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An Algorithm for Fat Points on $\mathbf{P}^2$

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Abstract. Let F be a divisor on the blow-up X of $\mathbf{P}^2$ at r general points $p_1, \dots, p_r$ and let L be the total transform of a line on $\mathbf{P}^2$. An approach is presented for reducing the computation of the dimension of the cokernel of the natural map $\mu_F\colon \Gamma\big(\mathcal{O}_X(F)\big) \otimes \Gamma\big(\mathcal{O}_X(L)\big) \to \Gamma\big(\mathcal{O}_X(F) \otimes \mathcal{O}_X(L)\big)$ to the case that F is ample. As an application, a formula for the dimension of the cokernel of μ_F is obtained when $r = 7$, completely solving the problem of determining the modules in minimal free resolutions of fat point subschemes $m_1p_1 + \cdots + m_7p_7 \subset \mathbf{P}^2$. All results hold for an arbitrary algebraically closed ground field k.

1 Introduction

Let $p_1, \dots, p_r \in \mathbf{P}^N$ be general points in projective space, let $m_1, \dots, m_r$ be nonnegative integers and let $I(p_i)$ be the homogeneous ideal (in the homogeneous coordinate ring $R = k[\mathbf{P}^N]$ of $\mathbf{P}^N$) generated by all homogeneous polynomials vanishing at p_i. A *fat point* subscheme $Z = m_1p_1 + \cdots + m_rp_r \subset \mathbf{P}^N$ is the subscheme corresponding to the homogeneous ideal $I(Z) = I(p_1)^{m_1} \cap \cdots \cap I(p_r)^{m_r}$ (which it is easy to see is generated by all homogeneous polynomials vanishing at each point p_i to order at least m_i). If $m_i \leq 1$ for all i, we say Z is a *thin point* subscheme. We denote by $I(Z)_t$ the homogeneous component of $I(Z)$ of degree t.

The first module in any minimal free homogeneous resolution of $I(Z)$ is, up to graded isomorphism, $\bigoplus_t R[-t]^{\nu_t}$, where ν_t (or $\nu_t(Z)$ if for clarity Z needs to be specified) is the dimension of the cokernel of the obvious multiplication map $\mu_{t-1}(Z)\colon I(Z)_{t-1} \otimes R_1 \to I(Z)_t$. (More concretely, ν_t is the number of generators in degree t of any minimal set of homogeneous generators of $I(Z)$.)

In the case of a thin point subscheme $Z = p_1 + \cdots + p_r$ (with p_i general), the dimensions of the homogeneous components $I(Z)_j$ are known so one can determine $\nu_t(Z)$ from the rank of $\mu_{t-1}(Z)$, and the maximal rank conjecture of [3], [4] is that μ_t should be of maximal rank for all t (meaning that μ_t should always be either injective or surjective). Although this conjecture has been verified in a number of cases (including $N = 2$), it remains open in general. For the more general but analogous situation of fat points, no conjecture has been put forward. This is partly because the multiplication maps often fail to have maximal rank, and partly because little is known about how otherwise the ranks and numbers of generators should behave, but also because one typically first wants to understand Hilbert functions, and Hilbert functions of fat point ideals are themselves not yet well understood.

However, understanding of Hilbert functions for $N = 2$, although not complete, is much better than in higher dimensions. Indeed, there are comprehensive conjectures (see

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[6], [7], [13]) which in various situations are known to hold. Thus some attention has begun to be paid to the behavior of generators and resolutions of ideals of fat point subschemes for $N = 2$, both for its own interest and as an initial means of developing one's understanding in general.

So, for the rest of this paper we will assume $N = 2$, in which case, since a fat points subscheme of $\mathbf{P}^2$ is arithmetically Cohen-Macaulay, a minimal free graded resolution of $I(Z)$ is of the form $0 \to F_1 \to F_0 \to I(Z) \to 0$; the values ν_t determine $F_0 = \bigoplus_t R[-t]^{\nu_t}$, which with the Hilbert function of $I(Z)$ then determines F_1. Thus, for $N = 2$, given the numbers of generators and the Hilbert function of $I(Z)$, one also has the modules in a minimal free resolution of $I(Z)$.

I.1 The Particular Interest of $\nu_{\beta+1}$ and $r = 7$

Denote by $\alpha(Z)$ (or by just α when Z is understood) the least degree t such that $I(Z)_t \neq 0$ and by $\beta(Z)$ the least degree t such that the base locus of $I(Z)_t$ is 0-dimensional (said alternately, $\beta(Z)$ is the least degree t such that the elements of $I(Z)_t$ have no nontrivial common divisor). Given $Z = m_1 p_1 + \cdots + m_r p_r$, if the points p_i are sufficiently general, conjecturally (see [6], [7], [13]) the regularity of $I(Z)$ is at most $\beta(Z) + 1$, assuming which the general problem of finding ν_t reduces by Lemmas 2.9 and 2.10 of [10] to computing Hilbert functions and $\nu_{\beta+1}$, and thus the case $t = \beta + 1$ is of particular interest. In [2] Fitchett develops a means of handling $\nu_{\beta+1}$ in the case that $\alpha < \beta$, but it remains unclear what to do when $\alpha = \beta$. The naive hope that in this situation μ_β might have maximal rank is quashed by examples from [11] showing that maximal rank can fail.

It is for $r = 7$ general points of $\mathbf{P}^2$ that we can first hope to begin to understand the source of such failures, since for $r \leq 6$ there are none. For example, for $r \leq 5$, it follows from [1] that $\nu_{\beta+1} = 0$ always holds (or see [10]), and hence that μ_β has maximal rank, and for $r = 6$, although $\nu_{\beta+1}$ need not always vanish, [2] shows that μ_β always has maximal rank.

I.2 The Geometric Translation

Thus in this paper we resolve the problem of determining $\nu_{\beta+1}$ when $r = 7$, thereby working out the resolution of ideals defining fat point subschemes involving $r = 7$ general points of $\mathbf{P}^2$, taking, as did [2], [10], [11], a geometric approach in which we obtain results for line bundles on certain rational surfaces. Those readers unfamiliar with the by-now standard translation of questions about fat points in $\mathbf{P}^2$ to questions about line bundles on blow ups of $\mathbf{P}^2$ may find it helpful to refer to [10]. In particular, for any fat point subscheme $Z = m_1 p_1 + \cdots + m_r p_r \subset \mathbf{P}^2$, there is for each degree t a corresponding divisor F (which is effective and numerically effective when $t \geq \beta$) on the blow up X of $\mathbf{P}^2$ at the points p_i such that the dimension of $I(Z)_t$ is $h^0\big(X, \mathcal{O}_X(F)\big)$, and ν_{t+1} is the dimension of the cokernel $\mathcal{S}(F, L)$ of the natural map $\mu_F \colon \Gamma\big(\mathcal{O}_X(F)\big) \otimes \Gamma\big(\mathcal{O}_X(L)\big) \to \Gamma\big(\mathcal{O}_X(F) \otimes \mathcal{O}_X(L)\big)$, where $L \subset X$ is the total transform to X of a line on $\mathbf{P}^2$.

For the reader's convenience, we recall some standard notions from geometry. A divisor (as always, on a given smooth projective surface) which is a nonnegative (integer) linear combination of curves is said to be *effective*. A divisor F (or its linear equivalence class $[F]$)

is *numerically effective* if $F \cdot C \geq 0$ for every effective divisor C, while an *ample* divisor is one whose intersection with every effective divisor is positive. On the other hand, an *exceptional curve* is a smooth rational curve of self-intersection -1; for example, the curve obtained by blowing up a smooth point on a projective surface is an exceptional curve. The exceptional curves on a smooth projective rational surface are known; see [16], [14].

Now let F be a divisor on a surface X obtained by blowing up distinct points $p_1, \dots, p_r$ of $\mathbf{P}^2$. Let L be the total transform to X of a line on $\mathbf{P}^2$. We denote by $\mathrm{Cl}(X)$ the group of divisors on X modulo linear equivalence. This quotient, the *divisor class group*, is a free abelian group. The classes $[L]$, $[E_1], \dots, [E_r]$ (where E_i is the exceptional curve obtained by blowing up p_i) give a basis of $\mathrm{Cl}(X)$ which we refer to as an *exceptional configuration*. (In this notation, the divisor F referred to above, corresponding in degree t to $Z = m_1 p_1 + \cdots + m_r p_r \subset \mathbf{P}^2$, is $F = tL - m_1 E_1 - \cdots - m_r E_r$.) Also, there is a bilinear form, the intersection form, on $\mathrm{Cl}(X)$, in which the basis elements $[L]$, $[E_1], \dots, [E_r]$ are orthogonal and such that $-[L]^2 = [E_1]^2 = \cdots = [E_r]^2 = -1$.

I.3 Discussion of Results

So, in fact, in this paper we solve the problem of computing the dimension of the cokernel of μ_F for arbitrary divisors F on a blow up X of $\mathbf{P}^2$ at 7 general points. Our solution is, first, algorithmically to reduce to the case that F is ample, and second, to show that μ_F is surjective when F is ample. This approach mimics what was already known concerning the determination of the Hilbert function of $I(Z)$ involving $r \leq 9$ general points of $\mathbf{P}^2$. As mentioned above, given t, there is a corresponding divisor F on the blow up X of $\mathbf{P}^2$ at the r points such that the dimension of $I(Z)_t$ is equal to $h^0\big(X, \mathcal{O}_X(F)\big)$. But as shown in [5] and [9] and as is discussed in [8], one can for $r \leq 9$ general points algorithmically reduce the computation of $h^0\big(X, \mathcal{O}_X(F)\big)$ to the case that F is numerically effective, in which case $h^0\big(X, \mathcal{O}_X(F)\big) = (F^2 - F \cdot K_X)/2 + 1$.

For computing the dimension of the cokernel of μ_F, the reduction to ample F depends on three hypotheses, (A1), (A2) and (A3), which we explicitly mention below and which are known to hold for a divisor on a surface obtained by blowing up $r \leq 8$ general points. If we consider r generic points, these hypotheses continue to hold for $r = 9$ and they have been conjectured to hold for all r. (Alternatively, since (A2) and (A3) pose conditions on possibly infinite sets of divisors, only finitely many of which are relevant at any one time, (A2) and (A3) are slightly stronger than needed for our purposes. So in fact slightly weaker but more complicated versions of (A2) and (A3) can also be used which are known to hold for $r = 9$ general points, and which are conjectured to hold for any r general points.)

Thus the main difficulty of working out resolutions of fat point subschemes involving $r > 7$ general points is not the reduction to ampleness. It is rather that for $r > 7$, μ_F need not be surjective when F is ample. If $r < 7$, the surjectivity of μ_F for an ample divisor F already follows from [1], [2], [10]. In this paper, Theorem IV.1 extends this to $r = 7$. But in both cases, the tools used to show that the cokernel of μ_F vanishes for an ample divisor F only bound the dimension of the cokernel in general, and the bounds obtained are not always delicate enough to pin down the dimension of the cokernel completely when $r > 7$. If the set of problematical cases were not too large, one could hope to handle them *ad hoc*, and this seems possible in case $r = 8$, but as of this writing this does not seem workable for

$r > 8$. (This is related to the fact that $K_X^\perp$ is negative definite for $r < 9$, but indefinite for all $r > 8$, where K_X denotes the canonical class of X and $K_X^\perp$ denotes the subspace of $\mathrm{Cl}(X)$ of classes of divisors D with $D \cdot K_X = 0$.)

Our main result is Theorem I.6.1 (but see also Corollary IV.5), which explicitly determines the dimension of the cokernel of μ_F for any numerically effective divisor F on X. We regard this as our main result because, from the point of view of a homogeneous ideal defining a fat points subscheme $Z = \sum_i m_i p_i \subset \mathbf{P}^2$, numerically effective divisors are more natural than ample divisors. For example, if $I(Z)_t \neq 0$, let $V \subset R$ be the subspace of elements obtained from $I(Z)_t$ by dividing out by a greatest divisor common to all of the elements of $I(Z)_t$. Then, under the standard translation, V corresponds to $|F|$ for some numerically effective divisor F on the blow up of $\mathbf{P}^2$ at the points p_i. From the dimensions of the linear system $|F|$ and of the cokernel of μ_F we can find the dimension of the kernel of μ_F, which is the same as that of $V \otimes R_1 \to R$, whose kernel has the same dimension as that of $\mu_t(Z)$, which with the dimensions of $I(Z)_t$ and $I(Z)_{t+1}$, allows us to compute the dimension of the cokernel of $\mu_t(Z)$, and hence $\nu_{t+1}(Z)$.

Putting it all together, we obtain an algorithm for determining $\nu_t(Z)$ for each t for any fat point subscheme Z involving $r \leq 7$ general points of $\mathbf{P}^2$. As discussed above, since the Hilbert function of $I(Z)$ is known, this gives an algorithm for determining up to graded isomorphism the modules in the minimal free resolution of the ideal $I(Z)$. (As of this writing, an implementation of this algorithm can be run via the World-Wide Web at the author's web site; the specific web address is
http://www.math.unl.edu/∼bharbour/cgi-bin/7fatpts.cgi .)

I.4 Algorithm's Underlying Assumptions

Our algorithm assumes that:

(A1) X is obtained by blowing up r distinct points of $\mathbf{P}^2$, that
(A2) the only curves of negative self-intersection on X are exceptional curves, and that
(A3) $h^1\big(X, \mathcal{O}_X(F)\big) = 0$ for any effective, numerically effective divisor F.

By [8], (A3) holds for any $r \leq 8$ points, general or not. For $r \leq 8$ general points, X is Del Pezzo, so $-K_X$ is ample, so by adjunction (A2) holds too.

More generally, since $K_X = -3[L] + [E_1 + \cdots + E_r]$, it follows for all r by duality that $h^2\big(X, \mathcal{O}_X(F)\big) = 0$ whenever $F \cdot L > -3$ (such as is the case if F is numerically effective or effective). In addition, [8] shows $F^2 \geq 0$ for any numerically effective divisor F. Moreover, for an arbitrary divisor F it is true that $h^0\big(X, \mathcal{O}_X(F)\big) = h^0\big(X, \mathcal{O}_X(F - E)\big)$ if E is effective, reduced and irreducible with $F \cdot E < 0$. By iteratively replacing F by $F - E$ whenever E is an exceptional curve and $F \cdot E < 0$ (see Remark I.4.2), we thus eventually obtain a divisor F such that either $F \cdot L < 0$, and hence $h^0\big(X, \mathcal{O}_X(F)\big) = 0$, or such that $F \cdot L \geq 0$ and $F \cdot E \geq 0$ for every exceptional curve E.

But in the latter case, $h^0\big(X, \mathcal{O}_X(F)\big) > 0$ if and only if $(F^2 - K_X \cdot F)/2 + 1 > 0$. This is because if $h^0\big(X, \mathcal{O}_X(F)\big) > 0$, then F would be numerically effective (because F meets all exceptional curves nonnegatively and by (A2) there are no other curves of negative self-intersection), and hence by (A3) $h^0\big(X, \mathcal{O}_X(F)\big) = (F^2 - K_X \cdot F)/2 + 1$. Conversely, $F \cdot L \geq 0$ means $h^2\big(X, \mathcal{O}_X(F)\big) = 0$ so $(F^2 - K_X \cdot F)/2 + 1 > 0$ implies $h^0\big(X, \mathcal{O}_X(F)\big) > 0$ by Riemann-Roch (and then, as before, $h^0\big(X, \mathcal{O}_X(F)\big) = (F^2 - K_X \cdot F)/2 + 1$).

In any case, we end up knowing $h^0\big(X, \mathcal{O}_X(F)\big)$. Thus it is not an additional assumption to assume that $h^0\big(X, \mathcal{O}_X(F)\big)$ is always available for any divisor class $[F]$ on X.

But since we can assume that we always can compute $h^0\big(X, \mathcal{O}_X(F)\big)$, we can also assume that given the class of any effective divisor F, we can determine the classes of the divisors occurring as fixed components of $|F|$. This is because for any effective divisor F there is in terms of the classes $[L], [E_1], \dots, [E_r]$ a finite list of classes such that if C is an effective divisor for which $F - C$ is effective, then the class of C is on the list: if $[F] = d[L] - \sum_{i>0} m_i[E_i]$ and $[C] = d'[L] - \sum_{i>0} m'_i[E_i]$, where F, C and $F - C$ are all effective, then $d \geq d' \geq 0$, $d' \geq m'_i$ and $d - d' - m_i \geq -m'_i$ for all $i > 0$. Thus we have only finitely many classes to test, the test being that C is a fixed component of $|F|$ if and only if $h^0\big(X, \mathcal{O}_X(F)\big) = h^0\big(X, \mathcal{O}_X(F - C)\big)$.

We also have:

Lemma I.4.1 *Given (A1), (A2) and (A3), let $F \neq 0$ be an effective divisor on X with $F \cdot E > 0$ for every exceptional curve E and such that $|F|$ is fixed component free. Then either $F^2 > 0$ and F is ample, or $F^2 = 0$ and $|F|$ is composed with a pencil $|D|$ where D is a smooth rational curve.*

Proof Note that F is numerically effective. If $F^2 > 0$, by the Hodge Index Theorem and (A2), there can be no effective divisors C with $F \cdot C = 0$, so F is ample.

If, instead, $F^2 = 0$, we want to show that $|F| = |tD|$ for some smooth rational curve D. By Riemann-Roch and (A3), $1 < h^0\big(X, \mathcal{O}_X(F)\big) = 1 - K_X \cdot F/2$, so $-K_X \cdot F > 0$. In any case, $|F|$ defines a morphism to $\mathbf{P}^1$. By Stein factorization, F is linearly equivalent to tG for some $t > 0$, where $|G|$ defines a morphism to $\mathbf{P}^1$ with connected fibers, $G^2 = 0$ and G is free. If G is irreducible, then $G = sC$ for some prime divisor C, so $C^2 = 0$ and $-K_X \cdot C > 0$ (since $-K_X \cdot F > 0$), hence by adjunction C is a smooth rational curve and $|G|$ and hence $|F|$ is composed with the pencil $|C|$. If G is not irreducible, then among the components of G are two distinct reduced and irreducible components B and C which meet. *I.e.*, $B \cdot C > 0$, and since $G^2 = 0$, we also have $B^2 < 0$ and $C^2 < 0$, so, by (A2), B and C are exceptional and, since $(B + C) \cdot G = 0$, also $B \cdot C = 1$. By (A3) it follows that $|B + C|$ is a pencil, all elements of which are either irreducible or sums of exceptional curves. As the latter can happen in only finitely many ways, a general element D of $|B + C|$ is irreducible, hence as before, a smooth rational curve. Moreover, $G - B - C$ is effective and $D + (G - B - C)$ is an element of $|G|$ and thus connected, but $D + (G - B - C) \neq G$ so D is disjoint from G; hence $G - B - C = 0$ so G is linearly equivalent to D and $|F|$ is composed with the pencil $|D|$. ∎

Remark I.4.2 The procedure described above for determining $h^0\big(X, \mathcal{O}_X(F)\big)$ depends on checking $F \cdot E$ for all exceptional curves E. This is no problem when $r \leq 8$, since then there are only finitely many exceptional curves. More generally, there is an action on $\mathrm{Cl}(X)$ by a Weyl group, W; see Section III. Given (A1) and (A2), in the terminology of the proof of Theorem 2.1 of [5], no nodal classes are effective, so W acts transitively on the exceptional configurations of X, and, for $r > 2$, the set of classes of exceptional curves is precisely a single W-orbit. For simplicity, let us say $r > 2$. Then the proof of [5, Theorem 2.1] gives an algorithm (under different hypotheses but still applicable here) for finding an element $w \in W$ such that either:

(i) $w(F) \cdot L < 0$;
(ii) $w(F) \cdot E_i < 0$ for some $i > 0$; or
(iii) $w(F)$ is a nonnegative sum of the classes $[L]$, $[L - E_1]$, $[2L - E_1 - E_2]$, $[3L - E_1 - E_2 - E_3], \dots, [3L - E_1 - \cdots - E_r]$.

But $w(F) \cdot L < 0$ means that $w(F) \cdot [C] < 0$, where C is either E_i for some $i > 0$ or $L - E_1 - E_2$, so in cases (i) or (ii) with C being one of the exceptional curves whose class is $[E_i]$ for some $i > 0$ or $[L - E_1 - E_2]$, we have $F \cdot w^{-1}C < 0$. In case (iii), it is easy to check that the classes $[L]$, $[L - E_1]$ and $[2L - E_1 - E_2]$ are numerically effective while $[3L - E_1 - \cdots - E_i]$ meets every exceptional class nonnegatively (since $[3L - E_1 - \cdots - E_i] = -K_X + [E_{i+1} + \cdots + E_r]$), and hence that $F \cdot E \geq 0$ for every exceptional curve E. Thus, for an arbitrary F, we have an effective means of finding an E with $F \cdot E < 0$, or of deciding none such exists. ∎

I.5 The Algorithm

So here is our algorithm. Assume (A1), (A2) and (A3) and let $[F]$ be a divisor class on X. Our goal is to compute $\dim \ker(\mu_F)$, from which we can obtain our ultimate goal of computing $\dim \operatorname{cok}(\mu_F)$ via the obvious formula $\dim \operatorname{cok}(\mu_F) = h^0\big(X, \mathcal{O}_X(F + L)\big) - 3h^0\big(X, \mathcal{O}_X(F)\big) + \dim \ker(\mu_F)$. The following algorithm reduces the problem of computing $\dim \ker(\mu_F)$ for an arbitrary F to the case that F is ample.

START Given F, compute $h^0\big(X, \mathcal{O}_X(F)\big)$.

I. If $h^0\big(X, \mathcal{O}_X(F)\big) \leq 1$, then clearly μ_F is injective: **STOP**.

II. Assume $h^0\big(X, \mathcal{O}_X(F)\big) > 1$:

1. If $|F|$ has a fixed component C, then clearly μ_F and μ_{F-C} have kernels of the same dimension, and we replace F by $F - C$. After a finite number of such subtractions, we reduce to the case that F is effective and $|F|$ is fixed component free, without changing $\dim \ker(\mu_F)$: go to step 2.
2. Assume $h^0\big(X, \mathcal{O}_X(F)\big) \geq 2$ and $|F|$ has no fixed components.
   a. If $F \cdot E = 0$ for some exceptional curve E, consider the following cases.
      i. If $E \cdot L \geq 2$, then replace F by $F - E$ and return to **START**. (Replacing F by $F - E$ reduces $h^0\big(X, \mathcal{O}_X(F)\big)$ by 1, but by Lemma II.4 does not change the dimension of the kernel of μ_F.)
      ii. If $E \cdot L = 1$, then Proposition II.2(e) gives the dimension of the kernel of μ_F: **STOP**.
      iii. If $E \cdot L = 0$, then contracting E gives a birational morphism $\pi \colon X \to X'$, with respect to which $\mathcal{O}_X(F) = \pi^*\mathcal{O}_{X'}(F')$ for some (in fact canonically determined) F', where F' is an effective divisor on X' and fixed component free and μ_F and $\mu_{F'}$ have kernels of the same dimension. But r has been reduced by 1, because X' is a blow up of $\mathbf{P}^2$ at $r - 1$ points. So replace X by X' and F by F', and return to step 2.
   b. We thus reduce to the case that $F \neq 0$ is effective, fixed component free and has $F \cdot E > 0$ for all exceptional curves E.
      i. If $F^2 = 0$, then Lemma II.5 applies by Lemma I.4.1, giving $\dim \ker(\mu_F)$: **STOP**.

ii. If $F^2 > 0$, then F is ample by Lemma I.4.1: **STOP**.

I.6 The Main Result

By Theorem IV.1, for $r \leq 7$ general points of $\mathbf{P}^2$, μ_F is surjective when F is ample. Thus, the algorithm above determines the rank of μ_F for an arbitrary F on a blowing up of $\mathbf{P}^2$ at $r \leq 7$ general points. But as mentioned above, it is also desirable to have an explicit result in the case that F is numerically effective. An analysis of our algorithm for numerically effective divisors leads to an especially simple such result, Theorem I.6.1.

So say $r = 7$. Denote $h^0\big(X, \mathcal{O}_X(F+L)\big) - 3h^0\big(X, \mathcal{O}_X(F)\big)$ by λ'_F and let λ_F be the maximum of 0 and λ'_F; note that $\lambda'_F = \dim \operatorname{cok}(\mu_F) - \dim \ker(\mu_F)$ and that μ_F has maximal rank if and only if $\dim \operatorname{cok}(\mu_F) = \lambda_F$. Let t_F be the number of exceptional curves E on X with $E \cdot L = 3$ such that $E \cdot F = 0$. It is well known (see [14], [16]) that $[E]$ is the class of an exceptional curve with $E \cdot L = 3$ if and only if $[E]$ is, up to permutation of the E_i, $[3L - 2E_1 - E_2 - \cdots - E_7]$. We denote these seven by $C_1 = 3L - 2E_1 - E_2 - \cdots - E_7$, $C_2 = 3L - E_1 - 2E_2 - \cdots - E_7$, *etc.* We now have:

Theorem I.6.1 *Let F be a numerically effective divisor on the blow up X of $\mathbf{P}^2$ at 7 general points, $[L], [E_1], \ldots, [E_7]$ being the corresponding exceptional configuration. Then* $\dim \operatorname{cok}(\mu_F) = \max(t_F, \lambda_F)$, *unless $[F]$ is, up to permutation of the E_i, either* 0, $[B]$, $[B + C_4]$, $[B + C_4 + C_5]$, $[B + C_4 + C_5 + C_6]$, $[B + C_4 + C_5 + C_6 + C_7]$, $[G]$ *or* $[G + C_7]$, *where* $B = 4L - 2E_1 - 2E_2 - 2E_3 - E_4 - \cdots - E_7$ *and* $G = 5L - 2E_1 - \cdots - 2E_6 - E_7$, *in which case μ_F is injective and* $\dim \operatorname{cok}(\mu_F) = \lambda_F$.

Although Theorem I.6.1 does not explicitly address the failure of μ_F to have maximal rank, it follows from Theorem I.6.1that μ_F fails to have maximal rank if and only if $t_F > \lambda_F$ with $[F]$ not among the stated exceptions. (For an explicit example, if H is ample, then μ_F fails to have maximal rank for $F = H + (H \cdot C_i)C_i$: by Theorem IV.1, μ_H and hence μ_F are not injective and thus $1 = t_F = \dim \operatorname{cok}(\mu_F)$ by Theorem I.6.1.)

On the other hand, as a corollary of Theorem I.6.1 we see for a numerically effective $F \subset X$ that μ_F never fails by much to have maximal rank: μ_F is never more than 7 short of maximal rank.

II Generalities

We first recall a useful exact sequence from [15]. For sheaves $\mathcal{F}$ and $\mathcal{G}$ on X, we will denote the kernel of the natural map $H^0(X, \mathcal{F}) \otimes H^0(X, \mathcal{G}) \to H^0(X, \mathcal{F} \otimes \mathcal{G})$ by $\mathcal{R}(\mathcal{F}, \mathcal{G})$ and the cokernel by $\mathcal{S}(\mathcal{F}, \mathcal{G})$. When $\mathcal{F} = \mathcal{O}_X(F)$ and $\mathcal{G} = \mathcal{O}_X(G)$ for divisors F and G on X, we will, if it is convenient, just write $\mathcal{R}(F, G)$ and $\mathcal{S}(F, G)$.

Proposition II.1 *Let $C \subset X$ be a curve on a smooth projective surface X, and let A and B be divisors on X, so we have the exact sequence* $0 \to \mathcal{O}_X(A - C) \to \mathcal{O}_X(A) \to \mathcal{O}_C \otimes \mathcal{O}_X(A) \to 0$. *Then there is an exact sequence*

$$0 \to \mathcal{R}\big(\mathcal{O}_X(A - C), \mathcal{O}_X(B)\big) \to \mathcal{R}\big(\mathcal{O}_X(A), \mathcal{O}_X(B)\big) \to \mathcal{R}\big(\mathcal{O}_C \otimes \mathcal{O}_X(A), \mathcal{O}_X(B)\big).$$

If the restriction homomorphisms $H^0\big(X, \mathcal{O}_X(A)\big) \to H^0\big(C, \mathcal{O}_X(A) \otimes \mathcal{O}_C\big)$ *and* $H^0\big(X, \mathcal{O}_X(A + B)\big) \to H^0\big(C, \mathcal{O}_X(A + B) \otimes \mathcal{O}_C\big)$ *are surjective (for example, if*

$h^1\big(X, \mathcal{O}_X(A-C)\big) = 0 = h^1\big(X, \mathcal{O}_X(A+B-C)\big)$), this extends to an exact sequence

$$\begin{aligned}0 &\to \mathcal{R}\big(\mathcal{O}_X(A-C), \mathcal{O}_X(B)\big) \to \mathcal{R}\big(\mathcal{O}_X(A), \mathcal{O}_X(B)\big) \to \mathcal{R}\big(\mathcal{O}_C \otimes \mathcal{O}_X(A), \mathcal{O}_X(B)\big)\\ &\to \mathcal{S}\big(\mathcal{O}_X(A-C), \mathcal{O}_X(B)\big) \to \mathcal{S}\big(\mathcal{O}_X(A), \mathcal{O}_X(B)\big) \to \mathcal{S}\big(\mathcal{O}_C \otimes \mathcal{O}_X(A), \mathcal{O}_X(B)\big) \to 0.\end{aligned}$$

It will be helpful to have bounds on the dimensions of $\mathcal{R}$ and $\mathcal{S}$.

Proposition II.2 *Let F be an effective divisor with $h^1\big(X, \mathcal{O}_X(F)\big) = 0$ on the blowing up X of $\mathbf{P}^2$ at r distinct points $p_1, \ldots, p_r$, let $[L], [E_1], \ldots, [E_r]$ be the corresponding exceptional configuration, and assume that $F \cdot E_1 \geq \cdots \geq F \cdot E_r$. Let $d = F \cdot L$, $h = h^0\big(X, \mathcal{O}_X(F)\big)$, $l_i = h^0\Big(X, \mathcal{O}_X\big(F - (L - E_i)\big)\Big)$, and $q_i = h^0\big(X, \mathcal{O}_X(F - E_i)\big)$.*

(a) Then μ_F has maximal rank if and only if $\max(0, 2h - d - 2) = \dim \mathcal{R}(F, L)$.

(b) If $h^1\Big(X, \mathcal{O}_X\big(F - (L - E_1)\big)\Big) = 0 = h^1\big(X, \mathcal{O}_X(F - E_1)\big)$, then $l_1 + q_1 = 2h - d - 2$.

(c) In any case, we have $\max(0, 2h - d - 2) \leq \dim \mathcal{R}(F, L) \leq l_1 + q_1$.

(d) We also have $l_1 + l_2 \leq \dim \mathcal{R}(F, L) \leq l_1 + l_2 + h^0\Big(X, \mathcal{O}_X\big(F + (L - E_1 - E_2)\big)\Big) - h^0\big(X, \mathcal{O}_X(F)\big)$.

(e) If $[L - E_1 - E_2]$ is the class of an irreducible curve with $F \cdot (L - E_1 - E_2) = 0$, then $\dim \mathcal{R}(F, L) = l_1 + l_2$ *and* $\dim \mathcal{S}(F, L) = h^1\Big(X, \mathcal{O}_X\big(F - (L - E_1)\big)\Big) + h^1\Big(X, \mathcal{O}_X\big(F - (L - E_2)\big)\Big)$.

Proof Proposition II.2 (a,b,c) is just Corollary 4.2 of [11]. Consider (d). If we choose coordinates x, y and z where x and y pass through p_1 and y and z through p_2, then (from the proof of Lemma 4.1 of [11]) l_i is just the dimension of the kernel of the restriction of μ_F to $H^0\big(X, \mathcal{O}_X(F)\big) \otimes V_i \to H^0\big(X, \mathcal{O}_X(F + L)\big)$, where V_1 is the vector space span of x and y in $H^0\big(X, \mathcal{O}_X(L)\big)$ and where V_2 is the vector space span of z and y. It is easy to see that these two kernels have only 0 in common; this gives the lower bound of (d). For the upper bound, it suffices to show $l_1+q_1 \leq l_1+l_2+\bigg(h^0\Big(X, \mathcal{O}_X\big(F+(L-E_1-E_2)\big)\Big)-h^0\big(X, \mathcal{O}_X(F)\big)\bigg)$. Since $[F-(L-E_2)] = [F-E_1-E]$, where E is the effective divisor in the class $[L-E_1-E_2]$, this follows from taking cohomology of $0 \to \mathcal{O}_X\big(F - (L - E_2)\big) \to \mathcal{O}_X(F - E_1) \to \mathcal{O}_E \otimes \mathcal{O}_X(F - E_1) \to 0$, using $\mathcal{O}_E \otimes \mathcal{O}_X(F - E_1) \cong \mathcal{O}_E \otimes \mathcal{O}_X(F + E)$ and the fact that $h^1\big(X, \mathcal{O}_X(F)\big) = 0$ implies that $h^0\Big(X, \mathcal{O}_X\big(F + (L - E_1 - E_2)\big)\Big) - h^0\big(X, \mathcal{O}_X(F)\big) = h^0\big(E, \mathcal{O}_E \otimes \mathcal{O}_X(F + E)\big)$.

Finally consider (e); then E is irreducible and hence a fixed component of $|F + E|$, so (d) gives us $l_1 + l_2 = \dim \mathcal{R}(F, L)$. From $\dim \mathcal{S}(F, L) = h^0\big(X, \mathcal{O}_X(F + L)\big) - 3h^0\big(X, \mathcal{O}_X(F)\big) + \dim \mathcal{R}(F, L)$, we thus obtain $\dim \mathcal{S}(F, L) = h^0\big(X, \mathcal{O}_X(F + L)\big) - 3h^0\big(X, \mathcal{O}_X(F)\big) + h^0\Big(X, \mathcal{O}_X\big(F - (L - E_1)\big)\Big) + h^0\Big(X, \mathcal{O}_X\big(F - (L - E_2)\big)\Big)$. But $h^1\big(X, \mathcal{O}_X(F)\big) = 0$ and hence $h^1\big(X, \mathcal{O}_X(F + L)\big) = 0$ so Riemann-Roch gives $h^0\big(X, \mathcal{O}_X(F + L)\big) = h^0\big(X, \mathcal{O}_X(F)\big) + F \cdot L + 2$. Riemann-Roch also gives $h^0\Big(X, \mathcal{O}_X\big(F - (L - E_i)\big)\Big) = h^0\big(X, \mathcal{O}_X(F)\big) + h^1\Big(X, \mathcal{O}_X\big(F-(L-E_i)\big)\Big) - 1 - F\cdot(L-E_i)$ for $i = 1, 2$. Now substituting into our expression

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for $\dim \mathcal{S}(F, L)$ and simplifying (using $F \cdot L - F \cdot (L - E_1) - F \cdot (L - E_2) = -F \cdot E = 0$) gives the result. ■

Remark II.3 Note that the conclusion $\dim \mathcal{R}(F, L) = l_1 + l_2$ of Proposition II.2 (e) does not need the hypothesis that $h^1\big(X, \mathcal{O}_X(F)\big) = 0$. The argument that $l_1 + l_2 \leq \dim \mathcal{R}(F, L)$ does not use $h^1\big(X, \mathcal{O}_X(F)\big) = 0$, and by Lemma 4.1 of [11] neither does $\dim \mathcal{R}(F, L) \leq l_1 + q_1$. Finally, with E as in the proof of Proposition II.2 (e), we have $h^0\Big(E, \mathcal{O}_E\big((F - E_1) \cdot E\big)\Big) = 0$, so $l_2 = q_1$ follows by taking cohomology of $0 \to \mathcal{O}_X\big(F - (L - E_2)\big) \to \mathcal{O}_X(F - E_1) \to \mathcal{O}_E\big((F - E_1) \cdot E\big) \to 0$. ■

Lemma II.4 *Let $F \neq 0$ be an effective divisor on a smooth projective surface X, and let E be an exceptional curve with $F \cdot E = 0$.*

(a) Say $|F|$ is fixed component free. Then $h^0\big(X, \mathcal{O}_X(F - E)\big) > 0$, and if $h^1\big(X, \mathcal{O}_X(F)\big) = 0$, then $h^1\big(X, \mathcal{O}_X(F - E)\big) = 0$.

(b) Say X is a blowing up of points of $\mathbf{P}^2$ and L is the total transform of a line. If $E \cdot L \geq 2$, then the kernels of μ_F and μ_{F-E} have the same dimension.

Proof (a) Since $F \cdot E = 0$, we have an exact sequence $0 \to \mathcal{O}_X(F - E) \to \mathcal{O}_X(F) \to \mathcal{O}_E \to 0$. Since $|F|$ has no fixed components, $h^0\big(X, \mathcal{O}_X(F)\big) > 1$ and $H^0\big(X, \mathcal{O}_X(F)\big) \to H^0(E, \mathcal{O}_E)$ is surjective. From the latter, our sequence is exact on global sections, so our conclusions follow.

(b) Because $E \cdot L \geq 2$, it follows that $h^0\big(X, \mathcal{O}_X(L - E)\big) = 0$, but clearly $H^0\big(X, \mathcal{O}_X(L - E)\big) = \mathcal{R}\big(\mathcal{O}_E, \mathcal{O}_X(L)\big)$, so $\mathcal{R}\big(\mathcal{O}_E, \mathcal{O}_X(L)\big) = 0$. Now apply Proposition II.1 to the exact sequence in the proof of (a) to get an isomorphism $\mathcal{R}(F - E, L) \to \mathcal{R}(F, L)$; *i.e.*, the kernels of μ_F and μ_{F-E} have the same dimension. ■

Lemma II.5 *Let X be a blowing up of distinct points of $\mathbf{P}^2$ with corresponding exceptional configuration $[L], [E_1], \ldots, [E_r]$. Let $D \subset X$ be a smooth rational curve with $D^2 = 0$ and let $m \geq 0$ be a nonnegative integer. Then $\dim \mathcal{R}(mD, L) = m$ if $D \cdot L = 1$ and $\mathcal{R}(mD, L) = 0$ if $D \cdot L > 1$.*

Proof If $L \cdot D > 1$, then (as in the proof of Lemma II.4 (b)) $0 = H^0\big(X, \mathcal{O}_X(L - D)\big) = \mathcal{R}\big(\mathcal{O}_D, \mathcal{O}_X(L)\big)$. Applying Proposition II.1 and induction on s to $0 \to \mathcal{O}_X(sD) \to \mathcal{O}_X\big((s + 1)D\big) \to \mathcal{O}_D \to 0$ gives $\mathcal{R}(mD, L) = 0$.

If $L \cdot D = 1$, then $[D]$ must be $[L - E_i]$ for some i. By [10] (or directly), $\mathcal{S}(mD, L) = 0$, so $\dim \mathcal{R}(mD, L) = 3h^0\big(X, \mathcal{O}_X(mD)\big) - h^0\big(X, \mathcal{O}_X(mD + L)\big) = 3(m + 1) - (2m + 3) = m$. ■

III Particularities

Now let X be obtained by blowing up $r \leq 8$ general points $p_1, \ldots, p_r$ of $\mathbf{P}^2$ and let $[L], [E_1], \ldots, [E_r]$ be the corresponding exceptional configuration. We recall some facts for which we refer to [5], [9], [12] and [16].

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The exceptional configuration $[L], [E_1], \dots, [E_r]$ is determined by and in turn determines a birational morphism $X \to \mathbf{P}^2$ with a factorization into monoidal transformations. Since X can have more than one birational morphism to $\mathbf{P}^2$, each of which typically factors in several ways, X can also have more than one exceptional configuration. For example, if $\pi_1 \colon X \to \mathbf{P}^2$ is the morphism determined by $[L], [E_1], \dots, [E_r]$, and if $\pi_2 \colon X \to \mathbf{P}^2$ is the morphism such that $\pi_2\pi_1^{-1}$ is the quadratic Cremona transformation centered at p_1, p_2 and p_3 (*i.e.*, $\pi_2\pi_1^{-1}$ is the birational map from $\mathbf{P}^2$ to $\mathbf{P}^2$ given by the linear system of conics with base points at p_1, p_2 and p_3), then the exceptional configuration determined by π_2 (after an appropriate factorization) is $[2L - E_1 - E_2 - E_3], [L - E_2 - E_3], [L - E_1 - E_3], [L - E_1 - E_2], [E_4], \dots, [E_r]$.

Any two exceptional configurations are related by an element of the orthogonal group on $\mathrm{Cl}(X)$. Inside the orthogonal group on $\mathrm{Cl}(X)$, the subgroup W generated by the reflections s_i, $0 \leq i < r$, where $s_0(x) = x + (x \cdot [L - E_1 - E_2 - E_3])[L - E_1 - E_2 - E_3]$ and $s_i(x) = x + (x \cdot [E_i - E_{i+1}])[E_i - E_{i+1}]$, is known as the *Weyl* group. For $i > 0$, the action of s_i on $[a_0L + a_1E_1 + \cdots + a_rE_r]$ is just to transpose the coefficients a_i and a_{i+1}, while s_0 takes $[L], [E_1], \dots, [E_r]$ to $[2L - E_1 - E_2 - E_3], [L - E_2 - E_3], [L - E_1 - E_3], [L - E_1 - E_2], [E_4], \dots, [E_r]$. More generally, given any pair of exceptional configurations there is an element of W taking one to the other, and any $w \in W$ takes $[L], [E_1], \dots, [E_r]$ to another exceptional configuration. This gives a bijection between exceptional configurations and elements of W.

If $[F_1]$ and $[F_2]$ are divisor classes in the same orbit of W, then $h^i\big(X, \mathcal{O}_X(F_1)\big) = h^i\big(X, \mathcal{O}_X(F_2)\big)$ holds for all i. In addition, if F is effective, then $h^2\big(X, \mathcal{O}_X(F)\big) = 0$, while if F is numerically effective, then $h^1\big(X, \mathcal{O}_X(F)\big) = 0$ and $|F|$ is nonempty and fixed component free.

Given any effective divisor D, we can write $[D] = [H] + [N]$, where H is numerically effective and $N = -\sum(E \cdot D)E$, where the sum is over all exceptional curves E with $E \cdot D < 0$; note that the summands E which appear in N are disjoint. Since, as noted above, $h^1\big(X, \mathcal{O}_X(H)\big) = 0$, it is easy to verify that $h^1\big(X, \mathcal{O}_X(D)\big) = 0$ if and only if no summand in N occurs with a coefficient of 2 or more (and hence if and only if $D \cdot E \geq -1$ for every exceptional curve E).

For $8 \geq r \neq 2$, the classes of exceptional curves comprise one orbit, $W[E_r]$. (If $r = 2$, there are only three classes of exceptional curves, $[L - E_1 - E_2]$, $[E_1]$ and $[E_2]$, split between two W-orbits: $\{[L - E_1 - E_2]\}$ is one orbit, and $\{[E_1], [E_2]\}$ is the other.) For $r = 7$, up to permutations of the E_i, the classes of the exceptional curves are just $[E_7]$, $[L - E_1 - E_2]$, $[2L - E_1 - \cdots - E_5]$, and $[3L - 2E_1 - E_2 - \cdots - E_7]$.

Also for $r = 7$, the classes of numerically effective divisors are precisely the W-orbits of nonnegative linear combinations of the classes of L, $L - E_1$, $2L - E_1 - E_2$, $3L - E_1 - E_2 - E_3$, $\dots$, $3L - E_1 - \cdots - E_7$. By excluding elements which can be obtained from others (for example, exclude $[2L - E_1 - E_2]$, since $[2L - E_1 - E_2] = [L - E_1] + [L - E_2]$), we can give a more efficient list of generators for the cone of numerically effective divisor classes. We thereby get the following list of divisors, whose classes give a set of generators (complete up to permutation of the E_i) for the numerically effective cone:

$$G_1 = 1L - 0E_1 - 0E_2 - 0E_3 - 0E_4 - 0E_5 - 0E_6 - 0E_7,$$
$$G_2 = 2L - 1E_1 - 1E_2 - 1E_3 - 0E_4 - 0E_5 - 0E_6 - 0E_7,$$
$$G_3 = 3L - 2E_1 - 1E_2 - 1E_3 - 1E_4 - 1E_5 - 0E_6 - 0E_7,$$

$$G_4 = 4L - 2E_1 - 2E_2 - 2E_3 - 1E_4 - 1E_5 - 1E_6 - 0E_7,$$
$$G_5 = 4L - 3E_1 - 1E_2 - 1E_3 - 1E_4 - 1E_5 - 1E_6 - 1E_7,$$
$$G_6 = 5L - 3E_1 - 2E_2 - 2E_3 - 2E_4 - 1E_5 - 1E_6 - 1E_7,$$
$$G_7 = 5L - 2E_1 - 2E_2 - 2E_3 - 2E_4 - 2E_5 - 2E_6 - 0E_7,$$
$$G_8 = 6L - 3E_1 - 3E_2 - 2E_3 - 2E_4 - 2E_5 - 2E_6 - 1E_7,$$
$$G_9 = 7L - 3E_1 - 3E_2 - 3E_3 - 3E_4 - 2E_5 - 2E_6 - 2E_7,$$
$$G_{10} = 8L - 3E_1 - 3E_2 - 3E_3 - 3E_4 - 3E_5 - 3E_6 - 3E_7,$$
$$G_{11} = 1L - 1E_1 - 0E_2 - 0E_3 - 0E_4 - 0E_5 - 0E_6 - 0E_7,$$
$$G_{12} = 2L - 1E_1 - 1E_2 - 1E_3 - 1E_4 - 0E_5 - 0E_6 - 0E_7,$$
$$G_{13} = 3L - 2E_1 - 1E_2 - 1E_3 - 1E_4 - 1E_5 - 1E_6 - 0E_7,$$
$$G_{14} = 4L - 2E_1 - 2E_2 - 2E_3 - 1E_4 - 1E_5 - 1E_6 - 1E_7,$$
$$G_{15} = 5L - 2E_1 - 2E_2 - 2E_3 - 2E_4 - 2E_5 - 2E_6 - 1E_7,$$
$$G_{16} = 3L - 1E_1 - 1E_2 - 1E_3 - 1E_4 - 1E_5 - 1E_6 - 0E_7,$$
$$G_{17} = 4L - 2E_1 - 2E_2 - 1E_3 - 1E_4 - 1E_5 - 1E_6 - 1E_7,$$
$$G_{18} = 5L - 2E_1 - 2E_2 - 2E_3 - 2E_4 - 2E_5 - 1E_6 - 1E_7,$$
$$G_{19} = 6L - 3E_1 - 2E_2 - 2E_3 - 2E_4 - 2E_5 - 2E_6 - 2E_7 \text{ and}$$
$$G_{20} = 3L - 1E_1 - 1E_2 - 1E_3 - 1E_4 - 1E_5 - 1E_6 - 1E_7.$$

Since $[G_1]$ is clearly the class of a smooth rational curve, so are $[G_2], \dots, [G_{10}]$, since in fact they all are in the same orbit of W. Likewise, $[G_{11}], \dots, [G_{15}]$ is each the class of a smooth rational curve, and $[G_{16}], \dots, [G_{20}]$ is each the class of a smooth elliptic curve.

It is also easy to check that each class $[G_i]$ is a sum of classes of exceptional curves and hence for $r = 7$ the class of any effective divisor is a sum of classes of exceptional curves. It now follows for $r = 7$ (and in any case is well known) that $[3L - E_1 - \cdots - E_7] = -K_X$ is ample and hence so is any class of the form $[D] - K_X$, where D is numerically effective. Conversely, for $r = 7$ any ample class $[F]$ is of this form: as noted above, for some $w \in W$, $w[F]$ is a nonnegative linear combination of the classes of the divisors $L, L-E_1, 2L-E_1-E_2, 3L - E_1 - E_2 - E_3, \dots, 3L - E_1 - \cdots - E_7$. But $w[F] \cdot [E_7] = [F] \cdot w^{-1}[E_7] > 0$ since $w^{-1}[E_7]$ is the class of an exceptional curve and F is ample, so this linear combination involves $-K_X$ and hence is of the form $[D] - K_X$. *I.e.*, $[F] = w^{-1}([D] - K_X)$, but W preserves the numerically effective cone, so in particular $w^{-1}[D]$ is numerically effective. Finally, $w^{-1}(-K_X) = -K_X$ since $-K_X$ is stabilized by W, so $[F]$ has the required form.

The same argument works for $3 \le r < 7$; *i.e.*, every ample divisor class is $-K_X$ plus a numerically effective class. The argument fails for $0 \le r \le 2$ (for one thing, $-K_X$ is itself no longer needed as a generator of the numerically effective cone if $0 \le r \le 2$, and, if $r < 2$, the class of an effective divisor need not be the sum of classes of exceptional curves). However, an easy *ad hoc* argument shows that the conclusion is still true for $r = 2$, while for $r = 1$ the ample divisor classes are $[dL - mE_1]$, where $d > m$, and for $r = 0$ they are $[dL]$, where $d > 0$.

IV Application to 7 Points

As an application of our results above, we will prove Theorem I.6.1. To do so, we need some additional results. We begin by considering ample divisors.

Theorem IV.1 *Let F be an ample divisor on the blowing up X of* $\mathbf{P}^2$ *at* $t \le 7$ *general points,*

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with L the total transform of a line in $\mathbf{P}^2$. Then $\mathcal{R}(F, L) \neq 0$ and $\mathcal{S}(F, L) = 0$.

Proof Let $[L], [E_1], \ldots, [E_t]$ be the exceptional configuration corresponding to the t points blown up to obtain X. After reindexing, we may assume that $F \cdot E_1 \geq F \cdot E_2 \geq \cdots F \cdot E_t > 0$. If $t \leq 5$, then, in fact, $\mathcal{S}(F, L) = 0$ for any numerically effective F by [10], while for $t \leq 2$, as follows from a discussion above, every ample class F is of the form $[L]$ plus a numerically effective class. But $\mathcal{R}(L, L) \neq 0$, so of course $\mathcal{R}(F, L) \neq 0$, too.

So now we may assume $t \geq 3$. Since F is ample, as pointed out above we have $[F] = [D] - K_X$, where D is numerically effective. But $-K_X = [3L - E_1 - \cdots - E_t]$, so $[F - E_1] = [D] + [3L - 2E_1 - \cdots - E_t] = [D + C_1]$. For $t < 7$, $[C_1]$ is numerically effective and hence $0 < h^0\big(X, \mathcal{O}_X(F - E_1)\big) = q_1$ and $0 = h^1\big(X, \mathcal{O}_X(F - E_1)\big)$. If $t = 7$, then $[C_1]$ is the class of an exceptional curve, so $[F - E_1]$ is the class of an effective divisor, so $0 < h^0\big(X, \mathcal{O}_X(F - E_1)\big) = q_1$. Moreover, $E \cdot (D + C_1) \geq -1$ for every exceptional curve E, so $F - E_1$ is regular (*i.e.*, $h^1\big(X, \mathcal{O}_X(F - E_1)\big) = 0$).

Since $h^1\big(X, \mathcal{O}_X(F - E_1)\big) = 0$, if we show $h^1\Big(X, \mathcal{O}_X\big(F - (L - E_1)\big)\Big) = 0$, then by Proposition II.2 we will know that μ_F has maximal rank and, using Proposition II.2 and $q_1 > 0$ to see that $\mathcal{R}(F, L) \neq 0$, that $\mathcal{S}(F, L)$ must vanish.

From $[F] = [D] - K_X$ we obtain $[F - (L - E_1)] = [D] + [Q]$, where $Q = 2L - E_2 - \cdots - E_t$. Arguing as for $h^1\big(X, \mathcal{O}_X(F - E_1)\big)$, $h^1\Big(X, \mathcal{O}_X\big(F - (L - E_1)\big)\Big)$ also vanishes if $t < 7$, so we are reduced to the case that $t = 7$. Now, $[D]$ is a sum of classes $[U_i]$, where each divisor U_i is, up to permutation of the E_i, one of the divisors G_j of Section III. Recall each of the classes $[G_j]$ is the class of a smooth curve, either rational or elliptic; by considering all permutations of the E_i for each G_j, we explicitly check that $U_i \cdot (U_i + Q) \geq 2$ in each case that $[U_i]$ is the class of an elliptic curve and $U_i \cdot (U_i + Q) \geq -1$ in each case that $[U_i]$ is the class of a rational curve, unless $U_i = 5L - 1E_1 - 2E_2 - 2E_3 - 2E_4 - 2E_5 - 2E_6 - 2E_7$, in which case $U_i \cdot (U_i + Q) = -2$. Thus, letting A_i be a smooth curve with $[A_i] = [U_i]$, we have $h^1\big(A_i, \mathcal{O}_{A_i}(U_i + Q)\big) = 0$ unless $U_i = 5L - 1E_1 - 2E_2 - 2E_3 - 2E_4 - 2E_5 - 2E_6 - 2E_7$. Moreover, $(5L - 1E_1 - 2E_2 - 2E_3 - 2E_4 - 2E_5 - 2E_6 - 2E_7) \cdot U_i > 0$ for all i with $U_i \neq 5L - 1E_1 - 2E_2 - 2E_3 - 2E_4 - 2E_5 - 2E_6 - 2E_7$. Thus, unless each U_i is $5L - 1E_1 - 2E_2 - 2E_3 - 2E_4 - 2E_5 - 2E_6 - 2E_7$, we may assume that U_1 is not $5L - 1E_1 - 2E_2 - 2E_3 - 2E_4 - 2E_5 - 2E_6 - 2E_7$, and then from $h^1\big(X, \mathcal{O}_X(Q)\big) = 0$ it follows inductively by taking cohomology of

$$0 \to \mathcal{O}_X(Q + U_1 + \cdots + U_{i-1}) \to \mathcal{O}_X(Q + U_1 + \cdots + U_i) \to \mathcal{O}_{A_i}(Q + U_1 + \cdots + U_i) \to 0$$

that $h^1\big(X, \mathcal{O}_X(D + Q)\big) = 0$, as desired.

There remains the case that $F = m(5L - 1E_1 - 2E_2 - 2E_3 - 2E_4 - 2E_5 - 2E_6 - 2E_7) - K_X$, for $m > 0$. But our assumption that $F \cdot E_1 \geq F \cdot E_2 \geq \cdots \geq F \cdot E_7 > 0$ rules out this case. ∎

Lemma IV.2 *Let X be a blowing up of $\mathbf{P}^2$ at 7 general points $p_1, \ldots, p_7$, with $[L], [E_1], \ldots, [E_7]$ the corresponding exceptional configuration. Let J_i, $i = 1, 2$, be smooth curves whose classes are $[L - E_i]$. Let $0 \neq [F]$ be numerically effective with $F \cdot (L - E_1 - E_2) = 0$ and $F \cdot E_1 \geq \cdots \geq F \cdot E_7$. Then μ_F fails to have maximal rank if and only if $h^0\big(X, \mathcal{O}_X(F - J_1)\big) > 0$ and $h^1\big(X, \mathcal{O}_X(F - J_2)\big) > 0$.*

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Proof By Proposition II.2 (e), $l_1 > 0$ implies that μ_F is not injective, while $h^1\big(X, \mathcal{O}_X(F - J_2)\big) > 0$ implies that μ_F is not surjective.

Conversely, by Proposition II.2 (e), if μ_F is neither surjective nor injective, then $l_1+l_2 > 0$ and $h^1\big(X, \mathcal{O}_X(F - J_1)\big) + h^1\big(X, \mathcal{O}_X(F - J_2)\big) > 0$, so it suffices to check that $l_2 > 0$ implies $l_1 > 0$, and that $l_1 > 0$ and $h^1\big(X, \mathcal{O}_X(F - J_1)\big) > 0$ together imply $h^1\big(X, \mathcal{O}_X(F - J_2)\big) > 0$.

Suppose $l_2 > 0$. Thus $[F - J_2]$ is a sum of classes of exceptional curves T_i, and, since $F \cdot E_1 \geq F \cdot E_2$ and hence $(F - J_2) \cdot E_1 > (F - J_2) \cdot E_2$, some summand has $[T_i] \cdot (E_1 - E_2) > 0$, hence by Riemann-Roch and duality $h^0\Big(X, \mathcal{O}_X\big(T_i + (E_1 - E_2)\big)\Big) > 0$. Thus $l_1 = h^0\Big(X, \mathcal{O}_X\big(F - J_2 + (E_1 - E_2)\big)\Big) > 0$, as claimed.

Now assume $l_1 > 0$ and $h^1\big(X, \mathcal{O}_X(F - J_1)\big) > 0$. If $F \cdot E_1 = F \cdot E_2$, then $F - J_1$ and $F - J_2$ are the same, up to permutation of the E_i, hence in the same orbit of the Weyl group, so $h^1\big(X, \mathcal{O}_X(F - J_1)\big) = h^1\big(X, \mathcal{O}_X(F - J_2)\big)$. So suppose that $F \cdot E_1 > F \cdot E_2$, and hence that $(F - J_1) \cdot E_1 \geq \cdots \geq (F - J_1) \cdot E_7 \geq 0$. Since $[F - J_1]$ has an effective representative, $h^1\big(X, \mathcal{O}_X(F - J_1)\big) > 0$ implies that there is an exceptional curve E with $(F - J_1) \cdot E \leq -2$. Clearly, this E is not among the E_i, so we may assume that $[E]$ is either $[L - E_1 - E_2]$, $[2L - E_1 - \cdots - E_5]$ or $[3L - 2E_1 - E_2 - \cdots - E_7]$ (since up to permutation of the E_i, the class of every exceptional curve is one of these, and these are the permutations minimizing the intersection with $F - J_1$). But whichever of these is E, we have $(F - J_2) \cdot E = \big(F - J_1 - (E_1 - E_2)\big) \cdot E \leq (F - J_1) \cdot E \leq -2$, so from $0 \to \mathcal{O}_X(F - J_2 - E) \to \mathcal{O}_X(F - J_2) \to \mathcal{O}_E\big((F - J_2) \cdot E\big) \to 0$, it suffices to check that $h^2\big(X, \mathcal{O}_X(F - J_2 - E)\big) = 0$ to obtain that $h^1\big(X, \mathcal{O}_X(F - J_2)\big) > 0$, as required. But $F \cdot L \geq 1$ (since $[F]$ is nontrivial and numerically effective), so $(K_X - [F - J_2 - E]) \cdot L < 0$ (so $0 = h^0\Big(X, \mathcal{O}_X\big(K_X - (F - J_2 - E)\big)\Big) = h^2\big(X, \mathcal{O}_X(F - J_2 - E)\big)$, since L is numerically effective) unless $E \cdot L = 3$ and $F \cdot L = 1$. In this latter case $[E] = [3L - 2E_1 - E_2 - \cdots - E_7]$ and $[F] = [L - E_1]$. Then $K_X - [F - J_2 - E] = [-E_2]$, which again is not the class of an effective divisor, so again $h^2\big(X, \mathcal{O}_X(F - J_2 - E)\big) = 0$ by duality. ■

Lemma IV.3 *Let X be as in Lemma IV.2, let E be an exceptional curve with $E \cdot L = 1$, and let F be numerically effective such that $F \cdot E = 0$, but $F \cdot C > 0$ for every exceptional curve C with $C \cdot L \neq 1$. Then $\mathcal{S}(F, L) = 0$ but $\mathcal{R}(F, L) \neq 0$.*

Proof As usual, we may assume that $F \cdot E_1 \geq \cdots \geq F \cdot E_7$, and thus we may assume E is the exceptional curve whose class is $[L - E_1 - E_2]$.

First say that $F \cdot C > 0$ for every exceptional curve $C \neq E$. Choose an element w of the Weyl group W such that $w[F]$ is a sum of nonnegative multiples of the classes of L, $L - E_1$, $2L - E_1 - E_2$, $3L - E_1 - E_2 - E_3, \ldots, 3L - E_1 - \cdots - E_7$. Note that this sum cannot involve $-K_X = [3L - E_1 - \cdots - E_7]$. (If it did, then $w[F] = [D] - K_X$ for some numerically effective D, but $-K_X$ is ample and hence so would be $w[F]$ and thus $[F]$, contradicting $F \cdot E = 0$.) It follows that $w[F] \cdot E_7 = 0$ and hence that $w[E] = [E_7]$. Since $F \cdot C > 0$ for every exceptional curve $C \neq E$, we have $w[F] \cdot E_6 > 0$, hence the class of $H = 3L - E_1 - \cdots - E_6$ appears in the sum. Thus $[F] - w^{-1}[H]$ is numerically effective, so $F \cdot E = 0$ implies $w^{-1}[H] \cdot E = 0$. But looking over the W-orbit of $[H]$ shows it has only one element perpendicular to E;

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i.e., we must have $w^{-1}[H] = [4L - 2E_1 - 2E_2 - 1E_3 - 1E_4 - 1E_5 - 1E_6 - 1E_7]$. Thus $\big[(F - w^{-1}[H]) + \big(w^{-1}[H] - (L - E_1)\big)\big] = [D + C_2]$ for some numerically effective D. Since $[C_2] = [3L - 1E_1 - 2E_2 - 1E_3 - 1E_4 - 1E_5 - 1E_6 - 1E_7]$ is the class of an exceptional curve, we see that $l_1 = h^0\Big(X, \mathcal{O}_X\big(F - (L - E_1)\big)\Big) > 0$ and $h^1\Big(X, \mathcal{O}_X\big(F - (L - E_1)\big)\Big) = 0$; similarly, $h^1\Big(X, \mathcal{O}_X\big(F - (L - E_2)\big)\Big) = 0$. By Proposition II.2 (e), $\mathcal{R}(F, L) \neq 0$ and $\mathcal{S}(F, L) = 0$.

Now suppose that $F \cdot C = 0$ for some exceptional curve $C \neq E$. If we denote $L - E_i - E_j$ by C_{ij}, then by hypothesis $[C] = [C_{ij}]$ for some i and j, and, since $F \cdot E_1 \geq \cdots \geq F \cdot E_7$, either $F \cdot E_1 = F \cdot E_2$ and thus $[F]$ is of the form $[2a_1 L - a_1(E_1 + \cdots + E_i) - b_{i+1}E_{i+1} - \cdots - b_7 E_7]$ where $a_1 > b_{i+1} \geq \cdots \geq b_7 > 0$ and $i \geq 3$, or $F \cdot E_1 > F \cdot E_2$ and thus $[F]$ is of the form $[(a_1 + a_2)L - a_1 E_1 - a_2(E_2 + \cdots + E_i) - b_{i+1}E_{i+1} - \cdots - b_7 E_7]$ where $a_1 > a_2 > b_{i+1} \geq \cdots \geq b_7 > 0$ and $i \geq 3$.

For the former, $i = 3$, since otherwise $F \cdot (2L - E_1 - \cdots - E_5) \leq 0$, so the classes of the only exceptional curves that F is perpendicular to are $[C_{12}]$, $[C_{13}]$, and $[C_{23}]$. As above, $w\{C_{12}, C_{13}, C_{23}\} = \{E_5, E_6, E_7\}$ and $w[F]$ is a nonnegative sum of the classes of L, $L - E_1$, $2L - E_1 - E_2$, $3L - E_1 - E_2 - E_3$, and $3L - E_1 - E_2 - E_3 - E_4$, for some $w \in W$, and this sum involves $H = 3L - E_1 - E_2 - E_3 - E_4$. Thus $w^{-1}[H]$ is perpendicular to each of $[C_{12}]$, $[C_{13}]$, and $[C_{23}]$, but by examining the W-orbit of H, we see there is only one element of $W[H]$ perpendicular to each of $[C_{12}]$, $[C_{13}]$, and $[C_{23}]$; *i.e.*, $w^{-1}[H] = [6L - 3E_1 - 3E_2 - 3E_3 - 1E_4 - 1E_5 - 1E_6 - 1E_7]$. But $w^{-1}[H] - [L - E_1] = [C_{23}] + [4L - 2E_1 - 2E_2 - 2E_3 - 1E_4 - 1E_5 - 1E_6 - 1E_7]$ and $w^{-1}[H] - [L - E_2] = [C_{13}] + [4L - 2E_1 - 2E_2 - 2E_3 - 1E_4 - 1E_5 - 1E_6 - 1E_7]$; since $4L - 2E_1 - 2E_2 - 2E_3 - 1E_4 - 1E_5 - 1E_6 - 1E_7$ is numerically effective, we conclude that $l_1 = h^0\Big(X, \mathcal{O}_X\big(F - (L - E_1)\big)\Big) > 0$ and $h^1\Big(X, \mathcal{O}_X\big(F - (L - E_1)\big)\Big) = 0 = h^1\Big(X, \mathcal{O}_X\big(F - (L - E_2)\big)\Big)$ and hence $\mathcal{R}(F, L) \neq 0$ and $\mathcal{S}(F, L) = 0$ by Proposition II.2 (e).

For the latter, F is perpendicular to C_{1j} for all $2 \leq j \leq i$, where, we recall, $i \geq 3$. Reasoning as above, for some $w \in W$, $[F]$ is a sum of a numerically effective class $[D]$ and $w^{-1}[M_i]$, where $M_i = 3L - E_1 - \cdots - E_{8-i}$ for $3 \leq i \leq 6$ and $M_7 = 2L - E_1$, and where $w^{-1}[M_i]$ is perpendicular to each C_{1j} but to no other exceptional curves. As above, there is in each case a unique possibility for $w^{-1}[M_i]$: $w^{-1}[M_3] = [5L - 3E_1 - 2E_2 - 2E_3 - 1E_4 - 1E_5 - 1E_6 - 1E_7]$; $w^{-1}[M_4] = [6L - 4E_1 - 2E_2 - 2E_3 - 2E_4 - 1E_5 - 1E_6 - 1E_7]$; $w^{-1}[M_5] = [7L - 5E_1 - 2E_2 - 2E_3 - 2E_4 - 2E_5 - 1E_6 - 1E_7]$; $w^{-1}[M_6] = [8L - 6E_1 - 2E_2 - 2E_3 - 2E_4 - 2E_5 - 2E_6 - 1E_7]$; and $w^{-1}[M_7] = [5L - 4E_1 - 1E_2 - 1E_3 - 1E_4 - 1E_5 - 1E_6 - 1E_7]$.

In each of the cases $3 \leq i \leq 6$ one checks as above that $w^{-1}[M_i] - [(L - E_1)]$ and hence $[F - (L - E_1)]$ are classes of effective divisors, and similarly that $[F - (L - E_2)]$ is the class of an effective divisor with $[F - (L - E_2)] \cdot C \geq -1$ for every exceptional curve C. This implies that $l_1 > 0$ and $h^1\Big(X, \mathcal{O}_X\big(F - (L - E_2)\big)\Big) = 0$, as required.

We are left with the case $[H] = w^{-1}[M_7]$. We note that $h^0\Big(X, \mathcal{O}_X\big(H - (L - E_1)\big)\Big) > 0$ and $h^1\Big(X, \mathcal{O}_X\big(H - (L - E_2)\big)\Big) = 0$, but $h^0\Big(X, \mathcal{O}_X\big(H - (L - E_2)\big)\Big) = 0$. By Riemann-Roch, $h^0\Big(X, \mathcal{O}_X\big(H + D - (L - E_2)\big)\Big) \geq h^0\big(X, \mathcal{O}_X(D)\big) - 1 + D \cdot \big(H - (L - E_2)\big)$. By checking each of the generators $[G_i]$ of the numerically effective cone (including those obtained by permutations of the E_i), we see that $h^0\Big(X, \mathcal{O}_X\big(H + D - (L - E_2)\big)\Big)$ is positive unless $[D]$ is a nonnegative multiple of $[3L - 2E_1 - 0E_2 - 1E_3 - 1E_4 - 1E_5 - 1E_6 - 1E_7]$, in which case

$\big(H+D-(L-E_2)\big)\cdot(3L-2E_1-0E_2-1E_3-1E_4-1E_5-1E_6-1E_7)=-1$, so numerical effectivity of $[3L-2E_1-0E_2-1E_3-1E_4-1E_5-1E_6-1E_7]$ implies

$$h^0\Big(X,\mathcal{O}_X\big(H+D-(L-E_2)\big)\Big)=0,$$

and now Riemann-Roch gives $h^1\Big(X,\mathcal{O}_X\big(H+D-(L-E_2)\big)\Big)=0$, as required. If $[D]$ is not a multiple of $[3L-2E_1-0E_2-1E_3-1E_4-1E_5-1E_6-1E_7]$, then

$$h^0\Big(X,\mathcal{O}_X\big(H+D-(L-E_2)\big)\Big)>0.$$

But then $\big(H-(L-E_2)\big)\cdot B\geq -1$ for every exceptional curve B and hence the same is true for $D+H-(L-E_2)$ so again $h^1\Big(X,\mathcal{O}_X\big(H+D-(L-E_2)\big)\Big)=0$. ∎

Lemma IV.4 *Let X be a blowing up of $\mathbf{P}^2$ at 7 general points, $[L],[E_1],\ldots,[E_7]$ the corresponding exceptional configuration. Let $[F]$ be a nontrivial numerically effective class and let E be an exceptional curve with $E\cdot F=0$. If C is a reduced irreducible curve occurring as a fixed component of $|F-E|$, then C is an exceptional curve, $F^2=0$ and $[F]=m[E+C]$ for some $m>0$. In addition, if $L\cdot(E+C)>1$, then $\mathcal{R}(F,L)=0$.*

Proof Suppose C is a fixed component of $|F-E|$ (recall by Lemma II.4 (a) that $|F-E|$ is nonempty). Any integral curve C is either numerically effective or has $C^2<0$. But on a 7 point blow up, the former are never fixed and the latter are exceptional; thus C must be an exceptional curve.

Since C is in the base locus of $|F-E|$, we can write $[F-E]=[H]+[N]$, where N and H are the fixed and free parts, respectively, of $|F-E|$ and C is a component of N, hence $C\cdot(F-E)=C\cdot N<0$, but F is numerically effective so $C\cdot E>0$. On the other hand, $E\cdot(H+N+E)=E\cdot F=0$, so $E\cdot(H+N)=1$. Now, $E\cdot C>0$ implies that $|E+C|$ is positive dimensional, hence cannot be contained in N. Of course, C is in N, so E cannot be. Thus $E\cdot N>0$, so $E\cdot(H+N)=1$ tells us that $E\cdot H=0$ and $E\cdot N=1$. Therefore, E is perpendicular to components of N other than C while $E\cdot C=1$ (which means that $|E+C|$ is a pencil). Since this would mean components of N other than C would meet $[H+N+E]=[F]$ negatively, there can be no other components and we see that $N=C$. Thus H is perpendicular to both C and E, and therefore $|H|$ is composed with the pencil $|E+C|$; *i.e.*, $[H]$ is a multiple of $[E+C]$, so $[F]=m[E+C]$ for some $m>0$.

Now let $L\cdot(E+C)>1$; then apply Lemma II.5 with D a general element of $|E+C|$ to obtain $\mathcal{R}(F,L)=0$. ∎

We now give the proof of Theorem I.6.1.

Proof By the algorithm discussed in Section I, one can explicitly check that $t_F>\lambda_F=\dim\mathcal{S}(F,L)$ and $\mathcal{R}(F,L)=0$ for each exception F listed in the statement of the theorem.

We now show that otherwise $\dim\mathcal{S}(F,L)$ is the maximum of t_F and λ_F. So let F be a nontrivial numerically effective divisor.

It may be that $F \cdot E' = 0$ for some exceptional curve E' with $E' \cdot L \geq 2$. By Lemma II.4 (a), $|F'|$ is nonempty for $F' = F - E'$. We continue in this way, subtracting off exceptional curves meeting L at least twice, to obtain a sequence $F = F', F'', \ldots, F^{(j)}, \ldots$, as long as $F^{(j)}$ continues to be perpendicular to some such exceptional curve $E^{(j+1)}$ and as long as $F^{(j+1)} = F^{(j)} - E^{(j+1)}$ has a fixed component free linear system. Eventually, however, say for $j = t$, either $F^{(t)}$ is numerically effective with $F^{(t)} \cdot E > 0$ for every exceptional curve E with $E \cdot L \geq 2$, or $F^{(t)}$ is effective but $|F^{(t)}|$ has a fixed component. For convenience, we write Q_t for E', Q_{t-1} for E'', *etc.*, and also F_0 for $F^{(t)}$ so $[F^{(j)}] = [F_{t-j}]$, where $F_{t-j} = F_0 + Q_1 + \cdots + Q_{t-j}$ for $0 \leq j \leq t$. Thus F_i is numerically effective with $F_i \cdot Q_i = 0$ for $i > 0$, and F_0 is either numerically effective with $F \cdot E > 0$ for every exceptional curve E with $E \cdot L \geq 2$, or F_0 is effective but $|F_0|$ has a fixed component.

Since $0 = H^0\big(X, \mathcal{O}_X(L - Q_i)\big) = \mathcal{R}\big(\mathcal{O}_{Q_i}, \mathcal{O}_X(L)\big)$, we have $\dim \mathcal{S}\big(\mathcal{O}_{Q_i}, \mathcal{O}_X(L)\big) = h^0\big(Q_i, \mathcal{O}_{Q_i}(Q_i \cdot L)\big) - 3 = Q_i \cdot L - 2$, so applying Proposition II.1 and induction on i to $0 \to \mathcal{O}_X(F_{i-1}) \to \mathcal{O}_X(F_i) \to \mathcal{O}_{Q_i} \to 0$ we see $\dim \mathcal{S}(F, L) = \dim \mathcal{S}(F_0, L) + (Q_1 + \cdots + Q_t) \cdot L - 2t$. (Note that $(Q_1 + \cdots + Q_t) \cdot L - 2t$ is just the number of summands Q_i with $Q_i \cdot L = 3$.)

Consider first the case that F_0 is numerically effective. If $F_0 \cdot E_i = 0$ for some i, then we can regard F_0 as a divisor on a blowing up of $\mathbf{P}^2$ at 6 points. By [2], $\mathcal{S}(H, L) = 0$ for all numerically effective divisors H on a blowing up of $\mathbf{P}^2$ at 6 general points $p_1, \ldots, p_6$ unless H is $5L - 2E_1 - \cdots - 2E_6$ or a multiple of $3L - 2E_{i_1} - E_{i_2} - \cdots - E_{i_6}$. But F_0 cannot be any of these since they are perpendicular to exceptional curves meeting L at least twice. Thus $\mathcal{S}(F_0, L) = 0$ if $F_0 \cdot E_i = 0$ for some i. Otherwise, $F_0 \cdot E > 0$ for every exceptional curve E with $E \cdot L \neq 1$, and either F_0 is ample (whence $\mathcal{S}(F_0, L) = 0$ by Theorem IV.1) or $F_0 \cdot E = 0$ for some exceptional curve E with $E \cdot L = 1$ (whence $\mathcal{S}(F_0, L) = 0$ by Lemma IV.3). Either way, we have $F_0 \cdot Q_i > 0$ for all i and $\dim \mathcal{S}(F, L) = (Q_1 + \cdots + Q_t) \cdot L - 2t$. Since $F_0 \cdot Q_i > 0$ but $F_i \cdot Q_i = 0$, it follows inductively that $Q_i \cdot Q_j = 0$ for all $i \neq j$. It now follows easily that $t_F = (Q_1 + \cdots + Q_t) \cdot L - 2t$; since $\lambda_F \leq \dim \mathcal{S}(F, L)$ is always true, we have $\dim \mathcal{S}(F, L) = t_F = \max(t_F, \lambda_F)$, as claimed.

Now consider the case that $|F_0|$ has a fixed component. By Lemma IV.4, $[F_0 + Q_1]$ is $m[H]$, where $H^2 = 0$ and $|H|$ is a pencil. Thus, up to indexation, $[H]$ is among $[G_{11}] = [L - E_1]$, $[G_{12}] = [2L - E_1 - \cdots - E_4]$, $[G_{13}] = [3L - 2E_1 - E_2 - \cdots - E_6]$, $[G_{14}] = [4L - 2E_1 - 2E_2 - 2E_3 - 1E_4 - 1E_5 - 1E_6 - 1E_7]$, or $[G_{15}] = [5L - 2E_1 - 2E_2 - 2E_3 - 2E_4 - 2E_5 - 2E_6 - 1E_7]$, but $Q_1 \cdot L \geq 2$ rules out $[L - E_1]$. If $t > 1$, then we have $(F_0 + Q_1) \cdot Q_2 = 1$, hence $m = 1$. Thus $[F]$ is either $m[H]$ or $[H + Q_2 + \cdots + Q_t]$. In the former case $\mathcal{R}(F, L) = 0$ by Lemma IV.4, hence $\dim \mathcal{S}(F, L) = \lambda_F$, and we explicitly check that $t_F \leq \lambda_F$ unless $m = 1$ and $[H]$ is either $[4L - 2E_1 - 2E_2 - 2E_3 - 1E_4 - 1E_5 - 1E_6 - 1E_7]$ or $[5L - 2E_1 - 2E_2 - 2E_3 - 2E_4 - 2E_5 - 2E_6 - 1E_7]$.

This verifies the statement of Theorem I.6.1 unless $[F]$ is of the form $[H + Q_2 + \cdots + Q_t]$, as above, where $[H]$ is one of $[2L - E_1 - \cdots - E_4]$, $[3L - 2E_1 - E_2 - \cdots - E_6]$, $[4L - 2E_1 - 2E_2 - 2E_3 - 1E_4 - 1E_5 - 1E_6 - 1E_7]$, or $[5L - 2E_1 - 2E_2 - 2E_3 - 2E_4 - 2E_5 - 2E_6 - 1E_7]$. We consider each possibility for $[H]$ in turn. In each case we have $0 = \mathcal{R}(H, L) = \mathcal{R}(F, L)$, so $\dim \mathcal{S}(F, L) = \lambda_F$, and it is enough to check that $t_F \leq \dim \mathcal{S}(H, L) + \rho$, where $\rho = (Q_2 + \cdots + Q_t) \cdot L - 2t + 2$, when F is not one of the stated exceptions.

First consider $[H] = [2L - E_1 - \cdots - E_4]$. Since any exceptional E with $E \cdot L = 3$ has $E \cdot H > 0$, we can have $E \cdot F = 0$ only if E is among the Q_i. Thus $t_F \leq \rho$, settling this case.

Now let $[H] = [3L - 2E_1 - E_2 - \cdots - E_6]$. Any exceptional E with $E \cdot L = 3$ and $E \cdot F = 0$ must be among the Q_i, or must have $E \cdot H = 0$ (and hence $[E] = [3L - 2E_1 - E_2 \cdots - E_7]$).

Thus $t_F \leq \rho + 1$, but $\dim \mathcal{S}(H, L) = 1$ in this case, so this case is also settled.

Now suppose $[H] = [4L - 2E_1 - 2E_2 - 2E_3 - 1E_4 - 1E_5 - 1E_6 - 1E_7]$. Let us say that a divisor B is *cubic* if $B \cdot L = 3$ and *conic* if $B \cdot L = 2$. Now argue as in the preceding paragraph. This time there are exactly three cubic exceptionals perpendicular to H (in fact, their classes $[C_i]$ are exactly $-K_X - [E_i]$, $1 \leq i \leq 3$), so we see $t_F \leq \rho + 3$. Since now $\dim \mathcal{S}(H, L) = 2$, this case is settled unless $t_F = \rho + 3$ and hence the classes of C_1, C_2, C_3 and of all of the cubics among Q_i, $i \geq 2$, are distinct and perpendicular to F (else we certainly would have $t_F < \rho + 3$), in which case these t_F cubics are also perpendicular to each conic Q_i, $i \geq 2$. But the class of a conic exceptional curve perpendicular to C_1, C_2, and C_3 must be of the form $[2L - E_1 - E_2 - E_3 - E_{i_1} - E_{i_2}]$ and is therefore perpendicular to H, and hence the conic Q_j with least $j \geq 2$ must meet one of the cubics occurring among the Q_i, $i \geq 2$. To avoid this contradiction we conclude there is no conic Q_i, $i \geq 2$. Thus $[F]$ must be $[H]$ plus any of the four classes of cubic exceptionals not perpendicular to H, giving only the exceptions $[H]$, $[H + C_4]$, $[H + C_4 + C_5]$, $[H + C_4 + C_5 + C_6]$, $[H + C_4 + C_5 + C_6 + C_7]$ listed in the statement of Theorem I.6.1.

Finally, we have $[H] = [5L - 2E_1 - 2E_2 - 2E_3 - 2E_4 - 2E_5 - 2E_6 - 1E_7]$. Here we have $\lambda_F = \dim \mathcal{S}(F, L) = \rho + 3$, so for $t_F > \lambda_F$ we need $t_F \geq 4$. Let us look at all numerically effective classes perpendicular to at least four (say to $[C_4], \ldots, [C_7]$) cubic exceptionals. From the generators $[G_i]$ of the numerically effective cone given in Section III it is easy to verify that any numerically effective class perpendicular to each of $C_4, \ldots, C_7$ is a nonnegative sum of the classes of $A = 7L - 2E_1 - 2E_2 - 2E_3 - 3E_4 - 3E_5 - 3E_6 - 3E_7$, $D_i = 5L - 2E_1 - 2E_2 - 2E_3 - 2E_4 - 2E_5 - 2E_6 - 2E_7 + E_i$, $1 \leq i \leq 3$, and $B = 8L - 3E_1 - 3E_2 - 3E_3 - 3E_4 - 3E_5 - 3E_6 - 3E_7$. We will show that any sum F of these 5 divisors with $t_F > \lambda_F$ and $\mathcal{R}(F, L) = 0$ must be among the list of exceptions given in the statement of Theorem I.6.1.

First note that the class of each of these five divisors is on the list of exceptions. Now let F be A plus any one of A, D_1, D_2, D_3 and B. In each case we check that $h^0\bigl(X, \mathcal{O}_X(F - C_4 - C_5 - C_6 - C_7)\bigr) > 0$ and $h^1\bigl(X, \mathcal{O}_X(F - C_4 - C_5 - C_6 - C_7)\bigr) = 0$. Thus the same is true for any sum F of two or more of the divisors A, D_1, D_2, D_3 and B such that at least one summand is A. Applying Proposition II.1 to $0 \to \mathcal{O}_X(F - Y) \to \mathcal{O}_X(F) \to \mathcal{O}_Y \to 0$, where Y is the disjoint union of the $t_F = 4$ cubic exceptional curves perpendicular to F (so $[Y] = [C_4] + \cdots + [C_7]$), we conclude that $\dim \mathcal{S}(F, L) = \dim \mathcal{S}(F - Y, L) + t_F$. Thus, whenever we have $\mathcal{R}(F, L) = 0$ for such an F, we also have $\lambda_F = \dim \mathcal{S}(F, L) \geq t_F$.

Now consider the case that F is a sum with three or more summands taken from D_1, D_2, D_3 and B. In every such case of 3 summands (and hence also for more than 3 summands) except for pure multiples of some D_i (which were treated above), $[F]$ is, as in the preceding paragraph, the class of the sum of the cubic exceptionals perpendicular to F plus an effective regular divisor, and therefore as above $\mathcal{R}(F, L) = 0$ implies $t_F \leq \lambda_F$.

We are left to consider the case that F is a sum of any two of D_1, D_2, D_3 and B except pure multiples of some D_i: if $F = D_j + D_i$, $j \neq i$, then $t_F = 5$ and $\lambda_F = 4$; if $F = B + D_i$, then $t_F = 6$ and $\lambda_F = 5$; and if $F = 2B$, then $t_F = 7$ and $\lambda_F = 6$. But in each of these cases, F is one of the exceptions explicitly given in the statement of the theorem. ■

Although Theorem I.6.1 is well-suited for computational applications; our final result is more conceptually satisfying.

Corollary IV.5 *Let F be a numerically effective divisor on X, with X as in Theorem I.6.1. Let $D = C_{i_1} + \cdots + C_{i_{t_F}}$ be the sum of the cubic exceptional curves $C_{i_j} \in \{C_1, \ldots, C_7\}$ perpendicular to F. If μ_F fails to have maximal rank, then $\dim \mathcal{S}(F, L) = t_F$. Moreover, μ_F fails to have maximal rank if and only if: $t_F > 0$, $F-D$ is numerically effective, and $\lambda'_{F-D} < 0$.*

Proof If μ_F fails to have maximal rank, then $\dim \mathcal{S}(F, L) = t_F$ follows by Theorem I.6.1. We now consider the second claim.

If $t_F = 0$, then $\dim \mathcal{S}(F, L) = \lambda_F$ by Theorem I.6.1, so μ_F has maximal rank.

Consider the case that $t_F > 0$ but $F - D$ is not numerically effective. Since $F - D$ is not numerically effective, successively subtracting $C_{i_1}, C_{i_2}, \ldots$ from F, we eventually obtain by Lemma II.4 (as in the proof of Theorem I.6.1) a divisor F_0 whose class is the class of an effective but not numerically effective divisor. Now, by the proof of Theorem I.6.1, $S(F, L) = \lambda_F$ and thus μ_F has maximal rank.

Finally, say $t_F > 0$ and $F-D$ is numerically effective. Then $t_{F-D} = 0$, so as we saw above, μ_{F-D} has maximal rank; in particular, μ_{F-D} has a nontrivial kernel if and only if $\lambda'_{F-D} < 0$. Now apply Proposition II.1 to the exact sequence $0 \to \mathcal{O}_X(F-D) \to \mathcal{O}_X(F) \to \mathcal{D} \to 0$. Since $\mathcal{R}\big(\mathcal{O}_D, \mathcal{O}_X(L)\big) = 0$, we see that $\dim \mathcal{R}(F-D, L) = \dim \mathcal{R}(F, L)$ and $\dim \mathcal{S}(F, L) = \dim \mathcal{S}(F-D, L) + \dim \mathcal{S}\big(\mathcal{O}_D, \mathcal{O}_X(L)\big) = \dim \mathcal{S}(F-D, L) + t_F \geq t_F > 0$. Thus μ_F fails to have maximal rank if and only if μ_{F-D} fails to be injective, which we noted above holds if and only if $\lambda'_{F-D} < 0$. ■

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