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CHARACTERIZATIONS OF BERGER SPHERES FROM THE VIEWPOINT OF SUBMANIFOLD THEORY

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Abstract. In this paper, Berger spheres are regarded as geodesic spheres with sufficiently big radii in a complex projective space. We characterize such real hypersurfaces by investigating their geodesics and contact structures from the viewpoint of submanifold theory.

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1. Introduction. Klingenberg ([4]) proved the following: Let M be an even dimensional compact simply connected Riemannian manifold having the sectional curvature K with $0 < K \leqq L$ on M, where L is a constant. Then the length ℓ of every closed geodesic on M satisfies $\ell \geqq 2\pi/\sqrt{L}$.

Berger gave examples of metrics on S^3 for which this inequality does not hold. This 3-sphere is called a *Berger sphere* with a Riemannian metric from a one-parameter family, which can be obtained from the standard metric by shrinking along fibers of a Hopf fibration. Chavel constructed similar metrics on higher odd-dimensional spheres.

Weinstein ([10]) gave a description of these Berger and Chavel examples as geodesic hyperspheres $G(r)$ $(0 < r < \pi/\sqrt{c})$ with $\tan^2(\sqrt{c}\,r/2) > 2$ in a complex projective space $\mathbb{C}P^n(c)$, $n \geqq 2$ of constant holomorphic sectional curvature $c(> 0)$. Indeed, let $G(r)$ be a $(2n-1)$-dimensional geodesic sphere of radius r $(0 < r < \pi/\sqrt{c})$ with $\tan^2(\sqrt{c}\,r/2) > 2$ in $\mathbb{C}P^n(c)$. Then in this case there exists a closed geodesic on $G(r)$ whose length is shorter than $2\pi/\sqrt{L}$, where L is the maximal sectional curvature of $G(r)$. In general, the sectional curvature K of every geodesic sphere $G(r)$ of radius r $(0 < r < \pi/\sqrt{c})$ satisfies sharp inequalities $0 < (c/4)\cot^2(\sqrt{c}\,r/2) \leqq K \leqq c + (c/4)\cot^2(\sqrt{c}\,r/2)(= L)$ at each point (see Section 4).

In this paper, geodesic spheres of radius r $(0 < r < \pi/\sqrt{c})$ with $\tan^2(\sqrt{c}\,r/2) > 2$ in $\mathbb{C}P^n(c)$, $n \geqq 2$ are called *Berger spheres*. It is natural to characterize Berger spheres as real hypersurfaces isometrically immersed into $\mathbb{C}P^n(c)$.

The purpose of this paper is to prove the following two theorems.

THEOREM 1.1. *Let M^{2n-1} be a real hypersurface of $\mathbb{C}P^n(c)$, $n \geqq 2$ through an isometric immersion. Then M is locally congruent to a Berger sphere, namely a geodesic sphere $G(r)$ of radius r with $\tan^2(\sqrt{c}\,r/2) > 2$, with respect to the full isometry group $\mathrm{SU}(n+1)$ of the ambient space $\mathbb{C}P^n(c)$ if and only if at each point p of M there exists an orthonormal basis $v_1, \ldots, v_{2n-2}, \xi_p$ of T_pM such that all geodesics $\gamma_i = \gamma_i(s)$ $(1 \leqq i \leqq 2n-2)$ with initial condition that $\gamma_i(0) = p$ and $\dot{\gamma}_i(0) = v_i$ are mapped to circles of the same positive curvature $k(p)$ with $k(p) < \sqrt{c}/(2\sqrt{2})$ in the ambient space $\mathbb{C}P^n(c)$, where ξ_p is the characteristic vector of M at $p \in M$. In this case, the function $k = k(p)$ on M is automatically constant with $k = (\sqrt{c}/2)\cot(\sqrt{c}\,r/2)$.*

THEOREM 1.2. *Let M^{2n-1} be a real hypersurface of $\mathbb{C}P^n(c)$, $n \geqq 2$ through an isometric immersion. Then M is locally congruent to a Berger sphere if and only if M satisfies the following two conditions.*

(1) *There exists a positive constant k with $k < \sqrt{c}/(2\sqrt{2})$ such that the exterior derivative $d\eta$ of the contact form η on M satisfies either $d\eta(X, Y) = kg(\phi X, Y)$ for all $X, Y \in TM$ or $d\eta(X, Y) = -kg(\phi X, Y)$ for all $X, Y \in TM$, where g and ϕ are the Riemannian metric and the structure tensor on M, respectively.*
(2) *There exists a point x of M satisfying that every sectional curvature of M at x is positive.*

We here recall the definition of $d\eta$ on a real hypersurface M: $d\eta$ is given by $d\eta(X, Y) = (1/2)\{X(\eta(Y)) - Y(\eta(X)) - \eta([X, Y])\}$ for all $X, Y \in TM$. For further comments on Theorems 1.1 and 1.2, see Section 5.

We note for comparison the recent paper of Li, Vrancken, and Wang ([**5**]), which gives a characterization of three-dimensional Bereger spheres as Lagrangian submanifolds of $\mathbb{C}P^3$. They showed the following (for details, see Theorem 1.2 in [**5**]): Let ϕ be a Lagrangian isometric immersion (an open part of) one of the homogeneous 3-manifolds M^3 into a complex space form $M_3(c)(= \mathbb{C}P^3(c), \mathbb{C}H^3(c)$ or $\mathbb{C}^3)$. Then $c > 0$ and ϕ is minimal and M^3 is locally congruent to the Berger sphere.

2. Preliminaries.

Let M^{2n-1} be a real hypersurface with a unit normal local vector field $\mathcal{N}$ of an $n(\geqq 2)$-dimensional complex projective space $\mathbb{C}P^n(c)$ of constant holomorphic sectional curvature $c(> 0)$ through an isometric immersion. The ambient space $\mathbb{C}P^n(c)$ is furnished with the standard Riemannian metric g and the canonical Kähler structure J. The Riemannian connections $\widetilde{\nabla}$ of $\mathbb{C}P^n(c)$ and ∇ of M are related by the following formulas of Gauss and Weingarten:

$$\widetilde{\nabla}_X Y = \nabla_X Y + g(AX, Y)\mathcal{N}, \tag{2.1}$$

$$\widetilde{\nabla}_X \mathcal{N} = -AX \tag{2.2}$$

for arbitrary vector fields X and Y on M, where g is the Riemannian metric of M induced from the standard metric of the ambient space $\mathbb{C}P^n(c)$ and A is the shape operator of M in $\mathbb{C}P^n(c)$. An eigenvector of the shape operator A is called a *principal curvature vector* of M in $\mathbb{C}P^n(c)$ and an eigenvalue of A is called a *principal curvature* of M in $\mathbb{C}P^n(c)$. We denote by V_λ the eigenspace associated with the principal curvature λ, namely we set $V_\lambda = \{v \in TM | Av = \lambda v\}$.

On M it is well-known that an almost contact metric structure (ϕ, ξ, η, g) associated with $\mathcal{N}$ is canonically induced from the Kähler structure (J, g) of the ambient space $\mathbb{C}P^n(c)$, which is defined by

$$g(\phi X, Y) = g(JX, Y), \ \xi = -J\mathcal{N} \quad \text{and} \quad \eta(X) = g(\xi, X) = g(JX, \mathcal{N}).$$

ϕ, ξ and η are called the *structure tensor*, the *characteristic vector field* and the *contact form* on M, respectively. It follows from (2.1), (2.2) and $\widetilde{\nabla} J = 0$ that

$$(\nabla_X \phi)Y = \eta(Y)AX - g(AX, Y)\xi, \tag{2.3}$$

$$\nabla_X \xi = \phi AX. \tag{2.4}$$

Needless to say, M has two almost contact metric structures (ϕ, ξ, η, g) associated with $\mathcal{N}$ and $(\phi, -\xi, -\eta, g)$ associated with $-\mathcal{N}$.

Denoting the curvature tensor of M by R, we have the equation of Gauss given by

$$\begin{aligned} g\big((R(X, Y)Z, W\big) = (c/4)\big\{&g(Y, Z)g(X, W) - g(X, Z)g(Y, W) \\ &+ g(\phi Y, Z)g(\phi X, W) - g(\phi X, Z)g(\phi Y, W) - 2g(\phi X, Y)g(\phi Z, W)\big\} \\ &+ g(AY, Z)g(AX, W) - g(AX, Z)g(AY, W). \end{aligned}$$

Hence, the sectional curvature $K(X, Y)$ of the real plane spanned by a pair (X, Y) of orthonormal vectors is given by

$$K(X, Y) = (c/4)\big(1 + 3g(\phi X, Y)^2\big) + g(AX, X)g(AY, Y) - g(AX, Y)^2. \tag{2.5}$$

We usually call M a *Hopf hypersurface* if the characteristic vector ξ is a principal curvature vector at each point of M. The following is a key lemma in this paper.

LEMMA 2.1. ([6]). *Let M be a Hopf hypersurface of $\mathbb{C}P^n(c)$, $n \geqq 2$. Then the following hold.*

(1) *If a nonzero vector $v \in TM$ orthogonal to ξ satisfies $Av = \lambda v$, then $A\phi v = \big((\delta\lambda + (c/2))/(2\lambda - \delta)\big)\phi v$, where δ is the principal curvature associated with ξ.*
(2) *The principal curvature δ associated with ξ is locally constant.*

REMARK 2.1. For every Hopf hypersurface M in $\mathbb{C}P^n(c)$ we find easily that $2\lambda - \delta \neq 0$ because if $2\lambda - \delta = 0$, we have $\delta\lambda + (c/2) = 0$ (see Lemma 2.1(1)), which contradicts to $c > 0$.

THEOREM 2.1. ([8, 3]). *For a real hypersurface M^{2n-1} of $\mathbb{C}P^n(c)$ $(n \geqq 2)$, M is homogeneous in the ambient space $\mathbb{C}P^n(c)$, that is, M is an orbit of a subgroup of the full isometry group* $\mathrm{I}(\mathbb{C}P^n(c))(= \mathrm{SU}(n+1))$ *of $\mathbb{C}P^n(c)$ if and only if M^{2n-1} is a Hopf hypersurface all of whose principal curvatures are constant on M in $\mathbb{C}P^n(c)$. Moreover, M is locally congruent to one of the following*:

(A_1) *A geodesic sphere of radius r, where $0 < r < \pi/\sqrt{c}$;*
(A_2) *A tube of radius r around a totally geodesic $\mathbb{C}P^\ell(c)$ $(1 \leqq \ell \leqq n-2)$, where $0 < r < \pi/\sqrt{c}$;*
(*B*) *A tube of radius r around a complex hyperquadric $\mathbb{C}Q^{n-1}$, where $0 < r < \pi/(2\sqrt{c})$;*
(*C*) *A tube of radius r around the Segre embedding of $\mathbb{C}P^1(c) \times \mathbb{C}P^{(n-1)/2}(c)$, where $0 < r < \pi/(2\sqrt{c})$ and n $(\geqq 5)$ is odd;*
(*D*) *A tube of radius r around the Plücker embedding of a complex Grassmannian $\mathbb{C}G_{2,5}$, where $0 < r < \pi/(2\sqrt{c})$ and $n = 9$;*
(*E*) *A tube of radius r around a Hermitian symmetric space* $\mathrm{SO}(10)/\mathrm{U}(5)$*, where $0 < r < \pi/(2\sqrt{c})$ and $n = 15$.*

These real hypersurfaces are said to be of types (A_1), (A_2), (B), (C), (D) and (E). Unifying real hypersurfaces of types (A_1) and (A_2), we call them hypersurfaces of type (A).

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The numbers of distinct principal curvatures of these real hypersurfaces are 2, 3, 3, 5, 5, 5, respectively. The principal curvatures of these real hypersurfaces in $\mathbb{C}P^n(c)$ are given as follows:

	(A_1)	(A_2)	(B)	(C, D, E)
λ_1	$\frac{\sqrt{c}}{2}\cot\left(\frac{\sqrt{c}}{2}r\right)$	$\frac{\sqrt{c}}{2}\cot\left(\frac{\sqrt{c}}{2}r\right)$	$\frac{\sqrt{c}}{2}\cot\left(\frac{\sqrt{c}}{2}r-\frac{\pi}{4}\right)$	$\frac{\sqrt{c}}{2}\cot\left(\frac{\sqrt{c}}{2}r-\frac{\pi}{4}\right)$
λ_2	—	$-\frac{\sqrt{c}}{2}\tan\left(\frac{\sqrt{c}}{2}r\right)$	$\frac{\sqrt{c}}{2}\cot\left(\frac{\sqrt{c}}{2}r+\frac{\pi}{4}\right)$	$\frac{\sqrt{c}}{2}\cot\left(\frac{\sqrt{c}}{2}r+\frac{\pi}{4}\right)$
λ_3	—	—	—	$\frac{\sqrt{c}}{2}\cot\left(\frac{\sqrt{c}}{2}r\right)$
λ_4	—	—	—	$-\frac{\sqrt{c}}{2}\tan\left(\frac{\sqrt{c}}{2}r\right)$
δ	$\sqrt{c}\cot\left(\sqrt{c}\,r\right)$	$\sqrt{c}\cot\left(\sqrt{c}\,r\right)$	$\sqrt{c}\cot\left(\sqrt{c}\,r\right)$	$\sqrt{c}\cot\left(\sqrt{c}\,r\right)$

The principal curvatures and their multiplicities of all homogeneous real hypersurfaces in $\mathbb{C}P^n(c)$ are given in [**8**, **9**].

At the end of this section we review the definition of circles in Riemannian geometry. A smooth real curve $\gamma=\gamma(s)$ parametrized by its arclength s on a Riemannian manifold M with Riemannian connection ∇ is called a *circle of curvature* k if there exist a nonnegative constant k and the unit vector field Y_s orthogonal to the tangential vector $\dot\gamma$ along the curve γ satisfying the ordinary differential equations $\nabla_{\dot\gamma}\dot\gamma=kY_s$ and $\nabla_{\dot\gamma}Y_s=-k\dot\gamma$. It is well-known that a curve γ is a circle if and only if it satisfies the following differential equation:

$$\nabla_{\dot\gamma}\nabla_{\dot\gamma}\dot\gamma+g(\nabla_{\dot\gamma}\dot\gamma,\nabla_{\dot\gamma}\dot\gamma)\dot\gamma=0, \tag{2.6}$$

where g is the Riemannian metric on M. A circle of null curvature is nothing but a geodesic.

3. Proof of Theorem 1.1. $(\Longrightarrow)$ By assumption we can regard our real hypersurface M as a geodesic sphere $G(r)$ of radius r with $\tan^2(\sqrt{c}\,r/2)>2$ in the ambient space $\mathbb{C}P^n(c)$, $n\geqq 2$. So the tangent bundle TM of M is decomposed as: $TM=\{\xi\}_{\mathbb{R}}\oplus V_\lambda$, where $A\xi=\sqrt{c}\,\cot(\sqrt{c}\,r)\xi$ and $\lambda=(\sqrt{c}/2)\cot(\sqrt{c}\,r/2)$. We take an arbitrary geodesic $\gamma=\gamma(s)$ on M with an initial vector $\dot\gamma(0)$ perpendicular to the characteristic vector $\xi_{\gamma(0)}$. Then, using a fact that $\phi A=A\phi$ holds on M and equation (2.4), we obtain the following:

$$\begin{aligned}\dot\gamma(s)\big(g(\dot\gamma(s),\xi_{\gamma(s)})\big)&=\nabla_{\dot\gamma(s)}\big(g(\dot\gamma(s),\xi_{\gamma(s)})\big)=g\big(\dot\gamma(s),\nabla_{\dot\gamma(s)}\xi_{\gamma(s)}\big)\\&=g\big(\dot\gamma(s),\phi A\dot\gamma(s)\big)=g\big(\dot\gamma(s),A\phi\dot\gamma(s)\big)\\&=g\big(A\dot\gamma(s),\phi\dot\gamma(s)\big)=-g\big(\phi A\dot\gamma(s),\dot\gamma(s)\big)=0,\end{aligned}$$

so that $g(\dot\gamma(s),\xi_{\gamma(s)})$ is constant along the curve $\gamma=\gamma(s)$, which, together with $g(\dot\gamma(0),\xi_{\gamma(0)})=0$, yields that $g(\dot\gamma(s),\xi_{\gamma(s)})=0$ for each s. Hence we can find that $A\dot\gamma(s)=\lambda\dot\gamma(s)$ for every s. This, combined with equations (2.1) and (2.2), shows that $\widetilde\nabla_{\dot\gamma(s)}\dot\gamma(s)=\lambda\mathcal{N}_{\dot\gamma(s)}$ and $\widetilde\nabla_{\dot\gamma(s)}\mathcal{N}_{\dot\gamma(s)}=-\lambda\dot\gamma(s)$. Thus we know that the curve γ, considered as a curve in the ambient space $\mathbb{C}P^n(c)$, is a circle of the same positive curvature $\lambda(=(\sqrt{c}/2)\cot(\sqrt{c}\,r/2))$ which is independent of the choice of γ. Moreover, by the assumption that $\tan(\sqrt{c}\,r/2) > \sqrt{2}$ we get $\lambda < \sqrt{c}/(2\sqrt{2})$. Therefore we can obtain the "only if" part.

($\Longleftarrow$) Let $\gamma_i = \gamma_i(s)$ $(1 \leqq i \leqq 2n-2)$ be geodesics on M satisfying the hypothesis. Then it follows from (2.6) that

$$\widetilde{\nabla}_{\dot{\gamma}_i(s)}\left(\widetilde{\nabla}_{\dot{\gamma}_i(s)}\dot{\gamma}_i(s)\right) = -k^2(p)\dot{\gamma}_i(s). \tag{3.1}$$

On the other hand, from (2.1) and (2.2) we obtain

$$\widetilde{\nabla}_{\dot{\gamma}_i(s)}\left(\widetilde{\nabla}_{\dot{\gamma}_i(s)}\dot{\gamma}_i(s)\right) = g\big((\nabla_{\dot{\gamma}_i(s)}A)\dot{\gamma}_i(s),\, \dot{\gamma}_i(s)\big)\mathcal{N} - g(A\dot{\gamma}_i(s),\, \dot{\gamma}_i(s))A\dot{\gamma}_i(s). \tag{3.2}$$

Comparing the tangential components of (3.1) and (3.2), we have

$$g(A\dot{\gamma}_i(s),\, \dot{\gamma}_i(s))A\dot{\gamma}_i(s) = k^2(p)\dot{\gamma}_i(s) \quad \text{for } 1 \leqq i \leqq 2n-2,$$

which, combined with $k(p) \neq 0$, yields that $Av_i = k(p)v_i$ or $Av_i = -k(p)v_i$ for $1 \leqq i \leqq 2n-2$ at the point $p = \gamma_i(0)$. Note that ξ is principal. Indeed, $g(A\xi,\, v_i) = g(\xi,\, Av_i) = 0$ for $1 \leqq i \leqq 2n-2$. Thus, we know that our real hypersurface M is a Hopf hypersurface having at most three distinct principal curvatures δ, k and $-k$. We here note that the function $k = k(p)$ is automatically constant on M. In fact, from Lemma 2.1(1), we see that

$$k = \frac{k\delta + (c/2)}{2k - \delta} \quad \text{or} \quad k = -\frac{k\delta + (c/2)}{2k - \delta}.$$

However the latter case does not hold, since $c > 0$. Then we know that our real hypersurface M is a Hopf hypersurface having at most three distinct *constant* principal curvatures δ, k and $-k$. So, by virtue of Theorem 2.1 and the table of principal curvatures we find that our real hypersurface M is locally congruent to either a geodesic sphere $G(r)$ $(0 < r < \pi/\sqrt{c})$ or a real hypersurface of type (A_2) of $r = \pi/(2\sqrt{c})$. In the latter case, the constant function k is expressed as $k = \sqrt{c}/2$, which contradicts to the hypothesis $k < \sqrt{c}/(2\sqrt{2})$. Again, using the hypothesis $k < \sqrt{c}/(2\sqrt{2})$ for geodesic spheres $G(r)$ $(0 < r < \pi/\sqrt{c})$, we can see that $\tan^2(\sqrt{c}\,r/2) > 2$, so that we can obtain the "if" part.

4. Proof of Theorem 1.2. Before proving Theorem 1.2 we first compute the sectional curvature K of every geodesic sphere $G(r)$ $(0 < r < \pi/\sqrt{c})$. We take a pair $(X,\, Y)$ of orthonormal vectors that are orthogonal to ξ. In order to estimate the sectional curvature K of M, it suffices to calculate $K(\sin\theta \cdot X + \cos\theta \cdot \xi,\, Y)$. It follows from (2.5) that

$$K(\sin\theta \cdot X + \cos\theta \cdot \xi,\, Y) = (c/4)\{\sin^2\theta\big(1 + 3g(\phi X,\, Y)^2\big) + \cot^2(\sqrt{c}\,r/2)\}.$$

This gives the following inequalities:

$$(c/4)\cot^2(\sqrt{c}\,r/2) \leqq K \leqq c + (c/4)\cot^2(\sqrt{c}\,r/2). \tag{4.1}$$

We remark that $K(X,\, \phi X) = c + (c/4)\cot^2(\sqrt{c}\,r/2)$ and $K(X,\, \xi) = (c/4)\cot^2(\sqrt{c}\,r/2)$ for each unit vector X orthogonal to ξ.

We are now in a position to prove Theorem 1.2.

($\Longrightarrow$) Let M be a geodesic sphere $G(r)$ of radius r with $\tan^2(\sqrt{c}\,r/2) > 2$ in $\mathbb{C}P^n(c)$, $n \geqq 2$. Since M is totally η-umbilical in the ambient space $\mathbb{C}P^n(c)$ and $\sqrt{c}\cdot\cot(\sqrt{c}\,r) = (\sqrt{c}/2)\cot(\sqrt{c}\,r/2) - (\sqrt{c}/2)\tan(\sqrt{c}\,r/2)$, the shape operator A of M is expressed as follows:

$$AX = (\sqrt{c}/2)\cot(\sqrt{c}\,r/2)X - (\sqrt{c}/2)\tan(\sqrt{c}\,r/2)\eta(X)\xi \quad \text{for all } X \in TM. \tag{4.2}$$

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In view of (2.4), (4.2) and a fact that $\phi A = A\phi$ on M we shall compute $d\eta$.

$$\begin{aligned} d\eta(X, Y) &= (1/2)\{X(\eta(Y)) - Y(\eta(X)) - \eta([X, Y])\} \\ &= (1/2)\{X(g(\xi, Y)) - Y(g(\xi, X)) - \eta(\nabla_X Y - \nabla_Y X)\} \\ &= (1/2)\{g(\phi AX, Y) + g(\xi, \nabla_X Y) - g(\phi AY, X) - g(\xi, \nabla_Y X) \\ &\qquad - g(\nabla_X Y - \nabla_Y X, \xi)\} \\ &= (1/2) g\big((\phi A + A\phi)X, Y\big) = g(\phi AX, Y) \\ &= (\sqrt{c}/2)\cot(\sqrt{c}\,r/2) g(\phi X, Y), \end{aligned}$$

so that $k = (\sqrt{c}/2)\cot(\sqrt{c}\,r/2) > 0$, which, together with the assumption $\tan^2(\sqrt{c}\,r/2) > 2$, yields Theorem 1.2(1). Theorem 1.2(2) is an immediate consequence of (4.1). Thus we have proved the "only if" part.

($\Longleftarrow$) Let M be a real hypersurface satisfying Conditions (1) and (2) in Theorem 1.2 in the ambient space $\mathbb{C}P^n(c)$, $n \geqq 2$. By virtue of the above calculation we can make use of the following equality:

$$d\eta(X, Y) = (1/2) g\big((\phi A + A\phi)X, Y\big) \quad \text{for all } X, Y \in TM. \tag{4.3}$$

It follows from Condition (1) and (4.3) that

$$(\phi A + A\phi)X = \pm 2k\phi X \quad \text{for all } X \in TM. \tag{4.4}$$

The equality (4.4) shows $\phi A\xi = 0$, so that our real hypersurface M is a Hopf hypersurface. So we can set $A\xi = \delta\xi$. We take a principal curvature vector X orthogonal to ξ with $AX = \lambda X$. Then, by Lemma 2.1(1) and (4.4) we see that λ satisfies

$$\lambda + \frac{\delta\lambda + (c/2)}{2\lambda - \delta} = \pm 2k,$$

so that λ is a solution to the following quadratic equation with constant coefficients:

$$4\lambda^2 - 8k\lambda + c + 4\delta k = 0 \quad \text{or} \quad 4\lambda^2 + 8k\lambda + c - 4\delta k = 0.$$

Hence our Hopf hypersurface M has at most three distinct constant principal curvatures. Thus, from Theorem 2.1 we find that M is locally congruent to one of homogeneous real hypersurfaces of types (A_1), (A_2) and (B). So, in the following we shall check (4.4) one by one for these three homogeneous real hypersurfaces. We here note that $(\phi A + A\phi)\xi = 0 = \pm 2k\phi\xi$, since ξ is principal.

Let M be of type (A_1) of radius $r\,(0 < r < \pi/\sqrt{c})$. Then the above calculation guarantees (4.4). Moreover, by Condition (1) we have $(\sqrt{c}/2)\cot(\sqrt{c}\,r/2) < \sqrt{c}/(2\sqrt{2})$, so that $\tan^2(\sqrt{c}\,r/2) > 2$. Hence, in this case our Hopf hypersurface M is locally congruent to a Berger sphere.

Let M be of type (A_2) of radius $r\,(0 < r < \pi/\sqrt{c})$. It is sufficient to check (4.4) for every $X(\in TM)$ perpendicular to ξ. But, from Lemma 2.1(1) we get both $\phi V_{\lambda_1} = V_{\lambda_1}$ and $\phi V_{\lambda_2} = V_{\lambda_2}$, where $\lambda_1 = (\sqrt{c}/2)\cot(\sqrt{c}\,r/2)$ and $\lambda_2 = -(\sqrt{c}/2)\tan(\sqrt{c}\,r/2)$. These imply that Equation (4.4) does *not* hold for each $r \in (0, \pi/\sqrt{c})$.

Let M be of type (B) of radius $r\,(0 < r < \pi/(2\sqrt{c}))$. Then M has three distinct constant principal curvatures:

$$\lambda_1 = \frac{\sqrt{c}}{2}\frac{1+x}{1-x}, \quad \lambda_2 = -\frac{\sqrt{c}}{2}\frac{1-x}{1+x} \quad \text{and} \quad \delta = \frac{\sqrt{c}}{2}\Big(x - \frac{1}{x}\Big),$$

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where $x = \cot(\sqrt{c}\,r/2) > 1$. Note that $\phi V_{\lambda_1} = V_{\lambda_2}$ and $\phi V_{\lambda_2} = V_{\lambda_1}$ (see Lemma 2.1(1)). Furthermore, we see $\lambda_1 + \lambda_2 = (2\sqrt{c}\,x)/(1 - x^2) < 0$. So, in this case for the purpose of checking (4.4) we must solve the equation: $\lambda_1 + \lambda_2 = -2k$. Then we find easily $x = (\sqrt{c} + \sqrt{c + 4k^2})/(2k)$. Moreover, it follows from

$$\frac{(-\lambda_1) + (-\lambda_2)}{2} = k < \frac{\sqrt{c}}{2\sqrt{2}} \quad \text{that} \quad x > \sqrt{2} + \sqrt{3}\,.$$

Thus we can see that our real hypersurface of type (B) of radius r with $\cot(\sqrt{c}\,r/2) = (\sqrt{c} + \sqrt{c + 4k^2})/(2k)$ and $\cot(\sqrt{c}\,r/2) > \sqrt{2} + \sqrt{3}$ satisfies Condition (1). We remark that such real hypersurfaces of type (B) do exist. In fact, since the constant k is written as: $k = \sqrt{c}\,x/(x^2 - 1)$, by setting x satisfying an inequality $x > \sqrt{2} + \sqrt{3}$ we can guarantee that the above real hypersurfaces of type (B) exist, where x is a solution to $kx^2 - \sqrt{c}\,x - k = 0$ and $x = \cot(\sqrt{c}\,r/2) > 1$.

Next, we shall check Condition (2) for Berger spheres and the above homogeneous real hypersurface of type (B). The former case is obvious (see (4.1)). We investigate the latter case. It follows from (2.5) that

$$K(X, \xi) = \frac{c}{4} + \lambda_1\delta = \frac{c}{4} - \frac{c}{4}\frac{(1 + \tan(\sqrt{c}\,r/2))^2}{\tan(\sqrt{c}\,r/2)} < 0 \quad \text{for each unit } X \in V_{\lambda_1}$$

and

$$K(Y, \xi) = \frac{c}{4} + \lambda_2\delta = \frac{c}{4} + \frac{c}{4}\frac{(1 - \tan(\sqrt{c}\,r/2))^2}{\tan(\sqrt{c}\,r/2)} > 0 \quad \text{for each unit } Y \in V_{\lambda_2}.$$

Thus we see that every homogeneous real hypersurface of type (B) does not satisfy Condition (2). Therefore we have proved the "if" part.

5. Comments on Theorems 1.1 and 1.2. (1) In the statement of Theorem 1.1, if we remove $k(p) < \sqrt{c}/(2\sqrt{2})$, this theorem is no longer true. All geodesic spheres $G(r)$ $(0 < r < \pi/\sqrt{c})$ and a certain homogeneous real hypersurface of type (A_2) satisfy Theorem 1.1 without an inequality $k(p) < \sqrt{c}/(2\sqrt{2})$. We here recall a fact that if every geodesic γ on a submanifold M^n into a Riemannian manifold $\widetilde{M}^{n+p}$ through an isometric immersion is mapped to a circle (of curvature k) in the ambient space $\widetilde{M}^{n+p}$, then the curvature k does *not* depend on the choice of γ. On the other hand, we know that $\mathbb{C}P^n(c)$ admits *no* real hypersurfaces all of whose geodesics are mapped to circles in this space. Hence it is natural to consider real hypersurfaces *some of whose geodesics are mapped to circles of the same curvature* in $\mathbb{C}P^n(c)$. In this context, we establish Theorem 1.1.

(2) In the statement of Theorem 1.2, if we remove Condition (2), this theorem does not hold. The Berger sphere and a certain homogeneous real hypersurface of type (B) satisfy Theorem 1.2(1). We here review a fact that $\mathbb{C}P^n(c)$ admits *no* real hypersurfaces with $d\eta = 0$ (see [7]). On the other hand, a complex Euclidean space $\mathbb{C}^n$ has real hypersurfaces M^{2n-1} with $d\eta = 0$ (e.g., the totally geodesic real hypersurface $\mathbb{R}^{2n-1}$ satisfies this condition). So, in some sense the geometry of real hypersurfaces of $\mathbb{C}P^n(c)$ is more complicated than that of $\mathbb{C}^n$. Motivated by them, we establish Theorem 1.2.

(3) We review the length spectrum of every geodesic sphere $G(r)$ $(0 < r < \pi/\sqrt{c})$ for readers (for details, see [2]). We first note that every integral curve γ_ξ of the characteristic vector field ξ is a closed geodesic on $G(r)$ with length $2\pi \sin(\sqrt{c}\,r)/\sqrt{c}$. In fact, the curve γ_ξ satisfies $\nabla_\xi \xi = 0$, $\widetilde{\nabla}_\xi \xi = \sqrt{c}\cot(\sqrt{c}\,r)\mathcal{N}$ and $\widetilde{\nabla}_\xi \mathcal{N} = -\sqrt{c}\cot(\sqrt{c}\,r)\xi$ with $\dot{\gamma}_\xi = \xi$, where ∇ and $\widetilde{\nabla}$ are the Riemannian connections of $G(r)$ and $\mathbb{C}P^n(c)$,

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respectively. These mean that the curve γ_ξ can be regarded as a small circle of positive curvature $\sqrt{c}\,|\cot(\sqrt{c}\,r)|$ on $S^2(c)(=\mathbb{C}P^1(c))$. Hence, the length ℓ of γ_ξ is represented as: $\ell = 2\pi/\sqrt{c\cot^2(\sqrt{c}\,r)+c} = 2\pi\,\sin(\sqrt{c}\,r)/\sqrt{c}$. We here consider an inequality $2\pi\,\sin(\sqrt{c}\,r)/\sqrt{c} < 2\pi/\sqrt{c+(c/4)\cot^2(\sqrt{c}\,r/2)}$, where $c+(c/4)\cot^2(\sqrt{c}\,r/2)$ is the maximal sectional curvature of $G(r)$. Solving this inequality, we get $\tan^2(\sqrt{c}\,r/2) > 2$.

Every geodesic sphere $G(r)\,(0 < r < \pi/\sqrt{c})$ admits countably many congruence classes of closed geodesics with respect to the full isometry group $\mathrm{I}(G(r))$ of $G(r)$. All integral curves of the characteristic vector field ξ are congruent to each other with respect to $\mathrm{I}(G(r))$ and the shortest closed geodesics (with common length $2\pi\,\sin(\sqrt{c}\,r)/\sqrt{c}$) on $G(r)$. Furthermore, the lengths of all closed geodesics except integral curves of the characteristic vector field ξ on $G(r)$ are longer than $2\pi/\sqrt{c+(c/4)\cot^2(\sqrt{c}\,r/2)}$.

(4) We consider all geodesic spheres $G(r)\,(0 < r < \pi/\sqrt{c})$ from the viewpoint of contact geometry (cf. [1]). A geodesic sphere $G(r)$ is a Sasakian manifold (with respect to the almost contact metric structure (ϕ, ξ, η, g) induced from the Kähler structure J of $\mathbb{C}P^n(c)$) if and only if $(\sqrt{c}/2)\cot(\sqrt{c}\,r/2) = 1$. This Sasakian manifold M has automatically constant ϕ-sectional curvature $c+1$, so that it is a Sasakian space form of constant ϕ-sectional curvature $c+1$. Since an inequality $1 < \sqrt{c}/(2\sqrt{2})$ leads to an inequality $c > 8$, by the discussion in our paper we find that all Sasakian space forms of constant ϕ-sectional curvature $\tilde{c}$ with $\tilde{c} > 9$ are Berger spheres.

(5) We comment on the sectional curvature K of Berger spheres, that is, geodesic spheres $G(r)$ with $\tan^2(\sqrt{c}\,r/2) > 2$ in $\mathbb{C}P^n(c)$. The sectional curvature K satisfies sharp inequalities $\delta L \leqq K \leqq L$ for some $\delta \in (0, 1/9)$ at its each point, where $L = c + (c/4)\cot^2(\sqrt{c}\,r/2)$. In this context, we recall the following, which is derived from direct computation.

LEMMA 5.1. *Let $G(r)$ be a geodesic sphere of radius $r\,(0 < r < \pi/\sqrt{c})$ in $\mathbb{C}P^n(c)$. Then the following three conditions are mutually equivalent*:

(1) *The radius r satisfies an inequality* $\tan^2(\sqrt{c}\,r/2) > 2$.
(2) *The sectional curvature K of $G(r)$ satisfies sharp inequalities $\delta L \leqq K \leqq L$ for some $\delta \in (0, 1/9)$ at its each point.*
(3) *The length of every integral curve of the characteristic vector field ξ on $G(r)$ is shorter than $2\pi/\sqrt{L}$, where L is the maximal sectional curvature of $G(r)$.*

Needless to say, for every geodesic sphere $G(r)\,(0 < r < \pi/\sqrt{c})$ in $\mathbb{C}P^n(c)$ every integral curve of the characteristic vector field ξ on $G(r)$ is a geodesic.

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