DIMENSIONALITY OF STABLE AND UNSTABLE DIRECTIONS IN THE GRAVITATIONAL N–BODY PROBLEM

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Abstract. The gravitational n–body problem is chaotic. Phase trajectories that start very near each other separate rapidly. The rate looks exponential over long times. At any instant, trajectories separated in certain directions move apart rapidly (unstable directions), while those separated in other directions stay about the same (stable directions). Unstable directions lie along eigenvectors that correspond to positive eigenvalues of the matrix of force gradients. The number of positive eigenvalues of that matrix gives the dimensionality of stable regions. This number has been studied numerically in a series of 100–body integrations. It continues to change as long as the integration continues because the matrix changes extremely rapidly. On average, there are about $1.2n$ unstable directions out of $3n$. Issues of dimensionality arise when the tools of ergodic studies are brought to bear on the problem of trajectory separation. A method of estimating the rate of trajectory separation based on matrix descriptions is presented in this note. Severe approximations are required.

Key words: Chaos, phase space, Lagrangian displacements

Extreme sensitivity to initial conditions, one of the accepted signatures of chaotic systems, was demonstrated for the gravitational n–body problem (Miller, 1964) at about the same time as Lorenz's famous paper (Lorenz, 1963) in meteorology. The effect is physical, an example of Krylov's instability (Krylov, 1979). It was demonstrated analytically (Miller, 1966), and the interaction between trajectory separation and integrals of the motion was explored numerically by integrating the matrix equations Eq. (3) explicitly as an n–body system developed (Miller, 1971). A plea was also made to the numerical analysis community for help in trying to understand how valid physical conclusions might be inferred from numerical experiments in view of the underlying chaotic properties of phase trajectories (Miller, 1974). Some help may be on the way. Quinlan and Tremaine (1992) constructed shadowing orbits in a restricted form of the gravitational n–body problem. Their results suggest that real phase trajectories are so wild that a few can be found which remain near almost any "reasonable" computed trajectory for surprisingly long times.

Several attempts have been made to determine growth rates for trajectory separation numerically ((Kandrup et al., 1994); (Goodman et al., 1993), with references to earlier work in each of these papers) and analytically (Gurzadyan and Savvidy, 1986), (Kandrup, 1990). The numerical results lead to a general consensus that the growth rate is a few per crossing time (e.g., some multiple of $(G\rho)^{1/2}$). This same dependence had been found analytically, based on rather crude arguments in (Miller, 1966).

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1. Formulation

Let the pair of $3n-$vectors $(\mathbf{p}(1), \mathbf{q}(1))$ be the instantaneous phase coordinates of an $n-$particle system in the $6n-$dimensional phase space ($\Gamma-$space), and let $(\mathbf{p}(2), \mathbf{q}(2))$ represent a second system in the same space. Presume the two phase points to be "near" each other and let $\delta\mathbf{p} = \mathbf{p}(2) - \mathbf{p}(1)$ and $\delta\mathbf{q} = \mathbf{q}(2) - \mathbf{q}(1)$ represent the difference-vectors in the same space. The motion of either system is determined by the usual Hamiltonian equations, governed by a Hamiltonian $\mathcal{H}(\mathbf{p}, \mathbf{q})$.

Equations of motion for $\delta\mathbf{p}$ and $\delta\mathbf{q}$ can be constructed by forming a Taylor series expansion for $\mathcal{H}(\mathbf{p}(2), \mathbf{q}(2))$ in terms of $\mathcal{H}(\mathbf{p}(1), \mathbf{q}(1))$ and powers of $\delta\mathbf{p}$ and $\delta\mathbf{q}$. The linearized form of these equations is

$$\begin{aligned} \frac{d}{dt}(\delta\mathbf{q}) &= \frac{\partial^2\mathcal{H}}{\partial\mathbf{q}\partial\mathbf{p}}\delta\mathbf{q} + \frac{\partial^2\mathcal{H}}{\partial\mathbf{p}^2}\delta\mathbf{p} \qquad \text{and} \\ \frac{d}{dt}(\delta\mathbf{p}) &= -\frac{\partial^2\mathcal{H}}{\partial\mathbf{q}^2}\delta\mathbf{q} - \frac{\partial^2\mathcal{H}}{\partial\mathbf{p}\partial\mathbf{q}}\delta\mathbf{p}, \end{aligned} \tag{1}$$

where the derivatives are to be evaluated at $(\mathbf{p}(1), \mathbf{q}(1))$. Some care is required to remain in a regime in which the linear terms of the expansion suffice.

Write the $3n$ $\delta\mathbf{p}$'s and the $3n$ $\delta\mathbf{q}$'s as elements of a $6n-$vector:

$$\xi = \begin{pmatrix} \delta\mathbf{q} \\ \delta\mathbf{p} \end{pmatrix}. \tag{2}$$

Then the equations (1) can be written in matrix form,

$$\frac{d\xi}{dt} = \mathcal{M}\xi, \tag{3}$$

where the elements of $\mathcal{M}$ can be read off from Eq. (1):

$$\mathcal{M} = \begin{pmatrix} \dfrac{\partial^2\mathcal{H}}{\partial\mathbf{q}\partial\mathbf{p}} & \dfrac{\partial^2\mathcal{H}}{\partial\mathbf{p}^2} \\ -\dfrac{\partial^2\mathcal{H}}{\partial\mathbf{q}^2} & -\dfrac{\partial^2\mathcal{H}}{\partial\mathbf{p}\partial\mathbf{q}} \end{pmatrix}. \tag{4}$$

This matrix has dimension $6n \times 6n$, and it breaks into four $3n \times 3n$ blocks.

The matrix equation (3) is to be thought of in a Lagrangian sense. The matrix $\mathcal{M}$ is to be evaluated at the phase point currently occupied by the unperturbed system (system 1), while ξ gives the displacement of the second system relative to the first. The matrix $\mathcal{M}$ depends implicitly on the time through the motion of the phase point of the unperturbed system. It changes pretty drastically and pretty rapidly – on the time scale of nearest particles moving past each other.

In Cartesian coordinates, all the elements in the blocks that contain mixed second derivatives vanish, leaving those two blocks completely filled with zeroes.

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The off-diagonal elements are another story: the upper right block is diagonal. With equal masses, it is just $1/m$ times the $3n \times 3n$ identity. Unequal masses present no problem of principle, but they mess up the notation, so they'll not be worked out in this note.

The remaining $3n \times 3n$ block, that in the lower left, contains the gradient of forces. (Minus the) first derivative of the Hamiltonian gives the force acting on each particle, so (minus the) second derivative gives the gradient of the forces, $\mathcal{G} =$ (grad **F**). Its structure is worked out in detail in §1.2, but for now we need only note that all its elements are real and that it is symmetrical. A small displacement in which every particle in the system is moved infinitesimally adds an increment to the force acting on each particle by an amount given by (grad **F**).

1.1. Reduce $\mathcal{M}$ to Diagonal Form

Since the $\mathcal{G}$ is real symmetric, it can be brought to diagonal form by a $3n \times 3n$ orthogonal matrix, $\mathcal{O}$:

$$\mathcal{O}^{\mathrm{T}} \mathcal{G} \mathcal{O} = \mathrm{Diag}$$

(superscript T denotes transpose: since $\mathcal{O}$ is orthogonal, its transpose is its inverse). However, the identity matrix remains an identity matrix under the same transformation, and the scalar multiple, $1/m$, does no harm. Thus we can apply $\mathcal{O}$ to *both* nonzero blocks, bringing both to diagonal form. In the language of $3n \times 3n$ blocks, this reads

$$\begin{pmatrix} \mathcal{O} & 0 \\ 0 & \mathcal{O} \end{pmatrix}^{\mathrm{T}} \begin{pmatrix} 0 & \frac{1}{m}\mathcal{I} \\ \mathcal{G} & 0 \end{pmatrix} \begin{pmatrix} \mathcal{O} & 0 \\ 0 & \mathcal{O} \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{m}\mathcal{I} \\ \mathrm{Diag} & 0 \end{pmatrix}.$$

The resulting matrix, which is diagonal in the upper right and lower left $3n \times 3n$ blocks, with null blocks on the diagonal, can be reduced to a block diagonal form with $3n$ blocks, each of dimension 2×2, by means of row and column interchanges. Each of the 2×2 blocks is of the form:

$$\begin{pmatrix} 0 & 1/m \\ \lambda_K & 0 \end{pmatrix}. \tag{5}$$

Here, λ_K is one of the eigenvalues of $\mathcal{G}$. Those 2×2 matrices can each be diagonalized to give

$$\Lambda_K = \pm\sqrt{\lambda_K/m}, \tag{6}$$

on the diagonal. A pair of eigenvalues of the full $6n \times 6n$ matrix corresponds to each λ_K. There are $6n$ in all – the required number.

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Positive λ's give real square roots, while negatives give pure imaginary square roots. The real roots lead to unstable solutions, while the pure imaginary roots give stable solutions. Thus the topic of determining the dimensionality of the stable and unstable domains reduces to counting the number of positive (or negative) eigenvalues of the $3n \times 3n$ matrix, $\mathcal{G} = (\text{grad } \mathbf{F})$. The matrix, $\mathcal{M}$, is not self–adjoint.

The original $6n \times 6n$ matrix has trace 0 (the Liouville theorem!), and so the diagonal form must as well. It's reassuring that it does.

Lagrangian displacements have allowed this problem to be reduced to an exercise in diagonalizing two matrices simultaneously. All the standard texts on classical mechanics show the procedure in their sections on small oscillations.

1.2. The Matrix of Force Gradients

Eigenvalues of $\mathcal{M}$ can be worked out once the eigenvalues of $\mathcal{G} = (\text{grad } \mathbf{F})$ are in hand. The matrix $\mathcal{G}$ has the explicit form

$$\mathcal{G}_{ij}^{(\alpha\beta)} = \frac{1}{r_{\alpha\beta}^3}\left[\delta_{ij} - 3\frac{\left(x_i^{(\beta)} - x_i^{(\alpha)}\right)\left(x_j^{(\beta)} - x_j^{(\alpha)}\right)}{r_{\alpha\beta}^2}\right], \tag{7}$$

where

$$r_{\alpha\beta}^2 = \left(x_k^{(\beta)} - x_k^{(\alpha)}\right)\left(x_k^{(\beta)} - x_k^{(\alpha)}\right), \tag{8}$$

$\alpha, \beta = 1, 2, 3, \cdots, n$ are particle indices, and $i, j = 1, 2, 3$ are coordinate indices. Summation over repeated i, j, k pairs is implied, but not over repeated (α, β) pairs. An i, α pair runs $1, 2, 3, \cdots 3n$; the matrix is best thought of as having i, α as its first index and j, β as the second index. These expressions apply for $\alpha \neq \beta$; the α-α element is simply the sum of the elements with a minus sign

$$\begin{aligned}\mathcal{G}_{ij}^{(\alpha\alpha)} &= -{\sum_{\beta\neq\alpha}}' \mathcal{G}_{ij}^{(\alpha\beta)} \\ &= -{\sum_{\beta\neq\alpha}}' \frac{1}{r_{\alpha\beta}^3}\left[\delta_{ij} - 3\frac{\left(x_i^{(\beta)} - x_i^{(\alpha)}\right)\left(x_j^{(\beta)} - x_j^{(\alpha)}\right)}{r_{\alpha\beta}^2}\right],\end{aligned} \tag{9}$$

so the row (or column) sums are zero. The matrix is symmetric under coordinate index (i, j) and under particle number (α, β) interchanges separately. The quantities $x_i^{(\alpha)}$ are the coordinates of the individual particles. All particles have been taken to have the same mass; generalization to different masses does not change the character of the problem.

The (i, α) notation facilitates decomposition into an $n \times n$ matrix of 3×3 blocks; each 3×3 block runs through $i, j = 1, 2, 3$, and the $n \times n$ block matrix has a block entry for each particle pair (with special entries on the block diagonal to make the

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row– or column-sums be zero). Each of the 3×3 blocks is real symmetric and has trace zero. The $n \times n$ block matrix is also symmetric, so the full $3n \times 3n$ matrix is real symmetric and has real eigenvalues. The trace of the $3n \times 3n$ is zero, since that of each of the 3×3 submatrices is zero. Thus the sum of the eigenvalues is zero. There are nonzero eigenvalues, so some must be positive and some negative.

1.3. EIGENVALUES OF $\mathcal{G}$

All eigenvalues of $\mathcal{G}$ are real, since the matrix is real symmetric. It has three zero eigenvalues, corresponding to parallel displacements of all particles:

$$x_i^{(\alpha)} \longrightarrow x_i^{(\alpha)} + X_i.$$

Nothing more is known analytically beyond this and the fact that all eigenvalues must sum to zero.

The eigenvalues have been studied experimentally. Some matrices were generated in which coordinates $x_i^{(\alpha)}$ were chosen randomly within the unit cube, the force gradient matrix computed according to Eq. (7) and Eq. (9), and the eigenvalues found. Cases were tried for various values of n. In all cases, the expected three zero eigenvalues were smaller than the nonzero values by a factor near 10^{10} (reasonable for the diagonalization routine used), and were easily distinguishable. No other zero eigenvalues were found.

Eigenvalues have also been studied experimentally by forming the matrix from configurations generated as snapshots of a 100–body systems advanced by direct integration. Properties of these sets of eigenvalues are essentially indistinguishable from those generated by the random number loads except that closest pairs tended to be closer in the actual integrations. Numerical results quoted in the remainder of this note are based on these 100–body integrations.

The number of positive eigenvalues changes as an n–body system develops. There is no clear trend toward more (or toward fewer) positive eigenvalues during an integration.

No repeated eigenvalues were found, apart from the three zeroes always present.

The number of positive eigenvalues averages about $1.2n$ for systems containing 10 particles or more.

The most positive eigenvalue is well approximated by $4/r_{12}^3$ and the most negative by $-2/r_{12}^3$, where r_{12} is the separation of the closest pair of particles.

2. Trajectory Separation

Configuration and momentum components remain distinct in the formulation so far presented. Further discussion is facilitated by eliminating the momenta in favor of velocities to get a coupled set of second-order ODE's:

$$\frac{d^2}{dt^2}(\delta \mathbf{q}) = \frac{1}{m}\mathcal{G}(\delta \mathbf{q}). \tag{10}$$

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These equations decouple if $\mathcal{G}$ is brought to diagonal form. This decoupling holds only instantaneously, since the matrix $\mathcal{G}$ changes very rapidly with time. Significant changes occur in the time it takes the closest pair of particles to pass each other. Eq. (10) is also valid only for the tiny displacements allowed in writing the equations in matrix form, Eq. (3). Nonetheless, the picture presents a useful way of looking at the physics of trajectory separation. The rest of this section refers to "snapshots" that last only for these very short times.

Eigenvectors of $\mathcal{G}$ represent displacements of all particles in the system such that the change in the force acting on each particle lies along the vector by which that particle was displaced. The particle is accelerated along its vector displacement if the corresponding eigenvalue is positive, decelerated if it is negative.

2.1. "Normal Coordinates"

Eigenvectors of $\mathcal{G}$ are orthogonal and span the configuration space. They can serve as a basis in which to express $\delta\mathbf{q}$ within the little patch on which we are concentrating. They are "normal coordinates," and each corresponds to a "normal mode." There are $3n$ of them. They would be normal modes in the usual sense if the matrix $\mathcal{G}$ did not vary with time. Two degrees of freedom are associated with each "mode," accounting for the required $6n$ degrees of freedom. Most are stable, but about 40% of them are unstable, and the three with zero eigenvalues are neutral.

Apart from the three zero eigenvalues that correspond to rigid displacement of all the particles together, the only degeneracies arise when $\mathcal{G}$ is evaluated at a point that lies on one of several special hypersurfaces of reduced dimension. These hypersurfaces are sets of measure zero. Degeneracies present no problem in practice. The three zero eigenvalues reflect the six first integrals of the centroid motion, which are conserved in this picture.

2.2. The Rate of Trajectory Separation

Much of the interest in trajectory separation is to estimate the rate at which trajectories separate. Special interest attaches to its dependence on the number of particles, with the goal of finding how it might act with very large numbers of particles. Any attempt to address rates through the formulation presented here involves lots of approximations. The linerization introduced at Eq. (1) is the only approximation used so far in this note.

Estimates for $e-$folding rates of trajectory separation entail averages of $\mathcal{G}$ over realistic trajectories as the unperturbed system explores the phase space. The principal problem is to find some suitable average. The present discussion is included to illustrate the kinds of approximations involved. The description provides useful insights into the physical processes at work in trajectory separation.

Several steps are involved. A new assumption is introduced at each step. One way to go about this is sketched here.

First, the separation vector $\delta\mathbf{q}$ is referred to the basis instantaneously provided by eigenvectors of the current matrix $\mathcal{G}$. While $\delta\mathbf{q}$ itself may be fairly stable in direction, changes in $\mathcal{G}$ make it appear to vary. Eigenvectors of $\mathcal{G}$ swing around without any special relation to the direction of $\delta\mathbf{q}$, so $\delta\mathbf{q}$ appears to take random directions relative to the changing basis. Thus the amplitude of each degree of freedom is approximated as $\pm|\xi|/\sqrt{6n}$. The trajectory separation has been analyzed into "normal modes" with amplitudes split between the two degrees of freedom within each "normal mode."

Next, growths in ξ can be approximated by a random walk with unequal steps. Only unstable "modes," those with positive eigenvalues of $\mathcal{G}$, contribute. Each unstable "mode" grows like $\exp(\Lambda_K t)$ (Eq. (6)). Both momentum and configuration parts in the K^{th} "normal mode" grow together. The stable and neutral "normal modes" don't grow at all. Since Λ_K's are as often negative as positive, the expected growth is zero, a consequence of microscopic reversibility. But the expected square is nonzero. When growths over all unstable "normal modes" are added together, the mean square growth during a short time Δt is

$$\left(\frac{|\Delta\xi|^2}{|\xi|^2}\right) = \left[\ln\left(\frac{|\xi_{i+1}|}{|\xi_i|}\right)\right]^2 \approx \frac{1}{6n}\sum_K [\Lambda_K \Delta t]^2 = \frac{2\,(\Delta t)^2}{6nm}\sum\nolimits^{+}\lambda_K. \qquad (11)$$

The second form indicates the change between successive "snapshots." The last form, through Eq. (6), involves the sum over positive eigenvalues of $\mathcal{G}$. The factor 2 appears because each λ_K appears twice, once for each sign in Eq. (6). Eq. (11) gives the variance, σ^2, of $\Delta \ln|\xi|$, since the mean is zero. Each "normal mode" acts like the free motion of an harmonic oscillator. Energy is conserved because there is no forcing term. Estimates according to Eq. (11) areexamples of the "spectrum of stretching numbers" of Contopoulos and Voglis (1996).

Third, the random walk is asymptotically Gaussian if enough steps are involved. Steps take the spatial directions of unstable "normal modes" within one snapshot. Since the logarithm tends to a normal distribution, mean growth over this interval is the mean from a log–normal distribution, which is $e^{\sigma^2/2}$. It is always greater than unity.

Fourth, the quadratic dependence on time interval (the $(\Delta t)^2$ in Eq. (11)) arises because the growth is treated as coherent over the interval Δt. After a coherence time, this must be terminated, to be replaced by a new randomization of the separation vector. The time for the nearest particles to orbit by a radian might be used for a coherence time. This gives $(\Delta t)^2 \approx d^3/(2Gm)$ where d is the separation of the nearest pair of particles. We then have $s = T_{\text{cr}}/\Delta t$ independent steps, each growing by $e^{\sigma^2/2}$, leading to a total estimated growth per crossing time of $e^{(s\sigma^2)/2}$.

Happily, growth rates estimated in this manner from our 100–body integrations are around 1.5 to 2 per crossing time, which agrees reasonably well with values found by Kandrup *et al*. (1994) and earlier papers referenced therein. The greatest positive eigenvalue contributed about 40% of the sum of positive eigenvalues, and

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it is very nearly $4/r_{12}^3$ for the nearest pair. This implies that the contributions of the "mean field" and of the nearest pair to the growth are nearly the same, as observed by Kandrup *et al*. (1992).

3. Discussion

1. The number of positive eigenvalues of $\mathcal{G}$ changes as particles move about. Changing numbers of positive and negative eigenvalues have been known for some time (Miller, 1972). They indicate a changing dimensionality of the stable and unstable regions. There are always three zero eigenvalues, which correspond to rigid displacements of the entire system in any of three orthogonal directions.
2. The number of positive eigenvalues is about $1.2n$, where n is the number of particles. This ratio is attained for $n > 10$.
3. There are no duplicate eigenvalues, save for the three zero cases. Occasional duplications require special configurations that reduce them to a set of measure zero.
4. The formulation presented here takes the discrete particle nature of $n-$body systems into account explicitly. Many-body interactions are included up to the full number of particles. Systems with large numbers of particles show exponential trajectory separation, up to and including the infinite numbers implied in some analytic models.
5. The sum of all eigenvalues is zero (trace of $\mathcal{G}$ is zero). This means that there are some positive, as well as some negative, eigenvalues for any $n-$body system.
6. The greatest positive eigenvalue is well approximated by $4/d^3$, where d is the separation of the nearest pair of particles anywhere within the configuration. It contributes about 40% of the sum of all positive eigenvalues. The most negative eigenvalue is near $-2/d^3$.
7. This formulation shows why separations projected onto the configuration space and onto the velocity space each grow at the same rate, as noted by Kandrup and Smith (1991). They're tightly coupled, since the full system is actually described by the $6n \times 6n-$matrix $\mathcal{M}$.
8. There is nothing intrinsic in the arguments presented here to indicate that basic $e-$folding rates should be a few per crossing time. Estimates for $e-$folding rates of trajectory separation entail averages of $\mathcal{G}$ over realistic trajectories as the unperturbed system explores the phase space.
9. An heuristic argument permits an estimate of the $e-$folding rate for trajectory separation. It yields estimates of a few per crossing time, like those of (Kandrup et al., 1994) and earlier papers cited there. Severe approximations are required, but it provides a different way of looking at the problem of trajectory separation.
10. The changing numbers of positive (and of negative) eigenvalues along an actual trajectory demonstrates that the gravitational $n-$body system is *not* an "Anosov system" (such systems are sometimes called "C-systems", (Arnold and Avez, 1968)).

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