambda}^{ij} \neq 0$ for some $y \in W$. Now, we do not only have the orthogonality relations already mentioned above, but also the Schur relations in [**12**, Corollary 7.2.2]. Thus, we have

$$\varepsilon^{2\gamma} \boldsymbol{c}_\lambda \equiv \sum_{w \in W} (\varepsilon^\gamma \rho_{ij}^\lambda(T_w))(\varepsilon^\gamma \rho_{ji}^\lambda(T_{w^{-1}})) \equiv \sum_{w \in W} \hat{c}_{w,\lambda}^{ij} \hat{c}_{w^{-1},\lambda}^{ji} \bmod \mathfrak{p}.$$

Now we multiply the relation $\Delta^\lambda \rho^\lambda(T_{w^{-1}}) = \rho^\lambda(T_w)^{\mathrm{tr}} \Delta^\lambda$ by ε^γ and consider constant terms. Taking into account the relation $\Delta^\lambda \equiv D^\lambda \bmod \mathfrak{p}$, we obtain

$$b_j \hat{c}_{w^{-1},\lambda}^{ji} = \hat{c}_{w,\lambda}^{ij} b_i \quad \text{for all } w \in W.$$

This yields

$$\sum_{w \in W} \hat{c}_{w,\lambda}^{ij} \hat{c}_{w^{-1},\lambda}^{ji} = b_i b_j^{-1} \sum_{w \in W} (\hat{c}_{w,\lambda}^{ij})^2,$$

which is a non-zero real number since $\hat{c}_{y,\lambda}^{ij} \neq 0$ for some $y \in W$. Thus, we conclude that $\varepsilon^{2\gamma} \boldsymbol{c}_\lambda$ lies in $\mathcal{O}$ and has a non-zero constant term. Comparing this with the relation $\varepsilon^{2\boldsymbol{a}_\lambda} \boldsymbol{c}_\lambda \equiv f_\lambda \bmod \mathfrak{p}$, we deduce that $\gamma = \boldsymbol{a}_\lambda$ as required. □

Remark 2.4. In [**8**, Proposition 4.3], we assumed that $F = \mathbb{R}$. This allowed us to go one step further in the above proof and take square roots of the numbers b_i. Consequently, by rescaling the basis vectors e_i, we can even assume that Δ^λ is diagonal with diagonal coefficients in $1 + \mathfrak{p}$. The resulting balanced representations were called *orthogonal representations* in [**8**]. The corresponding leading matrix coefficients satisfy the following additional property (see [**8**, Theorem 4.4]):

$$c_{w,\lambda}^{ij} = c_{w^{-1},\lambda}^{ji} \quad \text{for all } w \in W \text{ and } 1 \leqslant i, j \leqslant d_\lambda.$$

2.3. The Kazhdan–Lusztig basis and Lusztig's $\boldsymbol{a}$-function

We now recall the basic facts about the Kazhdan–Lusztig basis of $\boldsymbol{H}$, following [**17**,**20**]. Again, this relies on the choice of a monomial $\leqslant$ on Γ. Now, there is a unique ring involution $A \to A$, $a \mapsto \bar{a}$, such that $\overline{\varepsilon^g} = \varepsilon^{-g}$ for all $g \in \Gamma$. We can extend this map to a ring involution $\boldsymbol{H} \to \boldsymbol{H}$, $h \mapsto \bar{h}$, such that

$$\overline{\sum_{w \in W} a_w T_w} = \sum_{w \in W} \bar{a}_w T_{w^{-1}}^{-1}, \quad a_w \in A.$$

By [**14**, **17**, **20**], we have a 'new' basis $\{C'_w \mid w \in W\}$ of $\boldsymbol{H}$ (depending on $\leqslant$), where C'_w is characterized by the following two conditions:

- $\bar{C}'_w = C'_w$ and
- $C'_w = T_w + \sum_{y \in W} p_{y,w} T_y$, where $p_{y,w} \in \mathbb{Z}[\Gamma_{<0}]$ for all $y \in W$.

Here we follow the original notation in [**14**, **17**]; the element C'_w is denoted by c_w in [**20**, Theorem 5.2]. As in [**20**], it will be convenient to work with the following alternative version of the Kazhdan–Lusztig basis. We set $\boldsymbol{C}_w = (C'_w)^\dagger$ for all $w \in W$, where $\dagger : \boldsymbol{H} \to \boldsymbol{H}$ is the A-algebra automorphism defined by $T_s^\dagger = -T_s^{-1}$, $s \in S$ (see [**20**, § 3.5]). Note that $\bar{h} = j(h)^\dagger = j(h^\dagger)$ for all $h \in \boldsymbol{H}$, where $j : \boldsymbol{H} \to \boldsymbol{H}$ is the ring involution such that $j(a) = \bar{a}$ for $a \in A$ and $j(T_w) = (-1)^{l(w)} T_w$ for $w \in W$. Thus, we have

- $\bar{\boldsymbol{C}}_w = \boldsymbol{C}_w$ and
- $\boldsymbol{C}_w = j(C'_w) = (-1)^{l(w)} T_w + \sum_{y \in W} (-1)^{l(y)} \bar{p}_{y,w} T_y$, where $\bar{p}_{y,w} \in \mathbb{Z}[\Gamma_{>0}]$.

Since the elements $\{\boldsymbol{C}_w \mid w \in W\}$ form a basis of $\boldsymbol{H}$, we can write

$$\boldsymbol{C}_x \boldsymbol{C}_y = \sum_{z \in W} h_{x,y,z} \boldsymbol{C}_z \quad \text{for any } x, y \in W,$$

where $h_{x,y,z} = \bar{h}_{x,y,z} \in A$ for all $x, y, z \in W$. Note that either $h_{x,y,z} \in \mathbb{Z}$ or $h_{x,y,z}$ involves terms from both $\Gamma_{<0}$ and $\Gamma_{>0}$. For a fixed $z \in W$, we set

$$\boldsymbol{a}(z) := \min\{g \in \Gamma_{\geqslant 0} \mid \varepsilon^g h_{x,y,z} \in \mathbb{Z}[\Gamma_{\geqslant 0}] \text{ for all } x, y \in W\}.$$

This is Lusztig's function $\boldsymbol{a} : W \to \Gamma$ (see [**20**, Chapter 13]). Given $x, y, z \in W$, we have $\varepsilon^{\boldsymbol{a}(z)} h_{x,y,z} \in \mathbb{Z}[\Gamma_{\geqslant 0}]$. By [**20**, § 13.9], we have $\boldsymbol{a}(z) = \boldsymbol{a}(z^{-1})$. Then we define $\gamma_{x,y,z} \in \mathbb{Z}$ to be the constant term of $\varepsilon^{\boldsymbol{a}(z)} h_{x,y,z^{-1}} \in \mathbb{Z}[\Gamma_{\geqslant 0}]$, that is, we have

$$\varepsilon^{\boldsymbol{a}(z)} h_{x,y,z^{-1}} \equiv \gamma_{x,y,z} \bmod \mathbb{Z}[\Gamma_{>0}].$$

These constants appear as the structure constants in Lusztig's ring $\boldsymbol{J}$ [**20**, Chapter 18].

We can now state the following result, which relates the $\boldsymbol{a}$-function and $\gamma_{x,y,z}$ to leading matrix coefficients. Here we assume that, for each $\lambda \in \Lambda$, we have chosen a balanced representation ρ^λ afforded by E_ε^λ as in Remark 2.4. (We will see in Lemma 3.2 that the same statement holds for any choice of balanced representations.)

Proposition 2.5 (Geck [10, Proposition 3.6 and Remark 4.2]). *Assume that conjectures (P1)–(P15) in [**20**, § 14.2] hold. Let $z \in W$. If $\lambda \in \Lambda$ and $i, j \in \{1, \dots, d_\lambda\}$ are such that $c_{z,\lambda}^{ij} \neq 0$, then $\boldsymbol{a}(z) = \boldsymbol{a}_\lambda$. Furthermore, for all $x, y, z \in W$, we have*

$$\gamma_{x,y,z} = \sum_{\lambda \in \Lambda} \sum_{1 \leqslant i,j,k \leqslant d_\lambda} f_\lambda^{-1} c_{x,\lambda}^{ij} c_{y,\lambda}^{jk} c_{z,\lambda}^{ki}.$$

In the next section, we will use the expression on the right-hand side of the above identity to construct a ring $\tilde{\boldsymbol{J}}$, *without* assuming that (P1)–(P15) hold. Note also that not all of (P1)–(P15) are required for proving Proposition 2.5. For example, (P15) is not needed (see [**10**, Remark 3.9]).

Remark 2.6. The conjectures (P1)–(P15) are known to hold, for example, in the equal parameter case. For crystallographic W, see [**20**, Chapter 16] and the references therein; for W of type $I_2(m)$, H_3 or H_4, see [**7**]. (An alternative argument for proving

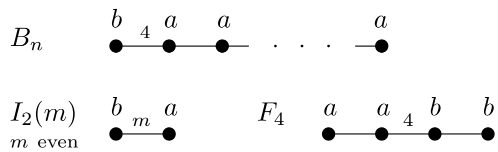

Figure 1.

(P1)–(P15) in types H_3 and H_4 is sketched in [**10**, Example 4.8].) Now let (W, S) be of type B_n, F_4 or $I_2(m)$ (m even). Let $L_0 : W \to \Gamma_0$ be the universal weight function as in Example 1.3. Thus, L_0 depends on two values $a, b \in \Gamma$, which are attached to the generators in S, as shown in Figure 1.

Choose a pure lexicographic order on Γ_0, such that $b > ra > 0$ for all $r \in \mathbb{Z}_{\geqslant 1}$. Then (P1)–(P15) are also known to hold (see [**10**, Theorem 5.3] and the references therein). In analogy to [**2**], this may be called the general 'asymptotic case'.

3. The ring $\tilde{\boldsymbol{J}}$

In this section, we show that the 'leading matrix coefficients' associated with balanced representations as in Definition 2.2 can be used to construct a ring $\tilde{\boldsymbol{J}}$. We keep the basic setting of § 2.2. Throughout this section we assume that, for each $\lambda \in \Lambda$, we are given a balanced representation ρ^λ afforded by E_ε^λ, with corresponding leading matrix coefficients $c_{w,\lambda}^{ij}$.

Definition 3.1. For $w, x, y, z \in W$, we set

$$\tilde{\gamma}_{x,y,z} := \sum_{\lambda \in \Lambda} \sum_{1 \leqslant i,j,k \leqslant d_\lambda} f_\lambda^{-1} c_{x,\lambda}^{ij} c_{y,\lambda}^{jk} c_{z,\lambda}^{ki},$$

$$\tilde{n}_w := \sum_{\lambda \in \Lambda} \sum_{1 \leqslant i \leqslant d_\lambda} f_\lambda^{-1} c_{w^{-1},\lambda}^{ii}.$$

Let $\tilde{\boldsymbol{J}}$ be the F-vector space with basis $\{t_w \mid w \in W\}$. We define a bilinear product on $\tilde{\boldsymbol{J}}$ by

$$t_x t_y = \sum_{z \in W} \tilde{\gamma}_{x,y,z^{-1}} t_z, \quad x, y \in W.$$

Let $\tilde{\mathcal{D}} := \{w \in W \mid \tilde{n}_w \neq 0\}$. We define an element of $\tilde{\boldsymbol{J}}$ by $1_{\tilde{\boldsymbol{J}}} := \sum_{w \in \tilde{\mathcal{D}}} \tilde{n}_w t_w$.

Lemma 3.2. *The constants $\tilde{\gamma}_{x,y,z}$ and $\tilde{n}_w$ do not depend on the choice of the balanced representations $\{\rho^\lambda \mid \lambda \in \Lambda\}$.*

Proof. For any $\lambda \in \Lambda$, the sum $\sum_{1 \leqslant i,j,k \leqslant d_\lambda} c_{x,\lambda}^{ij} c_{y,\lambda}^{jk} c_{z,\lambda}^{ki}$ (appearing in the definition of $\tilde{\gamma}_{x,y,z}$) is the constant term of

$$\varepsilon^{3\boldsymbol{a}_\lambda} \sum_{1 \leqslant i,j,k \leqslant d_\lambda} \rho_{ij}^\lambda(T_x) \rho_{jk}^\lambda(T_y) \rho_{ki}^\lambda(T_z) \in \mathcal{O}.$$

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But the latter expression just equals $\varepsilon^{3\boldsymbol{a}_\lambda}\chi^\lambda(T_xT_yT_z)$ and, hence, depends on the character of ρ^λ but not on the choice of ρ^λ itself. A similar argument applies to the sum $\sum_{1\leqslant i\leqslant d_\lambda} c^{ii}_{w^{-1},\lambda}$ (which appears in the definition of $\tilde{n}_w$): it is the constant term of $\varepsilon^{\boldsymbol{a}_\lambda}\chi^\lambda(T_{w^{-1}})$. □

Remark 3.3. Since $\boldsymbol{H}$ is symmetric, we have the following Schur relations (see [**12**, Corollary 7.2.2]):

$$\sum_{y\in W}\rho^\lambda_{ij}(T_w)\rho^\mu_{kl}(T_{w^{-1}}) = \delta_{il}\delta_{jk}\delta_{\lambda\mu}\boldsymbol{c}_\lambda,$$

where $\lambda,\mu\in\Lambda$, $1\leqslant i,j\leqslant d_\lambda$ and $1\leqslant k,l\leqslant d_\mu$. Multiplying by $\varepsilon^{\boldsymbol{a}_\lambda+\boldsymbol{a}_\mu}$ and taking constant terms on both sides, we obtain orthogonality relations for the leading matrix coefficients:

$$\sum_{w\in W} c^{ij}_{w,\lambda}c^{kl}_{w^{-1},\mu} = \delta_{il}\delta_{jk}\delta_{\lambda\mu}f_\lambda. \tag{3.1}$$

These relations can be 'inverted' and so we also have

$$\sum_{\lambda\in\Lambda}\sum_{1\leqslant i,j\leqslant d_\lambda} f_\lambda^{-1}c^{ij}_{x,\lambda}c^{ji}_{y^{-1},\lambda} = \delta_{xy} \quad \text{for all } x,y\in W. \tag{3.1$'$}$$

Lemma 3.4. *We have the following relations:*

(a) $\tilde{\gamma}_{x,y,z} = \tilde{\gamma}_{y,z,x}$ *for all* $x,y,z\in W$,

(b) $\sum_{w\in W}\tilde{\gamma}_{x^{-1},y,w}\tilde{n}_w = \delta_{xy}$ *for all* $x,y\in W$.

Proof. (a) Just note that the defining formula for $\tilde{\gamma}_{x,y,z}$ is symmetrical under cyclic permutations of x,y,z.

(b) Using the defining formulae for $\tilde{\gamma}_{x,y,z}$ and $\tilde{n}_w$, the left-hand side is evaluated as

$$\begin{aligned}\biggl(\sum_{\lambda\in\Lambda}\sum_{1\leqslant i,j,k\leqslant d_\lambda} f_\lambda^{-1}c^{ij}_{x^{-1},\lambda}c^{jk}_{y,\lambda}c^{ki}_{w,\lambda}\biggr)\biggl(\sum_{w\in W}\sum_{\mu\in\Lambda}\sum_{1\leqslant p\leqslant d_\mu} f_\mu^{-1}c^{pp}_{w^{-1},\mu}\biggr)\\ = \sum_{\lambda,\mu\in\Lambda}\sum_{1\leqslant i,j,k\leqslant d_\lambda}\sum_{1\leqslant p\leqslant d_\mu} f_\lambda^{-1}f_\mu^{-1}c^{ij}_{x^{-1},\lambda}c^{jk}_{y,\lambda}\biggl(\sum_{w\in W}c^{ki}_{w,\lambda}c^{pp}_{w^{-1},\mu}\biggr).\end{aligned}$$

By the relations in (3.1), the parenthesized sum evaluates to $\delta_{kp}\delta_{ip}\delta_{\lambda\mu}f_\lambda$. Inserting this into the above expression yields $\sum_{\lambda\in\Lambda}\sum_{1\leqslant i,j\leqslant d_\lambda} f_\lambda^{-1}c^{ij}_{x^{-1},\lambda}c^{ji}_{y,\lambda} = \delta_{xy}$, where the last equality holds by (3.1′). □

Proposition 3.5. $\tilde{\boldsymbol{J}}$ *is an associative algebra with identity element* $1_{\tilde{\boldsymbol{J}}}$.

Proof. Let $x,y,z\in W$. We must check that $(t_xt_y)t_z = t_x(t_yt_z)$, which is equivalent to

$$\sum_{u\in W}\tilde{\gamma}_{x,y,u^{-1}}\tilde{\gamma}_{u,z,w^{-1}} = \sum_{u\in W}\tilde{\gamma}_{x,u,w^{-1}}\tilde{\gamma}_{y,z,u^{-1}} \quad \text{for all } w\in W.$$

Using the defining formula, the left-hand side evaluates to

$$\sum_{u\in W}\Bigl(\sum_{\lambda\in\Lambda}\sum_{1\leqslant i,j,k\leqslant d_\lambda} f_\lambda^{-1}c_{x,\lambda}^{ij}c_{y,\lambda}^{jk}c_{u^{-1},\lambda}^{ki}\Bigr)\Bigl(\sum_{\mu\in\Lambda}\sum_{1\leqslant p,q,r\leqslant d_\mu} f_\mu^{-1}c_{u,\lambda}^{pq}c_{z,\lambda}^{qr}c_{w^{-1},\lambda}^{rp}\Bigr)$$
$$=\sum_{\lambda,\mu\in\Lambda}\sum_{1\leqslant i,j,k\leqslant d_\lambda}\sum_{1\leqslant p,q,r\leqslant d_\mu} f_\lambda^{-1}f_\mu^{-1}c_{x,\lambda}^{ij}c_{y,\lambda}^{jk}c_{z,\lambda}^{qr}c_{w^{-1},\lambda}^{rp}\Bigl(\sum_{u\in W}c_{u^{-1},\lambda}^{ki}c_{u,\lambda}^{pq}\Bigr).$$

By the relations in (3.1), the parenthesized sum evaluates to $\delta_{kq}\delta_{pi}\delta_{\lambda\mu}f_\lambda$. Hence, the above expression equals

$$\sum_{\lambda\in\Lambda}\sum_{1\leqslant i,j,k,r\leqslant d_\lambda} f_\lambda^{-1}c_{x,\lambda}^{ij}c_{y,\lambda}^{jk}c_{z,\lambda}^{kr}c_{w^{-1},\lambda}^{ri}.$$

By a similar computation, the right-hand side evaluates to

$$\sum_{u\in W}\Bigl(\sum_{\lambda\in\Lambda}\sum_{1\leqslant i,j,k\leqslant d_\lambda} f_\lambda^{-1}c_{x,\lambda}^{ij}c_{u,\lambda}^{jk}c_{w^{-1},\lambda}^{ki}\Bigr)\Bigl(\sum_{\mu\in\Lambda}\sum_{1\leqslant p,q,r\leqslant d_\mu} f_\mu^{-1}c_{y,\lambda}^{pq}c_{z,\lambda}^{qr}c_{u^{-1},\lambda}^{rp}\Bigr)$$
$$=\sum_{\lambda,\mu\in\Lambda}\sum_{1\leqslant i,j,k\leqslant d_\lambda}\sum_{1\leqslant p,q,r\leqslant d_\mu} f_\lambda^{-1}f_\mu^{-1}c_{x,\lambda}^{ij}c_{w^{-1},\lambda}^{ki}c_{y,\lambda}^{pq}c_{z,\lambda}^{qr}\Bigl(\sum_{u\in W}c_{u,\lambda}^{jk}c_{u^{-1},\lambda}^{rp}\Bigr)$$
$$=\sum_{\lambda\in\Lambda}\sum_{1\leqslant i,j,k,q\leqslant d_\lambda} f_\lambda^{-1}c_{x,\lambda}^{ij}c_{y,\lambda}^{jq}c_{z,\lambda}^{qk}c_{w^{-1},\lambda}^{ki}.$$

We see that both sides are equal; hence, $\tilde{\boldsymbol{J}}$ is associative. To show that $1_{\tilde{\boldsymbol{J}}}$ is the identity element of $\tilde{\boldsymbol{J}}$ we let $x\in W$ and note that

$$t_x 1_{\tilde{\boldsymbol{J}}}=\sum_{w\in W}\tilde{n}_w t_x t_w=\sum_{y\in W}\Bigl(\sum_{w\in W}\tilde{n}_w\tilde{\gamma}_{x,w,y^{-1}}\Bigr)t_y$$
$$=\sum_{y\in W}\Bigl(\sum_{w\in W}\tilde{n}_w\tilde{\gamma}_{y^{-1},x,w}\Bigr)t_y=t_x\quad\text{by Lemma 3.4 (a), (b).}$$

A similar argument shows that $1_{\tilde{\boldsymbol{J}}}t_x=t_x$. Thus, $1_{\tilde{\boldsymbol{J}}}$ is the identity element of $\tilde{\boldsymbol{J}}$. □

Proposition 3.6. *The linear map $\bar{\tau}:\tilde{\boldsymbol{J}}\to F$ defined by $\bar{\tau}(t_w)=\tilde{n}_{w^{-1}}$ is a symmetrizing trace such that $\bar{\tau}(t_x t_{y^{-1}})=\delta_{xy}$ for all $x,y\in W$.*

Proof. Let $x,y\in W$. Then, using Lemma 3.4 (b), we obtain

$$\bar{\tau}(t_{x^{-1}}t_y)=\sum_{w\in W}\tilde{\gamma}_{x^{-1},y,w^{-1}}\bar{\tau}(t_w)=\sum_{w\in W}\gamma_{x^{-1},y,w^{-1}}\tilde{n}_{w^{-1}}=\delta_{xy}.$$

This implies that $\bar{\tau}(t_x t_y)=\bar{\tau}(t_y t_x)$ for all $x,y\in W$, and hence $\bar{\tau}$ is a trace function. We also see that $\{t_w\mid w\in W\}$ and $\{t_{w^{-1}}\mid w\in W\}$ form a pair of dual bases; hence, $\bar{\tau}$ is non-degenerate. Thus, $\tilde{\boldsymbol{J}}$ is a symmetric algebra with trace form $\bar{\tau}$. □

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Proposition 3.7. *For $\lambda \in \Lambda$, define a linear map*

$$\bar{\rho}^\lambda : \tilde{\boldsymbol{J}} \to M_{d_\lambda}(F), \quad t_w \mapsto (c^{ij}_{w,\lambda})_{1 \leqslant i,j \leqslant d_\lambda}.$$

Then $\bar{\rho}^\lambda$ is an absolutely irreducible representation of $\tilde{\boldsymbol{J}}$, and all irreducible representations of $\tilde{\boldsymbol{J}}$ (up to equivalence) arise in this way. In particular, $\tilde{\boldsymbol{J}}$ is a split semisimple algebra. (Recall that F is any field containing $\mathbb{Z}_W$.)

Proof. We must show that $\bar{\rho}^\lambda(t_x t_y) = \bar{\rho}^\lambda(t_x)\bar{\rho}^\lambda(t_y)$ for all $x, y \in W$. Now, by the definition of $\tilde{\gamma}_{x,y,z}$, we have

$$\bar{\rho}^\lambda_{ij}(t_x t_y) = \sum_{z \in W} \tilde{\gamma}_{x,y,z^{-1}} c^{ij}_{z,\lambda} = \sum_{z \in W} \biggl(\sum_{\mu \in \Lambda} \sum_{1 \leqslant p,q,r \leqslant d_\mu} f_\mu^{-1} c^{pq}_{x,\mu} c^{qr}_{y,\mu} c^{rp}_{z^{-1},\mu} \biggr) c^{ij}_{z,\lambda}.$$

Using the relations in Remark 3.3, the right-hand side evaluates to

$$\sum_{\mu \in \Lambda} \sum_{1 \leqslant p,q,r \leqslant d_\mu} f_\mu^{-1} c^{pq}_{x,\mu} c^{qr}_{y,\mu} \delta_{rj} \delta_{pi} \delta_{\lambda\mu} f_\lambda = \sum_{1 \leqslant q \leqslant d_\lambda} c^{iq}_{x,\mu} c^{qj}_{y,\mu} = (\bar{\rho}^\lambda(t_x)\bar{\rho}^\lambda(t_y))_{ij},$$

as required. To show that $\bar{\rho}^\lambda$ is absolutely irreducible, we argue as follows. By Proposition 3.6, we have a symmetrizing trace where $\{t_w \mid w \in W\}$ and $\{t_{w^{-1}} \mid w \in W\}$ form a pair of dual bases. Consequently, the relations in Remark 3.3 can be interpreted as orthogonality relations for the coefficients of the representations $\bar{\rho}^\lambda$. Thus, we have

$$\sum_{w \in W} \bar{\rho}^\lambda_{ij}(t_w) \bar{\rho}^\lambda_{kl}(t_{w^{-1}}) = \delta_{il}\delta_{jk} f_\lambda \quad \text{for all } 1 \leqslant i,j,k,l \leqslant d_\lambda.$$

By [**12**, Remark 7.2.3], the validity of these relations implies that $\bar{\rho}^\lambda$ is absolutely irreducible. Finally, if $\lambda \neq \mu$ in Λ, then we also have the relations

$$\sum_{w \in W} \bar{\rho}^\lambda_{ij}(t_w) \bar{\rho}^\mu_{kl}(t_{w^{-1}}) = 0.$$

In particular, this implies that $\bar{\rho}^\lambda$ and $\bar{\rho}^\mu$ are not equivalent.

Since $\dim \tilde{\boldsymbol{J}} = |W| = \sum_{\lambda \in \Lambda} d_\lambda^2$, we can now conclude that $\tilde{\boldsymbol{J}}$ is split semisimple, and that $\{\bar{\rho}^\lambda \mid \lambda \in \Lambda\}$ are the irreducible representations of $\tilde{\boldsymbol{J}}$ (up to equivalence). □

Proposition 3.8. *The linear map $\tilde{\boldsymbol{J}} \to \tilde{\boldsymbol{J}}$ defined by $t_w \mapsto t_{w^{-1}}$ is an anti-involution, that is, we have $\tilde{\gamma}_{x,y,z} = \tilde{\gamma}_{y^{-1},x^{-1},z^{-1}}$ for all $x, y, z \in W$.*

Proof. By Lemma 3.2, we may assume that $F = \mathbb{R}$ and that our balanced representations ρ^λ are chosen such that they are orthogonal, as in Remark 2.4. Then the corresponding leading matrix coefficients have the additional property $c^{ij}_{w,\lambda} = c^{ji}_{w^{-1},\lambda}$. The defining formula then immediately shows that $\tilde{\gamma}_{x,y,z} = \tilde{\gamma}_{y^{-1},x^{-1},z^{-1}}$ for all $x, y, z \in W$. □

Lemma 3.9. *Assume that ρ^λ and σ^λ are balanced and equivalent over K. Then there exists a matrix $U^\lambda \in M_{d_\lambda}(\mathcal{O})$ such that*

$$\det(U^\lambda) \in \mathcal{O}^\times \quad \text{and} \quad U^\lambda \rho^\lambda(T_w) = \sigma^\lambda(T_w) U^\lambda \text{ for all } w \in W.$$

Denote the leading matrix coefficients with respect to σ^λ by $d_{w,\lambda}^{ij}$. Then, for a given element $w \in W$, we have

$$c_{w,\lambda}^{ij} \neq 0 \text{ for some } i,j \iff d_{w,\lambda}^{kl} \neq 0 \text{ for some } k,l.$$

Proof. Since ρ^λ and σ^λ are equivalent over K, there exists an invertible matrix $U^\lambda \in M_{d_\lambda}(K)$ such that $U^\lambda \rho^\lambda(T_w) = \sigma^\lambda(T_w) U^\lambda$ for all $\lambda \in \Lambda$. Multiplying U^λ by a suitable scalar, we may assume that all coefficients of U^λ lie in $\mathcal{O}$ and that at least one coefficient does not lie in $\mathfrak{p}$.

We show that $\det(U^\lambda) \in \mathcal{O}^\times$. For this purpose, let $\bar{U}^\lambda$ be the matrix whose (i,j)-coefficient is the constant term of the (i,j)-coefficient of U^λ. Multiplying the relation $U^\lambda \rho^\lambda(T_w) = \sigma^\lambda(T_w) U^\lambda$ by $\varepsilon^{\boldsymbol{a}_\lambda}$ and taking constant terms, we see that $\bar{U}^\lambda \in M_{d_\lambda}(F)$ is a non-zero matrix such that

$$\bar{U}^\lambda \bar{\rho}^\lambda(t_w) = \bar{\sigma}^\lambda(t_w) \bar{U}^\lambda \quad \text{for all } w \in W,$$

where $\bar{\sigma}^\lambda(t_w) := (d_{w,\lambda}^{ij})_{1 \leqslant i,j \leqslant d_\lambda}$. Now let $v \in F^{d_\lambda}$ be such that $\bar{U}^\lambda v = 0$. Then we also have

$$\bar{U}^\lambda(\bar{\rho}^\lambda(t_w)v) = \bar{\sigma}^\lambda(t_w)\bar{U}^\lambda v = 0,$$

and so the nullspace of $\bar{U}^\lambda$ is a $\bar{\rho}^\lambda$-invariant subspace of U^{d_λ}. Since $\bar{\rho}^\lambda$ is irreducible and $\bar{U}^\lambda \neq 0$, we conclude that the nullspace is 0 and, hence, $\bar{U}^\lambda$ is invertible, as claimed.

The assertion about the leading matrix coefficients is now clear. □

Remark 3.10. Let $\lambda \in \Lambda$ and $w \in W$. As in [**10**, Definition 3.1], we write $E^\lambda \leftrightsquigarrow_L w$ if $c_{w,\lambda}^{ij} \neq 0$ for some $i,j \in \{1,\dots,d_\lambda\}$. By Lemma 3.9, this relation does not depend on the choice of the balanced representations ρ^λ. In particular, choosing ρ^λ as in Remark 2.4, we see that

$$E^\lambda \leftrightsquigarrow_L w \iff E^\lambda \leftrightsquigarrow_L w^{-1}. \tag{3.2}$$

Now define a graph as follows: the vertices are in bijection with the elements of W; two vertices corresponding to elements $x \neq y$ in W are joined by an edge if there exists some $\lambda \in \Lambda$ such that $E^\lambda \leftrightsquigarrow_L x$ and $E^\lambda \leftrightsquigarrow_L y$. Considering the connected components of this graph, we obtain a partition of W; the pieces in this partition will be called the *L-blocks* of W. By [**10**, Remark 3.3], we have that

$$\text{each } L\text{-block is contained in a two-sided cell of } W. \tag{3.3}$$

(See [**20**, Chapter 8] for the definition of two-sided cells; if (P1)–(P14) hold, then one can show that the L-blocks are precisely the two-sided cells of W [**10**, Remark 3.9].)

For an L-block $\mathcal{F}$ of W, we define $\tilde{\boldsymbol{J}}_{\mathcal{F}} = \langle t_w \mid w \in \mathcal{F}\rangle_F \subseteq \tilde{\boldsymbol{J}}$. Then one may easily check that $\tilde{\boldsymbol{J}}_{\mathcal{F}}$ is a two-sided ideal of $\tilde{\boldsymbol{J}}$. (Indeed, let $x \in W$, $w \in \mathcal{F}$; we must show that $t_x t_w$

and $t_w t_x$ lie in $\tilde{\boldsymbol{J}}_{\mathcal{F}}$. Now, $t_x t_w = \sum_{y\in W} \tilde{\gamma}_{x,w,y^{-1}} t_y$. Assume that $\tilde{\gamma}_{x,w,y^{-1}} \neq 0$. Then, by the defining formula, there exists some $\lambda \in \Lambda$ such that $E^\lambda \leftrightsquigarrow_L x$, $E^\lambda \leftrightsquigarrow_L w$ and $E^\lambda \leftrightsquigarrow_L y^{-1}$. By (3.2), we also have $E^\lambda \leftrightsquigarrow_L y$. It follows that $x, y, y^{-1} \in \mathcal{F}$. Thus, $t_x t_w \in \tilde{\boldsymbol{J}}_{\mathcal{F}}$. The argument for $t_w t_x$ is similar.) We obtain a decomposition as a direct sum of two-sided ideals

$$\tilde{\boldsymbol{J}} = \bigoplus_{\mathcal{F}} \tilde{\boldsymbol{J}}_{\mathcal{F}} \quad \text{(sum over all } L\text{-blocks } \mathcal{F} \text{ of } W). \tag{3.4}$$

Now, given $\lambda \in \Lambda$, there will be a unique L-block $\mathcal{F}$ such that $\bar{\rho}^\lambda(t_w) \neq 0$ for some $w \in \mathcal{F}$. We denote this L-block by $\mathcal{F}_\lambda$.

4. Properties of balanced representations

The purpose of this section is to study in more detail balanced representations as in Definition 2.2. In particular, we wish to develop some methods for verifying whether or not a given matrix representation is balanced. The criterion in Proposition 4.3 will prove very useful in dealing with a number of examples. Proposition 4.10 exhibits some basic integrality properties.

We keep the general assumptions of the previous section. In particular, $\{\rho^\lambda \mid \lambda \in \Lambda\}$ is a fixed choice of balanced representations of $\boldsymbol{H}_K$.

Lemma 4.1. *Let $\{\delta^\lambda \mid \lambda \in \Lambda\}$ be a complete set of representatives for the equivalence classes of irreducible representations of $\tilde{\boldsymbol{J}}$. Then ρ^λ can be chosen such that $\bar{\rho}^\lambda(t_w) = \delta^\lambda(t_w)$ for all $w \in W$.*

Proof. First of all, we can assume without loss of generality that $\bar{\rho}^\lambda$ is equivalent to δ^λ for each $\lambda \in \Lambda$. Let $G^\lambda \in M_{d_\lambda}(F)$ be an invertible matrix such that $\delta^\lambda(t_w) = (G^\lambda)^{-1}\bar{\rho}^\lambda(t_w)G^\lambda$ for all $w \in W$. Now set $\hat{\rho}^\lambda(T_w) := (G^\lambda)^{-1}\rho^\lambda(T_w)G^\lambda$ for $w \in W$. Then $\hat{\rho}^\lambda$ is an irreducible representation of $\boldsymbol{H}_K$ equivalent to ρ^λ. Moreover, since the transforming matrix G^λ has all its coefficients in F, it is clear that $\hat{\rho}^\lambda$ is also balanced and that the leading matrix coefficients associated with $\hat{\rho}^\lambda(T_w)$ are given by $\delta^\lambda(t_w)$. It remains to use Lemma 3.2. □

Example 4.2. Assume that we are in the equal parameter case or, more generally, that Lusztig's (P1)–(P15) are known to hold (see Remark 2.6). Then, by Proposition 2.5, we have

$$\tilde{\gamma}_{x,y,z} = \gamma_{x,y,z} \in \mathbb{Z} \quad \text{for all } x, y, z \in W.$$

Assume further that $R := \mathbb{Z}_W$ is a principal ideal domain. Then, by a general argument (see, for example, [**12**, 7.3.7]), every irreducible representation of $\tilde{\boldsymbol{J}}$ can be realized over R. Hence, by Lemma 4.1, the balanced representations of $\boldsymbol{H}_K$ can be chosen such that

$$\bar{\rho}^\lambda(t_w) \in M_{d_\lambda}(\mathbb{Z}_W) \quad \text{for all } \lambda \in \Lambda \text{ and } w \in W.$$

This applies to all finite Weyl groups in the equal parameter case, where $\mathbb{Z}_W = W$. It also applies to (W, S) of type H_3 or H_4, where $\mathbb{Z}_W = \mathbb{Z}[\frac{1}{2}(-1+\sqrt{5})]$; note that $\mathbb{Z}_W$ is a

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principal ideal domain. By [**10**, Theorem 5.2], it also applies to (W, S) of type F_4 (where $\mathbb{Z}_W = \mathbb{Z}$), any weight function and any monomial order on Γ.

Proposition 4.3. *Assume that $F \subseteq \mathbb{R}$ (which we can do without loss of generality). Let $\lambda \in \Lambda$ and $\sigma^\lambda : \boldsymbol{H}_K \to M_{d_\lambda}(K)$ be any matrix representation afforded by E_ε^λ. Then σ^λ is balanced if and only if there exists a symmetric matrix $\Omega^\lambda \in M_{d_\lambda}(\mathcal{O})$ such that*

$$\det(\Omega^\lambda) \in \mathcal{O}^\times \quad \text{and} \quad \Omega^\lambda \sigma^\lambda(T_{w^{-1}}) = \sigma^\lambda(T_w)^{\mathrm{tr}} \Omega^\lambda \text{ for all } w \in W.$$

Proof. Assume first that σ^λ is balanced. Now σ^λ is obtained by choosing some basis of E_ε^λ. Let Ω^λ be the Gram matrix of $\langle \cdot\,, \cdot \rangle$ with respect to that basis, where $\langle \cdot\,, \cdot \rangle$ is a bilinear form on E_ε^λ as constructed in the proof of Proposition 2.3. Multiplying Ω^λ by a suitable scalar, we may assume without loss of generality that all coefficients of Ω^λ lie in $\mathcal{O}$ and that some coefficient of Ω^λ does not lie in $\mathfrak{p}$. Then $\Omega^\lambda \in M_{d_\lambda}(\mathcal{O})$ is a symmetric matrix such that

$$\Omega^\lambda \neq 0 \quad \text{and} \quad \Omega^\lambda \sigma^\lambda(T_{w^{-1}}) = \sigma^\lambda(T_w)^{\mathrm{tr}} \Omega^\lambda \text{ for all } w \in W.$$

Let $\bar{\Omega}^\lambda$ be the matrix whose (i,j)-coefficient is the constant term of the (i,j)-coefficient of Ω^λ. Now, multiplying the relation $\Omega^\lambda \sigma^\lambda(T_{w^{-1}}) = \sigma^\lambda(T_w)^{\mathrm{tr}} \Omega^\lambda$ by $\varepsilon^{\boldsymbol{a}_\lambda}$ and taking constant terms, we see that $\bar{\Omega}^\lambda$ is a non-zero symmetric matrix such that

$$\bar{\Omega}^\lambda \bar{\sigma}^\lambda(t_{w^{-1}}) = \bar{\sigma}^\lambda(t_w)^{\mathrm{tr}} \bar{\Omega}^\lambda \quad \text{for all } w \in W.$$

Thus, $\bar{\Omega}^\lambda$ defines a $\tilde{\boldsymbol{J}}$-invariant symmetric bilinear form on a representation space affording $\bar{\sigma}^\lambda$. The invariance implies that the radical of the form is a $\tilde{\boldsymbol{J}}$-submodule. Hence, since $\bar{\sigma}^\lambda$ is an irreducible representation, we conclude that the radical must be zero and so $\det(\bar{\Omega}^\lambda) \neq 0$.

Conversely, assume that a matrix Ω^λ with the above properties exists. Let $\bar{\Omega}^\lambda$ be the matrix whose (i,j)-coefficient is the constant term of the (i,j)-coefficient of Ω^λ. Then $\bar{\Omega}^\lambda \in M_{d_\lambda}(F)$ is a symmetric matrix such that $\det(\bar{\Omega}^\lambda) \neq 0$ (since $\det(\Omega^\lambda) \in \mathcal{O}^\times$). Thus, $\bar{\Omega}^\lambda$ defines a non-degenerate symmetric bilinear form. Now, since we are working over a field of characteristic 0, there will be an orthogonal basis with respect to that form. So we can find invertible matrices $Q^\lambda, D^\lambda \in M_{d_\lambda}(F)$ such that $\bar{\Omega}^\lambda = (Q^\lambda)^{\mathrm{tr}} D^\lambda Q^\lambda$ and D^λ is diagonal. Now let $P^\lambda := (Q^\lambda)^{-1}$ and define

$$\hat{\sigma}^\lambda(T_w) := (P^\lambda)^{-1} \sigma^\lambda(T_w) P^\lambda \quad \text{for all } w \in W,$$
$$\hat{\Omega}^\lambda := (P^\lambda)^{\mathrm{tr}} \Omega^\lambda P^\lambda.$$

Thus, $\hat{\sigma}^\lambda$ is an irreducible representation of $\boldsymbol{H}_K$ equivalent to σ^λ; furthermore, we have

$$\hat{\Omega}^\lambda \hat{\sigma}^\lambda(T_{w^{-1}}) = \hat{\sigma}^\lambda(T_w)^{\mathrm{tr}} \hat{\Omega}^\lambda \quad \text{for all } w \in W.$$

Since the transforming matrix P^λ has all its coefficients in F, it is clear that $\hat{\Omega}^\lambda \in M_{d_\lambda}(\mathcal{O})$ and $\det(\hat{\Omega}^\lambda) \in \mathcal{O}^\times$; furthermore, σ^λ is balanced if and only if $\hat{\sigma}^\lambda$ is balanced.

Thus, it remains to show that $\hat{\sigma}^\lambda$ is balanced. Now, the point about the above transformation is that we have $\hat{\Omega}^\lambda \equiv D^\lambda \bmod \mathfrak{p}$. We can now argue as in the proof of Proposition 2.3 to show that $\hat{\sigma}^\lambda$ is balanced. □

Remark 4.4. Note that, in order to verify that a matrix Ω^λ satisfies

$$\Omega^\lambda \sigma^\lambda(T_{w^{-1}}) = \sigma^\lambda(T_w)^{\mathrm{tr}} \Omega^\lambda \quad \text{for all } w \in W$$

it is sufficient to verify that $\Omega^\lambda \sigma^\lambda(T_s) = \sigma^\lambda(T_s)^{\mathrm{tr}} \Omega^\lambda$ for all $s \in S$. This remark, although almost trivial, is nevertheless useful in dealing with concrete examples.

Example 4.5. Let $3 \leqslant m < \infty$ and (W, S) be of type $I_2(m)$, with generators s_1, s_2 such that $(s_1 s_2)^m = 1$. We have $\mathbb{Z}_W = \mathbb{Z}[\zeta + \zeta^{-1}]$, where $\zeta \in \mathbb{C}$ is a root of unity of order m. We assume without loss of generality that $L(s_1) \geqslant L(s_2) > 0$. The irreducible representations of $\boldsymbol{H}_K$ are determined in [**12**, 8.3]. These representations have dimension 1 or 2. Notice that one-dimensional representations are automatically balanced. By [**12**, Theorem 8.3.1], the two-dimensional representations can be realized as

$$\rho_j : T_{s_1} \mapsto \begin{pmatrix} -v_{s_1}^{-1} & 0 \\ \mu_j & v_{s_1} \end{pmatrix}, \quad T_{s_2} \mapsto \begin{pmatrix} v_{s_2} & 1 \\ 0 & -v_{s_2}^{-1} \end{pmatrix},$$

where $\mu_j = v_{s_1} v_{s_2}^{-1} + \zeta^j + \zeta^{-j} + v_{s_1}^{-1} v_{s_2}$ and $1 \leqslant j \leqslant \frac{1}{2}(m-2)$ (if m is even) or $1 \leqslant j \leqslant \frac{1}{2}(m-1)$ (if m is odd). Note that the coefficients of the representing matrices lie in the ring $\mathbb{Z}_W[v_{s_1}^{\pm 1}, v_{s_2}^{\pm 1}]$. Now let

$$\Omega_j = \begin{pmatrix} v_{s_1} \mu_j (v_{s_2} + v_{s_2}^{-1}) & v_{s_1} \mu_j \\ v_{s_1} \mu_j & v_{s_1}^2 + 1 \end{pmatrix} \in M_2(\mathbb{Z}_W[v_{s_1}^{\pm 1}, v_{s_2}^{\pm 1}]).$$

Then Ω_j is a symmetric matrix satisfying $\Omega_j \rho_j(T_{w^{-1}}) = \rho_j(T_w)^{\mathrm{tr}} \Omega_j$ for all $w \in W$. (By Remark 4.4, it is sufficient to verify this for $w \in \{s_1, s_2\}$.) We see that

$$\Omega_j \equiv \begin{cases} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \bmod \mathfrak{p} & \text{if } L(s_2) > L(s_1) > 0, \\[2ex] \begin{pmatrix} 2 + \zeta^j + \zeta^{-j} & 0 \\ 0 & 1 \end{pmatrix} \bmod \mathfrak{p} & \text{if } L(s_2) = L(s_1) > 0. \end{cases}$$

Hence, by Proposition 4.3, ρ_j is a balanced representation, in all cases. Since the coefficients of the matrices $\rho_j(T_{s_1})$ and $\rho_j(T_{s_2})$ lie in $\mathbb{Z}_W[v_{s_1}^{\pm 1}, v_{s_2}^{\pm 1}]$, the same will be true for the matrices $\rho_j(T_w)$, where $w \in W$. Hence, all the corresponding leading matrix coefficients will also lie in $\mathbb{Z}_W$.

Example 4.6. The argument in Example 4.5 can be applied whenever the irreducible representations of $\boldsymbol{H}_K$ are explicitly known.

Assume, for example, that (W, S) is of type H_3 or H_4. In these cases, all elements in S are conjugate and so all v_s, $s \in S$, are equal; write $v = v_s$ for $s \in S$. We have $\mathbb{Z}_W = \mathbb{Z}[\alpha]$, where $\alpha = \frac{1}{2}(-1 + \sqrt{5})$. The irreducible representations of $\boldsymbol{H}_K$ are constructed by Lusztig [**15**, § 5] and Alvis and Lusztig [**1**] in terms of so-called *W-graphs*. (These W-graphs are reproduced in [**12**, Chapter 11].) Thus, we obtain explicit matrix representations $\rho^\lambda : \boldsymbol{H}_K \to M_{d_\lambda}(K)$ for all $\lambda \in \Lambda$. By inspection, one sees that

$$\rho^\lambda(T_w) \in M_{d_\lambda}(\mathbb{Z}_W[v, v^{-1}]) \quad \text{for all } w \in W.$$

Table 1. Invariant bilinear forms for H_3

$$\Omega^{1'_r} = \begin{bmatrix} 1 \end{bmatrix}, \qquad \Omega^{1_r} = \begin{bmatrix} 1 \end{bmatrix}, \qquad \Omega^{5'_r} = \begin{bmatrix} v^2+1 & 0 & -v & 0 & 0 \\ 0 & v^2+1 & 0 & -v & 0 \\ -v & 0 & v^2+1 & 0 & -v \\ 0 & -v & 0 & v^2+1 & -v \\ 0 & 0 & -v & -v & v^2+1 \end{bmatrix},$$

$$\Omega^{5_r} = \begin{bmatrix} v^8+v^6+v^4+v^2+1 & v^4 & v^7+v^5+v^3+v & v^5+v^3 & v^6+v^4+v^2 \\ v^4 & v^8+v^6+v^4+v^2+1 & v^5+v^3 & v^7+v^5+v^3+v & v^6+v^4+v^2 \\ v^7+v^5+v^3+v & v^5+v^3 & v^8+2v^6+2v^4+2v^2+1 & v^6+2v^4+v^2 & v^7+2v^5+2v^3+v \\ v^5+v^3 & v^7+v^5+v^3+v & v^6+2v^4+v^2 & v^8+2v^6+2v^4+2v^2+1 & v^7+2v^5+2v^3+v \\ v^6+v^4+v^2 & v^6+v^4+v^2 & v^7+2v^5+2v^3+v & v^7+2v^5+2v^3+v & v^8+2v^6+3v^4+2v^2+1 \end{bmatrix},$$

$$\Omega^{3_s} = \begin{bmatrix} v^2+1 & -v & 0 \\ -v & v^2+1 & \bar{\alpha}v \\ 0 & \bar{\alpha}v & v^2+1 \end{bmatrix}, \qquad \Omega^{3'_s} = \begin{bmatrix} v^4-\alpha v^2+1 & v^3+v & -\bar{\alpha}v^2 \\ v^3+v & v^4+2v^2+1 & -\bar{\alpha}v^3-\bar{\alpha}v \\ -\bar{\alpha}v^2 & -\bar{\alpha}v^3-\bar{\alpha}v & v^4+v^2+1 \end{bmatrix},$$

$$\Omega^{\bar{3}_s} = \begin{bmatrix} v^2+1 & -v & 0 \\ -v & v^2+1 & \alpha v \\ 0 & \alpha v & v^2+1 \end{bmatrix}, \qquad \Omega^{\bar{3}'_s} = \begin{bmatrix} v^4-\bar{\alpha}v^2+1 & v^3+v & -\alpha v^2 \\ v^3+v & v^4+2v^2+1 & -\alpha v^3-\alpha v \\ -\alpha v^2 & -\alpha v^3-\alpha v & v^4+v^2+1 \end{bmatrix},$$

$$\Omega^{4'_r} = \begin{bmatrix} v^4-v^3+2v^2-v+1 & v^3+v & v^3+v & v^2 \\ v^3+v & v^4+2v^2+1 & v^3+v^2+v & v^3+v \\ v^3+v & v^3+v^2+v & v^4+2v^2+1 & v^3+v \\ v^2 & v^3+v & v^3+v & v^4-v^3+2v^2-v+1 \end{bmatrix},$$

$$\Omega^{4_r} = \begin{bmatrix} v^4+v^3+2v^2+v+1 & -v^3-v & -v^3-v & v^2 \\ -v^3-v & v^4+2v^2+1 & -v^3+v^2-v & -v^3-v \\ -v^3-v & -v^3+v^2-v & v^4+2v^2+1 & -v^3-v \\ v^2 & -v^3-v & -v^3-v & v^4+v^3+2v^2+v+1 \end{bmatrix}$$

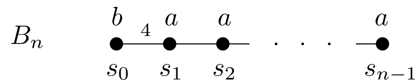

Figure 2.

For each $\lambda \in \Lambda$, we can work out a non-zero matrix $\Omega^\lambda \in M_{d_\lambda}(\mathbb{Z}_W[v, v^{-1}])$ such that $\Omega^\lambda \rho^\lambda(T_{w^{-1}}) = \rho^\lambda(T_w)^{\mathrm{tr}} \Omega^\lambda$ for all $w \in W$. (For example, with the help of a computer, we can simply compute $\Omega^\lambda := \sum_{w \in W} \rho^\lambda(T_w)^{\mathrm{tr}} \rho^\lambda(T_w)$.) Multiplying Ω^λ by a suitable scalar, we may assume that all coefficients lie in $\mathbb{Z}_W[v]$ and at least one coefficient does not lie in $v\mathbb{Z}_W[v]$. For type H_3, the matrices Ω^λ are given in Table 1, where we use the labelling of $\mathrm{Irr}(W)$ as in [**12**, Table C.1]. In this case, we notice that the diagonal coefficients lie in $1 + \mathfrak{p}$, while the off-diagonal coefficients lie in $\mathfrak{p}$. Hence, clearly we have $\det(\Omega^\lambda) \in 1 + \mathfrak{p}$. The situation in type H_4 is slightly more complicated, but one can check again that $\det(\Omega^\lambda) \in \mathcal{O}^\times$ for all $\lambda \in \Lambda$. Thus, by Proposition 4.3, the representations given by the W-graphs are balanced.

One may conjecture that every representation given by a W-graph is balanced.

Example 4.7. Let $W = W_n$ be a Coxeter group of type B_n, with generators $s_0, s_1, \ldots, s_{n-1}$ and relations given by the diagram in Figure 2; the 'weights' $a, b \in \Gamma$ attached to the generators of W_n uniquely determine a weight function $L = L_{a,b}$ on W_n.

Assume that $a > 0$. Then we claim that, for each $\lambda \in \Lambda$, there is a balanced representation ρ^λ with corresponding matrix Ω^λ (as in Proposition 4.3) such that

(a) all the leading matrix coefficients $c^{ij}_{w,\lambda}$ lie in $\mathbb{Z}$,

(b) $\Omega^\lambda \in M_{d_\lambda}(\mathbb{Z}[\Gamma])$ and $\det(\Omega^\lambda) \in 2^{n_\lambda} + \mathfrak{p}$ where $n_\lambda \in \mathbb{Z}$,

(c) $n_\lambda = 0$ if $b \notin \{a, 2a, \ldots, (n-1)a\}$.

This can be seen by an argument which is a variation of that in [**11**, Example 3.6]. Indeed, it is well known that we can take for Λ the set of all pairs of partitions of total size n. Furthermore, for each $\lambda \in \Lambda$, we have a corresponding Specht module $\tilde{S}^\lambda$ as constructed by Dipper *et al.* [**5**]. Let $\{e_t \mid t \in \mathbb{T}_\lambda\}$ be the standard basis of $\tilde{S}^\lambda$, where $\mathbb{T}_\lambda$ is the set of all standard bitableaux of shape λ. With respect to this basis, each T_w, $w \in W_n$, is represented by a matrix with coefficients in $\mathbb{Z}[\Gamma]$.

Let $\langle \cdot, \cdot \rangle_\lambda$ be the invariant bilinear form on $\tilde{S}^\lambda$ as constructed in [**5**, § 5]. Let Ψ^λ be the Gram matrix of this bilinear form with respect to the basis $\{e_t \mid t \in \mathbb{T}_\lambda\}$. All coefficients of Ψ^λ lie in $\mathbb{Z}[\Gamma]$. Let $\{f_t \mid t \in \mathbb{T}_\lambda\}$ be the orthogonal basis constructed in [**5**, Theorem 8.11]; this basis is obtained from the standard basis by a unitriangular transformation. Hence, we have

$$\det(\Psi^\lambda) = \prod_{t \in \mathbb{T}_\lambda} \langle f_t, f_t \rangle_\lambda \in \mathbb{Z}[\Gamma].$$

Using the recursion formula in [**6**, Proposition 3.8], it is straightforward to show that, for each basis element f_t, there exist integers $s_t, a_{ti}, b_{tj}, c_{tk}, d_{tl} \in \mathbb{Z}$ such that $a_{ti} \geqslant 0$,

$b_{tj} \geqslant 0$ and

$$\langle f_t, f_t \rangle_\lambda = \varepsilon^{2s_t a} \frac{\prod_i (1 + \varepsilon^{2a} + \cdots + \varepsilon^{2a_{ti}a})}{\prod_j (1 + \varepsilon^{2a} + \cdots + \varepsilon^{2b_{tj}a})} \frac{\prod_k (1 + \varepsilon^{2(b + c_{tk}a)})}{\prod_l (1 + \varepsilon^{2(b + d_{tl}a)})}.$$

So there exist $h_t, h'_t, m_{tk}, m'_{tl}, n_t, n'_t \in \mathbb{Z}$ such that

$$\prod_k (1 + \varepsilon^{2(b + c_{tk}a)}) = 2^{n_t} \varepsilon^{2h_t} \prod_k (1 + \varepsilon^{2m_{tk}}) \quad \text{where } m_{tk} > 0,$$
$$\prod_l (1 + \varepsilon^{2(b + d_{tl}a)}) = 2^{n'_t} \varepsilon^{2h'_t} \prod_l (1 + \varepsilon^{2m'_{tl}}) \quad \text{where } m'_{tl} > 0.$$

Hence, setting $\tilde{e}_t := \varepsilon^{-s_t a - h_t + h'_t} e_t$ and $\tilde{f}_t := \varepsilon^{-s_t a - h_t + h'_t} f_t$, we obtain $2^{n'_t - n_t} \langle \tilde{f}_t, \tilde{f}_t \rangle_\lambda \in 1 + \mathfrak{p}$ for all $t \in \mathbb{T}_\lambda$. Now let ρ^λ be the matrix representation afforded by $\tilde{S}^\lambda$ with respect to $\{\tilde{e}_t \mid t \in \mathbb{T}_\lambda\}$ and let Ω^λ be the Gram matrix of $\langle \cdot , \cdot \rangle_\lambda$ with respect to that basis. Then

$$\det(\Omega^\lambda) = \det(\Psi^\lambda) \prod_{t \in \mathbb{T}_\lambda} \varepsilon^{2(-s_t a - h_t + h'_t)} = \prod_{t \in \mathbb{T}_\lambda} (\varepsilon^{2(-s_t a - h_t + h'_t)} \langle f_t, f_t \rangle_\lambda) = \prod_{t \in \mathbb{T}_\lambda} \langle \tilde{f}_t, \tilde{f}_t \rangle_\lambda.$$

Hence, we can deduce that (a) and (b) hold. Finally, the cases in (c) correspond to the situations already considered in [**11**, Example 3.6] and [**3**, Proposition 2.3]; the special feature of these cases is that $n_t = 0$ for all t.

Definition 4.8. Recall that $\mathbb{Z}_W = \mathbb{Z}[2\cos(2\pi/m_{st}) \mid s, t \in S]$. We say that the subring $R \subseteq \mathbb{C}$ is *L-good* if the following conditions hold:

- $\mathbb{Z}_W \subseteq R$ and
- f_λ is contained and invertible in R for all $\lambda \in \Lambda$.

By Remark 2.1, this notion does not depend on the choice of the monomial order on Γ. Note that, if W is a finite Weyl group, i.e. we have $m_{st} \in \{2, 3, 4, 6\}$, then $2\cos(2\pi/m_{st}) \in \mathbb{Z}$ and $f_\lambda \in \mathbb{Z}$ for all $\lambda \in \Lambda$. Hence, in this case, $\mathbb{Z}_W = \mathbb{Z}$ and the only condition on R is that the integer f_λ is invertible in R for every $\lambda \in \Lambda$ (which is precisely the condition used in [**9**, § 2.2]).

Example 4.9. Assume that (W, S) is of type $I_2(m)$, where $m = 5$ or $m \geqslant 7$. Formulae for the elements $\boldsymbol{c}_\lambda$ can be found in [**12**, Theorem 8.3.4]. Using these formulae, one may check that R is L-good if and only if $2\cos(2\pi/m) \in R$ and the integer m is invertible in R.

Assume that (W, S) is of type H_3. Then [**12**, Table E.2] shows that R is L-good if and only if $\frac{1}{2}(1 + \sqrt{5}) \in R$ and the integers 2, 5 are invertible in R.

Assume that (W, S) is of type H_4. Then [**12**, Table E.3] shows that R is L-good if and only if $\frac{1}{2}(1 + \sqrt{5}) \in R$ and the integers 2, 3, 5 are invertible in R.

Proposition 4.10. *Let $R \subseteq \mathbb{C}$ be a subring which is L-good. Let $\lambda \in \Lambda$. Then the balanced representation ρ^λ can be chosen such that the following hold.*

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(a) $\bar{\rho}^{\lambda}_{ij}(t_w) = c^{ij}_{w,\lambda} \in \mathbb{Z}_W$ *for all* $w \in W$ *and* $1 \leqslant i, j \leqslant d_\lambda$.

In particular, we have $\tilde{\gamma}_{x,y,z} \in R$ *for all* $x, y, z \in W$. *Furthermore, there exists a symmetric, positive-definite matrix*

$$B^\lambda = (\beta^\lambda_{ij})_{1\leqslant i,j\leqslant d_\lambda}, \quad \textit{where } \beta^\lambda_{ij} \in \mathbb{Z}_W \textit{ for all } 1 \leqslant i, j \leqslant d_\lambda,$$

such that the following two conditions hold:

(b) $B^\lambda \bar{\rho}^\lambda(t_{w^{-1}}) = \bar{\rho}^\lambda(t_w)^{\mathrm{tr}} B^\lambda$ *for all* $w \in W$;

(c) $\det(B^\lambda) \neq 0$ *is invertible in* R.

Proof. By standard reduction arguments, one can assume that (W, S) is irreducible. Now (a) holds in all cases by Examples 4.2, 4.5 and 4.7. Once this is proved, we see (by the defining formula) that $\tilde{\gamma}_{x,y,z} \in R$ for all $x, y, z \in W$. We can now actually take R to be the ring generated by $\mathbb{Z}_W$ and f_λ^{-1}, $\lambda \in \Lambda$. Notice that, if $\mathbb{Z}_W$ is a principal ideal domain, then so is R.

Now (b) and (c) can be proved as in [**9**, Proposition 2.6], if $\mathbb{Z}_W$ is a principal ideal domain. (In the last step of [**9**, Proposition 2.6], instead of reducing modulo a prime number, one reduces modulo a prime ideal in R.) Hence, it only remains to prove (b) and (c) for (W, S) of type $I_2(m)$, $m \geqslant 3$. Note that the assertions are clear for one-dimensional representations, where we can just take $\Omega^\lambda = (1)$. For a two-dimensional representation ρ_j, let Ω_j be as in Example 4.5. Let B_j be the matrix obtained by taking the constant terms of the entries of Ω_j. We notice that all entries of B_j lie in $\mathbb{Z}_W$, and B_j satisfies (b). It remains to consider $\det(B_j)$. By Example 4.9, m is invertible in R, so it will be sufficient to show that $\det(B_j)$ divides m in R. Now, if $L(s_2) > L(s_1) > 0$, then $\det(B^\lambda) = 1$ and so there is nothing to prove. If $L(s_1) = L(s_2) > 0$, then $\det(B_j) = 2 + \zeta^j + \zeta^{-j}$. Now, we have

$$\prod_{1\leqslant j\leqslant (m-1)/2} (2 + \zeta^j + \zeta^{-j}) = 1 \qquad \text{if } m \text{ is odd,}$$

$$\prod_{1\leqslant j\leqslant (m-2)/2} (2 + \zeta^j + \zeta^{-j}) = \tfrac{1}{2}m \quad \text{if } m \text{ is even.}$$

Thus, $\det(B_j)$ divides m, as required. It follows that (c) holds. □

Corollary 4.11. *Let* $\mathbb{Q}_{(2)}$ *be the ring of all rational numbers of the form* $2^a b$, *where* $a, b \in \mathbb{Z}$. *Then* $\tilde{\gamma}_{x,y,z} \in \mathbb{Q}_{(2)}$ *for all* $x, y, z \in W$.

Proof. By standard reduction arguments, we can assume that (W, S) is irreducible. Now, if (P1)–(P15) hold, then $\tilde{\gamma}_{x,y,z} = \gamma_{x,y,z} \in \mathbb{Z}$ for all $x, y, z \in W$ (see Proposition 2.5). Hence, by Remark 2.6, the assertion holds in the equal parameter case. By [**10**, § 5], this also applies to (W, S) of type F_4 and $I_2(m)$ (for all choices of weight functions and monomial orders). If (W, S) if of type B_n, the result is covered by Example 4.7. □

5. Cellular bases

We are now ready to review the construction of a cellular basis of $\boldsymbol{H}$ and to extend this construction to further types of examples. We refer to [**20**, Chapter 8] for the definition of the Kazhdan–Lusztig preorder relation $\leqslant_{\mathcal{LR}}$. (Note that this depends on the weight function L and the monomial order on Γ.) For any $w \in W$, we have $\boldsymbol{H}\boldsymbol{C}_w\boldsymbol{H} \subseteq \sum_y A\boldsymbol{C}_y$, where the sum runs over all $y \in W$ such that $y \leqslant_{\mathcal{LR}} w$. Let $\sim_{\mathcal{LR}}$ be the associated equivalence relation; the equivalence classes are called the two-sided cells of W. Instead of Lusztig's (P1)–(P15) (see [**20**, § 14.2]), we shall only have to consider the following property, which is a variant of (P15).

($\widetilde{\text{P15}}$) If $x, x', y, w \in W$ satisfy $w \sim_{\mathcal{LR}} y$, then

$$\sum_{u \in W} \tilde{\gamma}_{w,x',u^{-1}} h_{x,u,y} = \sum_{u \in W} h_{x,w,u} \tilde{\gamma}_{u,x',y^{-1}}.$$

Remark 5.1. Assume that (P1)–(P15) in [**20**, § 14.2] hold. Then $\tilde{\gamma}_{x,y,z} = \gamma_{x,y,z}$ for all $x, y, z \in W$ (see Proposition 2.5). Now, if $x, x', y, w \in W$ satisfy $w \sim_{\mathcal{LR}} y$, then $\boldsymbol{a}(w) = \boldsymbol{a}(y)$ by (P4) and, hence, ($\widetilde{\text{P15}}$) follows from [**20**, Theorem 18.9 (b)], which itself is deduced from (P15). Thus, ($\widetilde{\text{P15}}$) holds if (P1)–(P15) hold.

Assume from now on that R is L-good (see Definition 4.8). By Proposition 4.10, all structure constants $\tilde{\gamma}_{x,y,z}$ lie in R. Let $\tilde{\boldsymbol{J}}_R$ be the R-span of $\{t_w \mid w \in W\}$. Then $\tilde{\boldsymbol{J}}_R$ is an R-subalgebra of $\tilde{\boldsymbol{J}}$ and $\tilde{\boldsymbol{J}} = F \bigotimes_R \tilde{\boldsymbol{J}}_R$. By the identification $\boldsymbol{C}_w \leftrightarrow t_w$, the natural left $\boldsymbol{H}$-module structure on $\boldsymbol{H}$ (given by left multiplication) can be transported to a left $\boldsymbol{H}$-module structure on $\tilde{\boldsymbol{J}}_A := A \bigotimes_R \tilde{\boldsymbol{J}}_R$. Explicitly, the action is given by

$$\boldsymbol{C}_x * t_y = \sum_{z \in W} h_{x,y,z} t_z \quad \text{for all } x, y \in W.$$

Now we have the following result which was first proved by Lusztig in [**19**] for the equal parameter case and in [**20**, Theorems 18.9 and 18.10] for the general case, assuming that (P1)–(P15) hold. Note that our proof is much less 'computational' than that in [**20**]; it is inspired by an analogous argument in [**15**].

Theorem 5.2 (Lusztig). *Assume that ($\widetilde{\text{P15}}$) holds. Then there is a unique unital A-algebra homomorphism $\phi : \boldsymbol{H} \to \tilde{\boldsymbol{J}}_A$ such that, for any $h \in \boldsymbol{H}$ and $w \in W$, the difference $\phi(h)t_w - h * t_w$ is an A-linear combination of terms t_y, where $y \leqslant_{\mathcal{LR}} w$ and $y \not\sim_{\mathcal{LR}} w$. Explicitly, ϕ is given by*

$$\phi(\boldsymbol{C}_w) = \sum_{\substack{z \in W,\, d \in \tilde{\mathcal{D}}, \\ z \sim_{\mathcal{LR}} d}} h_{w,d,z} \tilde{n}_d t_z, \quad w \in W.$$

Proof. Using the preorder $\leqslant_{\mathcal{LR}}$, we can define a left $\boldsymbol{H}$-module structure on $\tilde{\boldsymbol{J}}_A$ by the formula

$$\boldsymbol{C}_x \diamond t_y = \sum_{z \in W : z \sim_{\mathcal{LR}} y} h_{x,y,z} t_z \quad \text{for all } x, y \in W.$$

(More formally, one considers a graded module $\mathrm{gr}(E)$ with canonical basis $\{\bar{e}_w \mid w \in W\}$ as in [**15**, p. 492], and then transports the structure to $\tilde{\boldsymbol{J}}_A$ via the identification $\bar{e}_w \leftrightarrow t_w$. This immediately yields the above formula. Of course, one can also check directly that the above formula defines a left $\boldsymbol{H}$-module structure on $\tilde{\boldsymbol{J}}_A$.) For any $h \in \boldsymbol{H}$ and $w \in W$, the difference $h * t_w - h \diamond t_w$ is an A-linear combination of terms t_y, where $y \leqslant_{\mathcal{LR}} w$ and $y \not\sim_{\mathcal{LR}} w$.

On the other hand, we have a natural right $\tilde{\boldsymbol{J}}_A$-module structure on $\tilde{\boldsymbol{J}}_A$ (given by right multiplication). Then $(\widetilde{\mathrm{P15}})$ is equivalent to the statement that $\tilde{\boldsymbol{J}}_A$ is an $(\boldsymbol{H}, \tilde{\boldsymbol{J}}_A)$-bimodule. Indeed, we simply remark that $(\widetilde{\mathrm{P15}})$ is obtained by writing out the identity $\boldsymbol{C}_x \diamond (t_w t_{x'}) = (\boldsymbol{C}_x \diamond t_w) t_{x'}$, where we use the fact that, on both sides of $(\widetilde{\mathrm{P15}})$, the sum needs only to be extended over all $u \in W$ such that $u \sim_{\mathcal{LR}} w$. (This follows from the fact that each L-block is contained in a two-sided cell; see (3.3).)

Now we can argue as follows. The left $\boldsymbol{H}$-module structure on $\tilde{\boldsymbol{J}}_A$ gives rise to an A-algebra homomorphism

$$\psi : \boldsymbol{H} \to \mathrm{End}_A(\tilde{\boldsymbol{J}}_A) \quad \text{such that } \psi(h)(t_w) = h \diamond t_w.$$

Since the left action of $\boldsymbol{H}$ on $\tilde{\boldsymbol{J}}_A$ commutes with the right action of $\tilde{\boldsymbol{J}}_A$, the image of ψ lies in $\mathrm{End}_{\tilde{\boldsymbol{J}}_A}(\tilde{\boldsymbol{J}}_A)$. Now, we have a natural A-algebra isomorphism

$$\eta : \mathrm{End}_{\tilde{\boldsymbol{J}}_A}(\tilde{\boldsymbol{J}}_A) \to \tilde{\boldsymbol{J}}_A, \quad f \mapsto f(1_{\tilde{\boldsymbol{J}}_A}).$$

(This works for any ring with identity.) We define $\phi = \eta \circ \psi : \boldsymbol{H} \to \tilde{\boldsymbol{J}}_A$. Then ϕ is an A-algebra homomorphism such that

$$\phi(h) = \psi(h)(1_{\tilde{\boldsymbol{J}}_A}) = h \diamond 1_{\tilde{\boldsymbol{J}}_A} \quad \text{for all } h \in \boldsymbol{H}.$$

This yields $\phi(h)t_w = (h \diamond 1_{\tilde{\boldsymbol{J}}_A})t_w = h \diamond 1_{\tilde{\boldsymbol{J}}_A} t_w = h \diamond t_w$ or, in other words, the difference $\phi(h)t_w - h * t_w$ is an A-linear combination of terms t_y, where $y \leqslant_{\mathcal{LR}} w$ and $y \not\sim_{\mathcal{LR}} w$, as required. Finally, we immediately obtain the formula

$$\phi(\boldsymbol{C}_w) = \boldsymbol{C}_w \diamond 1_{\tilde{\boldsymbol{J}}_A} = \sum_{d \in \tilde{\mathcal{D}}} \tilde{n}_d \boldsymbol{C}_w \diamond t_d = \sum_{\substack{z \in W,\, d \in \tilde{\mathcal{D}},\\ z \sim_{\mathcal{LR}} d}} h_{w,d,z} \tilde{n}_d t_z.$$

Since $h_{1,d,z} = \delta_{d,z}$, this yields $\phi(\boldsymbol{C}_1) = 1_{\tilde{\boldsymbol{J}}_A}$; hence, ϕ is unital.

The unicity of ϕ is clear since the conditions on ϕ imply that $\phi(h)t_w = h \diamond t_w$ for all $w \in W$ and, hence, $\phi(h) = \phi(h)1_{\tilde{\boldsymbol{J}}_A} = h \diamond 1_{\tilde{\boldsymbol{J}}_A}$ for all $h \in \boldsymbol{H}$. □

Remark 5.3. Assume that (P1)–(P15) hold. Then $\tilde{\gamma}_{x,y,z} = \gamma_{x,y,z}$ for all $x, y, z \in W$ (see Proposition 2.5). Hence, $\tilde{\boldsymbol{J}}$ is Lusztig's ring $\boldsymbol{J}$ constructed in [**20**, Chapter 18]. Since the identity element is uniquely determined, we can also conclude that $\tilde{\mathcal{D}} = \mathcal{D}$ and $\tilde{n}_d = n_d$ for all $d \in \mathcal{D}$, where $\mathcal{D}$ and n_d are defined as in [**20**, Chapter 18]. Hence, the above result is a combination of [**20**, Theorems 18.9 and 18.10].

Note that the formula for ϕ in [**20**, Theorem 18.9] looks somewhat different: there is a factor $\hat{n}_z$ instead of $\tilde{n}_d = n_d$. However, by [**10**, Remark 2.10], one can easily see that the two versions are equivalent. And in view of the above proof, the version here seems more natural.

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Finally, we come to the construction of 'cell data' for $\boldsymbol{H}$ in the sense of Graham and Lehrer [**13**]. By [**13**, Definition 1.1], we must specify a quadruple $(\Lambda, M, C, *)$ satisfying the following conditions.

(C1) Λ is a partially ordered set (with partial order denoted by $\trianglelefteq$), $\{M(\lambda) \mid \lambda \in \Lambda\}$ is a collection of finite sets and

$$C : \coprod_{\lambda \in \Lambda} M(\lambda) \times M(\lambda) \to \boldsymbol{H}$$

is an injective map whose image is an A-basis of $\boldsymbol{H}$.

(C2) If $\lambda \in \Lambda$ and $\mathfrak{s}, \mathfrak{t} \in M(\lambda)$, write $C(\mathfrak{s}, \mathfrak{t}) = C^{\lambda}_{\mathfrak{s},\mathfrak{t}} \in \boldsymbol{H}$. Then $* : \boldsymbol{H} \to \boldsymbol{H}$ is an A-linear anti-involution such that $(C^{\lambda}_{\mathfrak{s},\mathfrak{t}})^* = C^{\lambda}_{\mathfrak{t},\mathfrak{s}}$.

(C3) If $\lambda \in \Lambda$ and $\mathfrak{s}, \mathfrak{t} \in M(\lambda)$, then for any element $h \in \boldsymbol{H}$ we have

$$hC^{\lambda}_{\mathfrak{s},\mathfrak{t}} \equiv \sum_{\mathfrak{s}' \in M(\lambda)} r_h(\mathfrak{s}', \mathfrak{s}) C^{\lambda}_{\mathfrak{s}',\mathfrak{t}} \mod \boldsymbol{H}(\lhd \lambda),$$

where $r_h(\mathfrak{s}', \mathfrak{s}) \in A$ is independent of $\mathfrak{t}$ and where $\boldsymbol{H}(\lhd \lambda)$ is the A-submodule of $\boldsymbol{H}$ generated by $\{C^{\mu}_{\mathfrak{s}'',\mathfrak{t}''} \mid \mu \trianglelefteq \lambda;\ \lambda \neq \mu;\ \mathfrak{s}'', \mathfrak{t}'' \in M(\mu)\}$.

We now define a required quadruple $(\Lambda, M, C, *)$ as follows.

As before, Λ is an indexing set for the irreducible representations of W. For $\lambda \in \Lambda$, we set $M(\lambda) = \{1, \dots, d_\lambda\}$. We define a partial order on Λ as follows. Recall that, in Remark 3.10, we have associated with $\lambda \in \Lambda$ an 'L-block' $\mathcal{F}_\lambda$ of W. Now, given $\lambda, \mu \in \Lambda$, let $x \in \mathcal{F}_\lambda$ and $y \in \mathcal{F}_\mu$; then we define

$$\lambda \trianglelefteq \mu \overset{\text{def}}{\iff} \lambda = \mu \quad \text{or} \quad x \leqslant_{\mathcal{LR}} y,\ x \not\sim_{\mathcal{LR}} y.$$

(This does not depend on the choice of x or y, since each L-block is contained in a two-sided cell of W; see (3.3).)

Remark 5.4. Assume that (P1)–(P15) in [**20**, § 14.2] hold. By Proposition 2.5, we then have $\boldsymbol{a}(z) = \boldsymbol{a}_\lambda$ if $\bar{\rho}^\lambda(t_z) \neq 0$. Furthermore, by (P4) and (P11), we have the implication '$x \leqslant_{\mathcal{LR}} y \Rightarrow \boldsymbol{a}(y) \leqslant \boldsymbol{a}(x)$', with equality only if $x \sim_{\mathcal{LR}} y$. Hence, we see that

$$\lambda \trianglelefteq \mu \implies \lambda = \mu \quad \text{or} \quad \boldsymbol{a}_\mu < \boldsymbol{a}_\lambda.$$

The partial order defined by the condition on the right-hand side is the one we used in [**9**].

Finally, we define an A-linear anti-involution $* : \boldsymbol{H} \to \boldsymbol{H}$ by $T_w^* = T_{w^{-1}}$ for all $w \in W$. Thus, $T_w^* = T_w^\flat$ in the notation of [**20**, 3.4]. We can now state the following result.

Theorem 5.5. *(Geck [**9**, Theorem 3.1].) Assume that $(\widetilde{\text{P15}})$ holds. Recall that $R \subseteq \mathbb{C}$ is assumed to be an L-good subring (see Definition 4.8). Let $(\bar{\rho}^\lambda_{\mathfrak{st}}(t_w))$ and $(\beta^\lambda_{\mathfrak{st}})$ be as in Proposition 4.10. For any $\lambda \in \Lambda$ and $\mathfrak{s}, \mathfrak{t} \in M(\lambda)$, define*

$$C^{\lambda}_{\mathfrak{s},\mathfrak{t}} = \sum_{w \in W} \sum_{\mathfrak{u} \in M(\lambda)} \beta^\lambda_{\mathfrak{tu}} \bar{\rho}^\lambda_{\mathfrak{us}}(t_{w^{-1}}) \boldsymbol{C}_w.$$

*Then $C^\lambda_{\mathfrak{s},\mathfrak{t}}$ is a $\mathbb{Z}_W$-linear combination of Kazhdan–Lusztig basis elements $\boldsymbol{C}_w$, where $w \in \mathcal{F}_\lambda$. The quadruple $(\Lambda, M, C, *)$ is a 'cell datum' in the sense of Graham–Lehrer [**13**].*

Proof. In all essential points, the argument is the same as in the proof of [**9**, Theorem 3.1]. Indeed, since $(\widetilde{\text{P15}})$ holds, we have the existence of Lusztig's homomorphism $\phi : \boldsymbol{H} \to \tilde{\boldsymbol{J}}_A$ as in Theorem 5.2. The statements in Proposition 4.10 are completely analogous to those in [**9**, Proposition 2.6]. Finally, by Theorem 5.2, we have the property that $\phi(h)t_w - h * t_w$ is an A-linear combination of terms t_y, where $y \leqslant_{\mathcal{LR}}$ and $y \not\sim_{\mathcal{LR}} w$. This is precisely what is needed in order to make step 3 of the proof of [**9**, Theorem 3.1] work with our stronger definition of the partial order $\trianglelefteq$ on Λ. □

The above result strengthens the main result of [**9**] in four ways:

- it works for finite Coxeter groups in general, and not just for Weyl groups;
- it only requires $(\widetilde{\text{P15}})$ to hold, and not all of (P1)–(P15) in [**20**, § 14.2];
- it uses a slightly stronger partial order on Λ (see Remark 5.4);
- it shows that the data required to define the cellular basis can be extracted from the balanced representations ρ^λ.

Corollary 5.6. *Let (W, S) be any Coxeter system where W is finite. Let $R \subseteq \mathbb{C}$ be a subring which is L_0-good, where L_0 is the 'universal' weight function in Example 1.3. Now let $L' : W \to \Gamma'$ be any weight function and $\boldsymbol{H}'$ the corresponding Iwahori–Hecke algebra over $A' = R[\Gamma']$. Then $\boldsymbol{H}'$ admits a cell datum in the sense of Graham–Lehrer [**13**].*

Proof. Let Γ_0, A_0 and $\boldsymbol{H}_0$ be as in Example 1.3. As pointed out in [**10**, Corollary 5.4], by combining all the known results about the validity of Lusztig's conjectures [**20**, § 14.2], we can choose a monomial order $\leqslant$ on Γ_0 such that (P1)–(P15) hold. Hence, by Remark 5.1 and Theorem 5.5, the algebra $\boldsymbol{H}_0$ admits a cell datum. Now, there is a group homomorphism $\alpha : \Gamma_0 \to \Gamma'$ such that $\alpha((n_s)_{s\in S}) = \sum_{s\in S} n_s L'(s)$. This extends to a ring homomorphism $A_0 \to A'$, which we denote by the same symbol. Extending scalars from A_0 to A' (via α), we obtain $\boldsymbol{H}' = A' \bigotimes_{A_0} \boldsymbol{H}_0$. By [**9**, Corollary 3.2], the images of the cellular basis elements of $\boldsymbol{H}_0$ in $\boldsymbol{H}'$ form a cellular basis in $\boldsymbol{H}'$. □

In type B_n, an alternative construction of a cell datum is given by Dipper *et al.* [**5**].

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