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Forms over number fields and weak approximation

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Abstract. Let K be a number field, and let $X \subseteq \mathbb{P}_K^{s-1}$ be a smooth complete intersection defined over K. In this paper, weak approximation is shown to hold for X provided s exceeds some function of the degree and codimension of X. This is a corollary of a more general result about the number of integral points on certain affine varieties in homogeneously expanding regions. This general result is established via a suitable adaptation of the Hardy-Littlewood Circle Method.

Key words: weak approximation, Hardy-Littlewood method, forms in many variables

1. Introduction

Let K be an algebraic number field of degree n over $\mathbb{Q}$. Let $X \subseteq \mathbb{P}_K^{s-1}$ be a variety defined over K. For any extension field L of K, let $X(L)$ denote the L-rational points on X. *Weak approximation* is said to hold for X if $X(K) \neq \emptyset$ and if the diagonal embedding $X(K) \to \prod X(K_v)$ is dense, where v runs through all places of K, and K_v is the completion of K at v. Here, $X(K_v)$ has the obvious v-adic topology.

It is known, for example, that weak approximation holds for smooth quadratic hypersurfaces of dimension at least two. However, Colliot–Thelene has remarked that it is not known whether weak approximation holds for smooth cubic hypersurfaces, not even for those of high dimension [M, Section 4, p. 39]. This note provides an affirmative answer to this problem. In fact, we establish that weak approximation holds for a large class of smooth complete intersections, provided that s exceeds some function of the degree and the codimension of X and that $X(K_v) \neq \emptyset$ for all v. This will be deduced from a general result about the number of zeros of certain (not nec. smooth) affine varieties in homogeneously expanding regions.

Let $\mathfrak{o}$ be the ring of integers of K, and let $\mathfrak{n}$ be an integral ideal. Put $V = K \otimes_{\mathbb{Q}} \mathbb{R}$. If $\omega_1, \ldots, \omega_n$ is a fixed $\mathbb{Z}$-basis for $\mathfrak{n}$, then the canonical embedding $K \to V$ makes the ω_i's an $\mathbb{R}$-basis for V. For any $\mathbf{b} \in \mathbb{R}^{ns}$ let $\mathcal{B}(\mathbf{b}) \subseteq V^s$ be the box

$$\mathcal{B}(\mathbf{b}) = \{(r_1, \ldots, r_s) \in V^s \colon b_{ij} - \tfrac{1}{2} \leqslant r_{ij} < b_{ij} + \tfrac{1}{2}\},$$

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where $r_i = r_{i1}\omega_1 + \cdots + r_{in}\omega_n$. Finally, suppose $\mathbf{G} = (g_1, \ldots, g_m)$ is a system of polynomials of total degree d in the variables $\mathbf{x} = (x_1, \ldots, x_s)$ having coefficients in $\mathfrak{o}$. For any box $\mathcal{B} = \mathcal{B}(\mathbf{b})$, we are interested in the quantity

$$N(\mathbf{G}, \mathcal{B}, P) = \#\{\mathbf{x} \in (P\mathcal{B} \cap \mathfrak{n}^s) : g_i(\mathbf{x}) = 0 \quad \text{for all } i\}.$$

For a system $\mathbf{G}$ as above, let $\mathbf{F} = (f_1, \ldots, f_m)$ with f_i the homogeneous part of degree d of g_i. For $\mathbf{m} \in \mathbb{C}^m$, let $V(\mathbf{m})$ be the affine variety defined by $f_i = m_i$ for $i = 1, \ldots, m$. Let V_{sing} be the union of the singular loci of the $V(\mathbf{m})$. Our main result is the following theorem.

THEOREM. *Let* $\mathbf{G}$ *be a system of* m *polynomials in the variables* $x_1, \ldots, x_s$ *with coefficients in* $\mathfrak{o}$ *and having the same total degree* d*. Suppose that*

$$s - \dim(V_{\text{sing}}) > m(m+1)(d-1)2^{d-1}.$$

Then for any box $\mathcal{B}$

$$N(\mathbf{G}, \mathcal{B}, P) = \mu P^{n(s-md)} + o(P^{n(s-md)}),$$

where $\mu = \mu(\mathbf{G}, \mathcal{B})$ *is a constant. Furthermore, if*

(i) *the system* $\mathbf{G}$ *has a non-singular solution in* $\mathfrak{n}_v^s$ *for every finite place* v *of* K*,*
(ii) $\mathbf{F}$ *has a non-singular solution over* K_v *for every infinite place* v *of* K*, and*
(iii) $\dim(V(\mathbf{0})) = s - m$*,*

then μ *is positive for some* $\mathcal{B}$*, whence the system* $\mathbf{G}$ *has a non-trivial zero in* $\mathfrak{n}^s$*. In particular, this holds for* $\mathcal{B}(\oplus \mathbf{z}_v)$ *with* $\mathbf{z}_v$ *any non-singular solution over* K_v *of* $\mathbf{F}$*,* v *running over the infinite places.*

From this we can deduce the following.

COROLLARY 1. *Let* $X \subset \mathbb{P}_K^{s-1}$ *be a smooth variety defined by a system* $\mathbf{F}$ *of* m *homogeneous forms satisfying the hypotheses of the Theorem. If* $X(K_v) \neq \emptyset$ *for all places* v *and if* $\dim X = s - m - 1$ *(i.e. if* X *is a complete intersection), then weak approximation holds for* X*.*

COROLLARY 2. *Let* $X \subset \mathbb{P}_K^{s-1}$ *be a smooth hypersurface of odd degree* d *defined over* K*. There exists a function* $\chi(d)$ *such that if* $s > \chi(d)$*, then weak approximation holds for* X*.*

Our Theorem is a generalization of the main result of [B1]. Of particular note is that the condition on s is independent of K. This is an improvement over Theorem 3 of [B1], which states a result for number fields, but with a great dependence on $n = [K : \mathbb{Q}]$.

Corollary 2 follows immediately from Corollary 1. By a result of Birch [B2] there is a function $\phi(d)$ such that if $s > \phi(d)$ then any hypersurface of odd degree d in $\mathbb{P}_K^{s-1}$ has a K_v-rational point for all v. Thus we may take

$$\chi(d) = \max(\phi(d), (d-1)2^d).$$

For example, in the case of cubic hypersurfaces, it is well known that $\phi(3) = 9$ and so $\chi(3) = 16$.

For $K = \mathbb{Q}$, our Theorem is essentially a consequence of Theorem II of [S2], and one expects that all of [S2] could be generalized to the number field setting. The only problem seems to lie in generalizing the results of [S2, Section 12].

As in [B1] and [S2], we deduce our Theorem via a suitable adaptation of the Hardy-Littlewood Circle Method. This occupies most of Sections 3 and 4 of this paper. The final section, Section 5, is devoted to deriving Corollary 1.

2. Notation and conventions

Along with the notation introduced in the introduction, the following will hold:

$\sigma_1, \dots, \sigma_{n_1}$ are the distinct real embeddings of K, and $\sigma_{n_1+1}, \dots, \sigma_{n_1+2n_2}$ are a complete set of distinct complex embeddings such that σ_{n_1+i} is conjugate to $\sigma_{n_1+i+n_2}$ $(i = 1, \dots, n_2)$.

K_i is the completion of K with respect to the embedding σ_i $(i = 1, \dots, n_1+n_2)$. Thus, $K_i = \mathbb{R}$ for $i \leqslant n_1$ and $K_i = \mathbb{C}$ for $i > n_1$.

V is the n-dimensional commutative $\mathbb{R}$-algebra $\oplus_{i=1}^{n_1+n_2} K_i \cong K \otimes_{\mathbb{Q}} \mathbb{R}$. For an element $x \in V$ we write $x^{(i)}$ for its projection onto the ith summand (i.e. $x = \oplus x^{(i)}$). There is a canonical embedding of K into V given by $\alpha \mapsto \oplus \sigma_i(\alpha)$. We identify K with its image in V. Under this identification $\mathfrak{n}$ forms a lattice in V, and $\omega_1, \dots, \omega_n$ form a real basis for V. Thus we may consider V to be the set $\{r_1\omega_1 + \cdots + r_n\omega_n : r_i \in \mathbb{R}\}$.

$R = \{r = r_1\omega_1 + \cdots + r_n\omega_n : 0 \leqslant r_i < 1\}$ and $R_K = R \cap K$. A box $\mathcal{B}$ is a translate of R^s by some element of V^s.

For $\gamma \in K$, $\mathfrak{a}_\gamma$ denotes the integral ideal $\{\alpha \in \mathfrak{o} : \alpha\gamma \in \mathfrak{n}\}$.

In summation, '(mod $\mathfrak{a}$)' is abbreviated '($\mathfrak{a}$)'. For example, $\sum_{\mathbf{x}(\mathfrak{a})}$ means summation over the set of s-tubles $\mathbf{x} = (x_1, \dots, x_s)$ with x_i running over a complete set of residues modulo $\mathfrak{a}$. Also, $\mathbf{x} \in \mathfrak{a}$ means $\mathbf{x} = (x_1, \dots x_s)$ with $x_i \in \mathfrak{a}$.

We will often associate to $\mathbf{x} \in V^s$ the vector $\mathbf{X}$ with the ns real entries $\mathbf{X} = (x_{11}, \dots, x_{sn})$, and *vice versa*.

Whenever used, the implied constants in $\ll$ and $\gg$ may, of course, depend on K, $\mathfrak{n}$, and the choice of the ω_i's. The actual dependence of the constant will usually be clear.

$f \asymp g$ means $f \ll g$ and $f \gg g$.

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The trace and norm maps on K extend to V

$$\mathrm{Tr}(x) = x^{(1)} + \cdots + x^{(r)} + 2\Re(x^{(r+1)}) + \cdots + 2\Re(x^{(r+s)}),$$

$$\mathrm{Nm}(x) = x^{(1)} \cdots x^{(r)} |x^{(r+1)}|^2 \cdots |x^{(r+s)}|^2.$$

Thus, $\mathrm{e}(\mathrm{Tr}(\cdot))$ is a character on V, which we denote by $\Phi(\cdot)$, where $\mathrm{e}(\cdot) = e^{2\pi i(\cdot)}$.

We define a distance function $|\cdot|$ on V as follows

$$|x| = |x_1\omega_1 + \cdots + x_n\omega_n| = \max_i |x_i|.$$

This extends to the nd-dimensional $\mathbb{R}$-space V^n in the obvious way: if $\mathbf{x} = (x_1, \ldots, x_n) \in V^n$, then

$$|\mathbf{x}| = \max_j |x_j|.$$

For $x \in V$ we have

$$\max_i |x^{(i)}| \asymp |x|,$$

where $|x^{(i)}|$ is the standard absolute value on K_i. Also

$$|x^{-1}| \ll \frac{|x|^{d-1}}{|\mathrm{Nm}(x)|}.$$

For $r \in \mathbb{R}$, $\|r\|$ denotes, as usual, the distance to the nearest integer.

We work with the volume form on V

$$dr = dr_1 \cdots dr_d,$$

the standard Lebesgue measure on $\mathbb{R}^d$ from the identification $V = \{r_1\omega_1 + \cdots + r_d\omega_d\}$.

To a polynomial $Q(\mathbf{x}) \in V[x_1, \ldots, x_s]$ we associate a number of related polynomials: $Q^*(\mathbf{X}) = \mathrm{Tr}(Q(\mathbf{x}))$ considered as a polynomial in $\mathbb{R}[\mathbf{X}] = \mathbb{R}[x_{11}, \ldots, x_{sn}]$, where $x_i = x_{i1}\omega_1 + \cdots + x_{in}\omega_n$. Similarly, $Q_j^*(\mathbf{X}) = \mathrm{Tr}(\omega_j Q(\mathbf{x}))$.

For a system of polynomials $\mathbf{Q} = (Q_1, \ldots, Q_m)$, $\mathbf{Q}^*$ is the system comprised of the polynomials Q_{ij}^* $(i = 1, \ldots, m; j = 1, \ldots, n)$.

If Q is homogeneous of total degree d, then we write $Q(\mathbf{x}_1|\mathbf{x}_2| \ldots |\mathbf{x}_d)$ for the associated multilinear form. The associated multilinear form for Q^* is therefore $\mathrm{Tr}(Q(\mathbf{x}_1|\mathbf{x}_2| \ldots |\mathbf{x}_d))$.

A final note about notation: P is a large positive number which we think of as tending to infinity, and, as usual, ε denotes a small positive constant which, in keeping with common practice, may change from one appearance to the next. This

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causes no problems as ε may be assumed to be as small as we like. Likewise, δ, η and ζ generally denote small constants.

3. Some exponential sums

In this section, we investigate some exponential sums which arise in our application of the Circle Method.

Let $\mathbf{G}$ and $\mathbf{F}$ be as in the introduction. For $\mathbf{r} \in R^m$, put

$$g = \mathbf{r} \cdot \mathbf{G} = \sum_{i=1}^{m} r_i g_i.$$

We are interested in the sum

$$\begin{aligned} S(\mathbf{r}) = S(\mathbf{G}, \mathcal{B}, \mathbf{r}) &= \sum_{\mathbf{x} \in P\mathcal{B}} \Phi(g(\mathbf{x})) \\ &= \sum_{\mathbf{X} \in P\mathcal{B}} \mathrm{e}(g^*(\mathbf{X})). \end{aligned}$$

Also, let $\{\mathbf{v}_i\}$ be the standard basis for the s-dimensional vector space V^s and let $\{\mathbf{E}_{ij}\}$ $(i = 1, \ldots, s; j = 1, \ldots, n)$ be the standard basis for $\mathbb{R}^{sn}$. Following the conventions of Section 2, $\mathbf{E}_{ij} = \omega_j \mathbf{v}_i$. For any $(d-1)$-tuple $(\mathbf{x}_1, \ldots, \mathbf{x}_{d-1})$ with $\mathbf{x}_i \in V^s$, put

$$M = (M_{ip}) = \left(f_i(\mathbf{x}_1 | \cdots | \mathbf{x}_{d-1} | \mathbf{v}_p)\right), \quad (i = 1, \ldots, m; p = 1, \ldots, n).$$

Let f be the homogeneous part of g of degree d, and let $N^*(P^\zeta, P^{-\eta}, \mathbf{r})$ denote the number of integral $(d-1)$-tuples $(\mathbf{X}_1, \ldots, \mathbf{X}_{d-1})$ such that

$$\mathbf{X}_k \in [-P^\zeta, P^\zeta]^{ns}$$

and

$$\|f^*(\mathbf{X}_1 | \cdots | \mathbf{X}_{d-1} | \mathbf{E}_{pq})\| \ll P^{-\eta},$$

for all p and q.

LEMMA 1. *Let* $0 < \Delta \leqslant 1$ *and let* $\varepsilon > 0$. *If* $S(\mathbf{r}) \gg P^{sn-\kappa}$, *then*

$$N^*(P^\Delta, P^{-d+(d-1)\Delta}, \mathbf{r}) \gg P^{(d-1)ns\Delta - 2^{d-1}\kappa - \varepsilon}.$$

Proof. This is just [B1, Lemma 2.4]. □

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LEMMA 2. *Let* $0 < \Delta \leqslant 1$ *and let* $\varepsilon > 0$. *Either*

(i) $S(\mathbf{r}) \ll P^{ns-\kappa}$,
(ii) *there exist* $0 \neq \alpha \in \mathfrak{n}$ *and* $\nu_i \in \mathfrak{n}$ *such that*

$$|\alpha| \ll P^{m(d-1)\Delta} \quad \text{and} \quad |\alpha r_i - \nu_i| \ll P^{-d+m(d-1)\Delta}$$

for $i = 1, \dots, m$, *or*
(iii) *the number of* $(d-1)$*-tuples of* $\mathfrak{n}$*-points of* $P^{\Delta}R$ *for which*

$$\operatorname{rank}(M) < m,$$

is

$$\gg P^{\Delta(d-1)ns-2^{d-1}\kappa-\varepsilon}.$$

Proof. If (i) does not hold, then by Lemma 1

$$N^*(P^{\Delta}, P^{-d+(d-1)\Delta}, \mathbf{r}) \gg P^{(d-1)ns\Delta-2^{d-1}\kappa-\varepsilon}.$$

If (iii) does not hold for all the $(d-1)$-tuples counted by N^*, then for some such $(d-1)$-tuple we have $\operatorname{rank}(M) = m$. WLG we may assume that the leading $m \times m$ minor has non-vanishing determinant. Denote this determinant by α'. Note that $\alpha' \in \mathfrak{n}$ and that

$$|\alpha'| \ll |\mathbf{x}_1|^m \cdots |\mathbf{x}_{d-1}|^m \ll P^{m(d-1)\Delta}.$$

Next, put

$$\beta_p = \sum_{i=1}^{m} r_i M_{ip} = \beta_{p1}\omega_1 + \cdots + \beta_{pn}\omega_n.$$

By assumption

$$\operatorname{Tr}(\omega_q\beta_p) = \operatorname{Tr}\left(\sum_{j=1}^{m} \omega_j\omega_q\beta_{pj}\right) = a_{pq} + d_{pq},$$

where $a_{pq} \in \mathbb{Z}$ and $|d_{pq}| \ll P^{-d+(d-1)\Delta}$ for $q = 1, \dots, n$. Let

$$\mathbf{B}_p = (\beta_{p1}, \dots, \beta_{pn}), \quad \mathbf{A}_p = (a_{p1}, \dots, a_{pn}), \quad \mathbf{D}_p = (d_{p1}, \dots, d_{pn}),$$

and

$$\Omega = (\operatorname{Tr}(\omega_q\omega_j)), \quad (q = 1, \dots, n; j = 1, \dots, n).$$

Then $\Omega\mathbf{B}_p = \mathbf{A}_p + \mathbf{D}_p$. Thus

$$\mathbf{B}_p = \Omega^{-1}\mathbf{A}_p + \Omega^{-1}\mathbf{D}_p = \mathbf{A}'_p + \mathbf{D}'_p.$$

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Observe that $|\det(\Omega)| = |\mathrm{Nm}(\mathfrak{n})\mathrm{Disc}(K)|$, whence

$$|d'_p| = |d'_{p1}\omega_1 + \cdots + d'_{pn}\omega_n| \ll \max_j |d_{pj}| \ll P^{-d+(d-1)\Delta}.$$

Let A be the leading $m \times m$ minor. Then

$$A\mathbf{r} = (\beta_p)_{1 \leqslant p \leqslant m}.$$

Let $(\nu_p)_{1 \leqslant p \leqslant m}$ be the solution of the linear equations

$$A\mathbf{n} = \det(\Omega)\alpha'(a'_p)_{1 \leqslant p \leqslant m}.$$

Then $\nu_p \in \mathfrak{n}$ and we have

$$A(\det(\Omega)\alpha'\mathbf{r} - \mathbf{n}) = (\det \Omega)\alpha'(d'_p)_{1 \leqslant p \leqslant m}.$$

Put $\alpha = \det(\Omega)\alpha'$. It follows that

$$|\alpha r_i - \nu_i| \ll |\mathbf{x}_1|^{m-1} \cdots |\mathbf{x}_{d-1}|^{m-1} \max_p |d'_p| \ll P^{-d+m(d-1)\Delta}$$

and

$$|\alpha| \ll |\alpha'| \ll P^{m(d-1)\Delta}.$$

Thus (ii) holds. $\square$

LEMMA 3. *Let $U \subset \mathbb{C}^s$ be an affine cone of dimension t defined over K. Then U contains $\ll P^{nt}$ elements of $PR^s \cap \mathfrak{n}^s$.*

Proof. Let $Y = \prod_{i=1}^{n_1+n_2} \sigma_i(U(K_i)) \subset \mathbb{R}^{n_1 s} \times \mathbb{C}^{n_2 s}$. Clearly, the topological dimension of Y is at most nt. Thus we can cover $Y \cap R^s$ by $\ll P^{nt}$ boxes of side length P^{-1}. Then P times any such box contains $\ll 1$ elements of $\mathfrak{n}^s$, whence $Y \cap PR^s = P(Y \cap R^s)$ contains $\ll P^{nt}$ elements of $\mathfrak{n}^s$. $\square$

The final lemma of this section shows that for the appropriate choice of κ, one may eliminate possibility (iii) of Lemma 2 for any system $\mathbf{G}$ satisfying the hypotheses of our Theorem.

LEMMA 4. *Let $\kappa = n(m(m+1)(d-1) + \delta)\Delta$. If Lemma* 2(iii) *holds, then*

$$s - \dim(V_{\mathrm{sing}}) \leqslant m(m+1)(d-1)2^{d-1},$$

provided $\delta > 0$ is sufficiently small.

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Proof. Let $\mathcal{S}$ denote the algebraic set defined by $\operatorname{rank}(M) < m$. The proof of [B1, Lemma 3.3] is entirely geometric, so it remains valid in our situation. Therefore, we have that

$$\dim(V_{\text{sing}}) \geqslant \dim(\mathcal{S}) - (d-2)s.$$

If Lemma 2(iii) holds, then by Lemma 3 we must have

$$\dim(\mathcal{S}) \geqslant (d-1)s - \frac{2^{d-1}\kappa}{n\Delta} - \varepsilon.$$

It follows that

$$s - \dim(V_{\text{sing}}) \leqslant \frac{2^{d-1}\kappa}{n\Delta} + \varepsilon \leqslant m(m+1)(d-1)2^{d-1} + \varepsilon + \delta.$$

Since $s - \dim(V_{\text{sing}})$ is an integer, the lemma follows easily. □

4. The circle method

In this section, we apply the Circle Method apparatus to our problem and deduce our Theorem. We now assume that **G** satisfies the hypotheses of the Theorem, namely, $s - \dim(V_{\text{sing}}) > m(m+1)(d-1)2^{d-1}$ and that κ is as in Lemma 4. Thus we can dispense with alternative (iii) of Lemma 2. First note that

$$N(P) = N(\mathbf{G}, \mathcal{B}, P) = \int_{R^m} S(\mathbf{r})\,\mathrm{d}\mathbf{r}.$$

As expected, we evaluate $N(P)$ by the standard major/minor arc estimates. To this end, we introduce the following sets. For $\Gamma = (\gamma_1, \ldots, \gamma_m) \in R_K^m$, put

$$\mathfrak{a}_\Gamma = \operatorname{lcm}(\mathfrak{a}_{\gamma_i}),$$

$$\mathfrak{M}_\Gamma(\theta) = \{\mathbf{r} \in R^m : |\mathbf{r}_i - \gamma_i| \leqslant P^{-d+\theta}\},$$

$$\mathfrak{M}(\theta) = \bigcup_{\substack{\Gamma \\ \operatorname{Nm}(\mathfrak{a}_\Gamma) \ll P^\theta}} \mathfrak{M}_\Gamma(\theta), \quad \mathfrak{m}(\theta) = R^m \backslash \mathfrak{M}(\theta),$$

and

$$\mathcal{E}(\Delta) = \{\mathbf{r} \in R^m \text{ such that alternative (ii) of Lemma 2 holds}\}.$$

Note that our R is certainly contained in the R of [Sk] expanded by the factor $\operatorname{Nm}(\mathfrak{n})$, whence Lemmas 4, 5, and 6 of [Sk] remain valid in our setting. We will need the following lemma.

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LEMMA 5.

(i) $\#\{\Gamma : \mathrm{Nm}(\mathfrak{a}_\Gamma) = N\} \ll N^{m+\varepsilon}$.
(ii) *If* $nm(d-1)\Delta \leqslant \theta$, *then* $\mathcal{E}(\Delta) \subseteq \mathfrak{M}(\theta)$.
(iii) $\mathrm{meas}(\mathcal{E}(\Delta)) \ll P^{n(-md+m(m+1)(d-1)\Delta)+\varepsilon}$.

Proof. (i) Observe that

$$\#\{\Gamma : \mathrm{Nm}(\mathfrak{a}_\Gamma) = N\} = \sum_{\substack{\mathfrak{a} \\ \mathrm{Nm}(\mathfrak{a})=N}} \#\{\Gamma : \mathfrak{a}_\Gamma = \mathfrak{a}\}.$$

Let $0 \neq \alpha \in \mathfrak{a}$ such that $|\alpha| \ll N^{1/n}$. Then by Lemma 6 of [Sk]

$$\begin{aligned}\#\{\Gamma : \mathfrak{a}_\Gamma = \mathfrak{a}\} &\ll \#\{(\alpha_1, \dots, \alpha_m) \in \mathfrak{o}^m : \alpha_i \in \alpha R_K\} \\ &\ll \#\{(\alpha_1, \dots, \alpha_m) \in \mathfrak{o}^m : |\alpha_i| \ll |\alpha|\} \\ &\ll (|\alpha|^{n+\varepsilon})^m \ll N^{m+\varepsilon},\end{aligned}$$

whence

$$\#\{\Gamma : \mathrm{Nm}(\mathfrak{a}_\Gamma) = N\} \ll N^{m+\varepsilon} \#\{\mathfrak{a} : \mathrm{Nm}(\mathfrak{a}) = N\} \ll N^{m+\varepsilon}.$$

(ii) If $\mathbf{r} \in \mathcal{E}(\Delta)$, then

$$\begin{aligned}\left| r_i - \frac{\nu_i}{\alpha} \right| &\leqslant |\alpha^{-1}| \cdot |\alpha \mathbf{r}_i - \nu_i| \\ &\ll |\alpha|^{n-1} P^{-d+m(d-1)\Delta} \\ &\ll P^{-d+nm(d-1)\Delta} \ll P^{-d+\theta},\end{aligned}$$

whence $\mathbf{r} \in \mathfrak{M}_\Gamma(\theta)$ with $\gamma_i = \frac{\nu_i}{\alpha}$.

(iii) For a fixed Γ

$$\mathrm{meas}\{\mathbf{r} : |r_i - \gamma_i| \ll P^{-d+m(d-1)\Delta}\} \ll P^{(-md+m(d-1)\Delta m)n}.$$

For $\alpha \in \mathfrak{a}_\Gamma$, put

$$\mathcal{E}_{\alpha,\Gamma} = \{\mathbf{r} \in R^m : |\alpha r_i - \alpha\gamma_i| \ll P^{-d+m(d-1)\Delta}\}.$$

Since multiplication by α is a linear map on V^m with determinant $|\mathrm{Nm}(\alpha)|^m$, we find that

$$\mathrm{meas}(\mathcal{E}_{\alpha,\Gamma}) \ll \frac{P^{(-md+m(d-1)\Delta m)n}}{|\mathrm{Nm}(\alpha)|^m}.$$

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By (i) and Lemmas 4 and 6 of [Sk], it follows that

$$\begin{aligned}
\operatorname{meas}(\mathcal{E}(\Delta)) &\ll \sum_{\substack{\Gamma \\ \operatorname{Nm}(\mathfrak{a}_\Gamma) \ll P^{nm(d-1)\Delta}}} \sum_{\substack{\alpha \in \mathfrak{a}_\Gamma \\ |\alpha| \ll P^{m(d-1)\Delta}}} \operatorname{meas}(\mathcal{E}_{\alpha,\Gamma}) \\
&\ll \sum_{\substack{\Gamma \\ \operatorname{Nm}(\mathfrak{a}_\Gamma) \ll P^{nm(d-1)\Delta}}} \frac{P^{(-md+m(d-1)\Delta m)n}}{\operatorname{Nm}(\mathfrak{a}_\Gamma)^m} \left(\frac{P^{m(d-1)\Delta n}}{\operatorname{Nm}(\mathfrak{a}_\Gamma)} \right)^{1+\varepsilon} \\
&\ll \sum_{N \ll P^{nm(d-1)\Delta}} \sum_{\substack{\Gamma \\ \operatorname{Nm}(\mathfrak{a}_\Gamma) = N}} \frac{P^{(-md+m^2(d-1)\Delta + m(d-1)\Delta)n+\varepsilon}}{N^{m+1+\varepsilon}} \\
&\ll \sum_{N \ll P^{nm(d-1)\Delta}} \frac{P^{(-md+m(m+1)(d-1)\Delta)n+\varepsilon}}{N} \\
&\ll P^{(-md+m(m+1)(d-1)\Delta)n+\varepsilon}. \qquad \square
\end{aligned}$$

We now estimate the contribution to the integral $N(P)$ from the regions $\mathfrak{m}(\theta)$.

LEMMA 6. *For Δ sufficiently small*

$$\int_{\mathfrak{m}(nm(d-1)\Delta)} |S(\mathbf{r})| \, \mathrm{d}\mathbf{r} \ll P^{n(s-md)-\eta},$$

where $\eta = \eta(\Delta)$ is a positive number.

Proof. Choose E so that $E(m+1)(d-1) = d$. Then $\operatorname{meas}(R^m \backslash \mathcal{E}(E)) \leqslant 1$, so by our choice of κ and Lemma 2(i)

$$\int_{R^m \backslash \mathcal{E}(E)} |S(\mathbf{r})| \, \mathrm{d}\mathbf{r} \ll P^{ns-\kappa} \ll P^{n(s-md-E\delta)}. \tag{4.1}$$

Choose a set of values

$$\Delta = \Delta_0 < \Delta_1 < \cdots < \Delta_g = E.$$

Put

$$\mathcal{F}_i = \mathcal{E}(\Delta_i) \backslash \mathcal{E}(\Delta_{i-1}), \quad (i = 1, \ldots, g),$$

$$\mathcal{F}_{g+1} = R^m \backslash \mathcal{E}(E).$$

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By Lemma 5(ii)

$$\mathfrak{m}(nm(d-1)\Delta) \subseteq \bigcup_{i=1}^{g+1} \mathcal{F}_i. \tag{4.2}$$

Now, for $i \leqslant g$ it follows from Lemma 5(iii) that

$$\text{meas}(\mathcal{F}_i) \leqslant \text{meas}(\mathcal{E}(\Delta_i)) \ll P^{-nmd+nm(m+1)(d-1)\Delta_i+\varepsilon}.$$

Thus, for $i \leqslant g$

$$\int_{\mathcal{F}_i} |S(\mathbf{r})| \, \mathrm{d}\mathbf{r} \ll P^{n(s-md)+nm(m+1)(d-1)(\Delta_i-\Delta_{i-1})-n\delta\Delta_{i-1}+\varepsilon}.$$

If the Δ_i are chosen sufficiently close, say

$$\Delta_i - \Delta_{i-1} < \tfrac{1}{2}\delta\Delta/m(m+1)(d-1),$$

then

$$\int_{\mathcal{F}_i} |S(\mathbf{r})| \, \mathrm{d}\mathbf{r} \ll P^{n(s-md)-\frac{1}{2}n\delta\Delta+\varepsilon}. \tag{4.3}$$

The lemma follows upon combining (4.1), (4.2), and (4.3) and recalling that we may take ε arbitrarily small. □

Next, we estimate the contribution to $N(P)$ from $\mathfrak{M}(\theta)$. Put

$$\begin{aligned} S(\Gamma) &= \text{Nm}(\mathfrak{a}_\Gamma)^{-s} \sum_{\mathbf{x}(\mathfrak{a}_\Gamma)} \Phi(\Gamma \cdot \mathbf{G}(\mathbf{x})), \\ \mathfrak{S}(\Delta) &= \sum_{\substack{\Gamma \\ \text{Nm}(\mathfrak{a}_\Gamma) \ll P^{nm(d-1)\Delta}}} S(\Gamma), \end{aligned}$$

and

$$\mathfrak{J}(\Delta) = \int_{|\mathbf{t}| \leqslant P^{nm(d-1)\Delta}} \int_{\mathcal{B}} \Phi(\mathbf{t} \cdot \mathbf{F}(\mathbf{y})) \, \mathrm{d}\mathbf{y} \, \mathrm{d}\mathbf{t}.$$

LEMMA 7. *For $\Delta > 0$ sufficiently small*

$$\int_{\mathfrak{M}(nm(d-1)\Delta)} S(\mathbf{r}) \, \mathrm{d}\mathbf{r} = \mathfrak{S}(\Delta)\mathfrak{J}(\Delta)P^{n(s-md)} + O(P^{n(s-md)-\eta}),$$

for some $\eta = \eta(\Delta) > 0$.

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Proof. First observe that that if $\mathbf{r} \in \mathfrak{M}_{\Gamma_1} \cap \mathfrak{M}_{\Gamma_2}$ with $\mathrm{Nm}(\mathfrak{a}_{\Gamma_i}) \ll P^{nm(d-1)\Delta}$, then

$$|\gamma_{1i} - \gamma_{2i}| \leqslant |r_i - \gamma_{1i}| + |r_i - \gamma_{2i}| \ll P^{-d+nm(d-1)\Delta}.$$

By Lemma 5 of [Sk], there exists an $\alpha \in \mathrm{lcm}(\mathfrak{a}_{\Gamma_i})$ such that $|\alpha| \ll P^{2m(d-1)\Delta}$, whence

$$|\alpha(\gamma_{1i} - \gamma_{2i})| \ll P^{-d+2m(d-1)\Delta+nm(d-1)\Delta}.$$

It follows that if Δ is sufficiently small, then we must have that $\gamma_{1i} = \gamma_{2i}$ for all i and therefore $\Gamma_1 = \Gamma_2$. Thus, if Δ is small, the $\mathfrak{M}_\Gamma(nm(d-1)\Delta)$ are disjoint, and we have

$$\begin{aligned} \int_{\mathfrak{M}(nm(d-1)\Delta)} S(\mathbf{r})\,\mathrm{d}\mathbf{r} &= \sum_{\substack{\Gamma \\ \mathrm{Nm}(\mathfrak{a}_\Gamma) \ll P^{nm(d-1)\Delta}}} \int_{\mathfrak{M}_\Gamma(nm(d-1)\Delta)} S(\mathbf{r})\,\mathrm{d}\mathbf{r} \\ &= \sum_{\substack{\Gamma \\ \mathrm{Nm}(\mathfrak{a}_\Gamma) \ll P^{nm(d-1)\Delta}}} I(\Gamma, \Delta). \end{aligned} \tag{4.4}$$

For $\mathbf{r} \in \mathfrak{M}_\Gamma$, put $\mathbf{r} = \mathbf{z} + \Gamma$. Put $N = \mathrm{Nm}(\mathfrak{a}_\Gamma)$. Then

$$\begin{aligned} S(\mathbf{r}) &= \sum_{\mathbf{x} \in P\mathcal{B}} \Phi((\mathbf{z} + \Gamma) \cdot \mathbf{G}(\mathbf{x})) \\ &= \sum_{\mathbf{a}(N)} \sum_{\mathbf{b}} \Phi((\mathbf{z} + \Gamma) \cdot \mathbf{G}(\mathbf{a} + N\mathbf{b})) \\ &= \sum_{\mathbf{a}(N)} \Phi(\Gamma \cdot \mathbf{G}(\mathbf{a})) \sum_{\mathbf{b}} \Phi(\mathbf{z} \cdot \mathbf{G}(\mathbf{a} + N\mathbf{b})), \end{aligned} \tag{4.5}$$

where the summation in $\mathbf{b}$ is over integral points $\mathbf{b}$ such that $\mathbf{a} + N\mathbf{b}$ is in $P\mathcal{B}$.

Observe that $\Gamma = (\nu_1/N, \ldots, \nu_m/N)$ where $\nu_i = \nu_{i1}\omega_1 + \cdots + \nu_{in}\omega_n \in \mathfrak{n}^{m_d}$. Put

$$\Gamma^* = (\nu_{11}/N, \ldots, \nu_{1n}/N, \ldots, \nu_{mn}/N).$$

Then by (4.5) we see that

$$S(\mathbf{r}) = \sum_{\mathbf{A}(N)} \mathrm{e}(\Gamma^* \cdot \mathbf{G}^*(\mathbf{A})) \sum_{\mathbf{B}} \mathrm{e}(\mathbf{Z} \cdot \mathbf{G}^*(\mathbf{A} + N\mathbf{B})).$$

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The arguments of [S1, Lemma 9] and [S2, Section 9] give

$$\begin{aligned} S(\mathbf{r}) &= N^{-sn} \sum_{\mathbf{A}\ (N)} \mathrm{e}(\Gamma^* \cdot \mathbf{G}^*(\mathbf{A})) \int_{P\mathcal{B}} \mathrm{e}(\mathbf{Z} \cdot \mathbf{F}^*(\mathbf{U}))\, \mathrm{d}\mathbf{U} \\ &\quad + O(NP^{ns-1+nm(d-1)\Delta}) \\ &= S(\Gamma) \int_{P\mathcal{B}} \Phi(\mathbf{z} \cdot \mathbf{F}(\mathbf{u})\, \mathrm{d}\mathbf{u} + O(NP^{ns-1+nm(d-1)\Delta}). \end{aligned} \tag{4.6}$$

Integrating (4.6) over $\mathfrak{M}_\Gamma(nm(d-1)\Delta)$ gives

$$\begin{aligned} I(\Gamma, \Delta) &= S(\Gamma) \int_{\mathfrak{M}_\Gamma(nm(d-1)\Delta)} \int_{P\mathcal{B}} \Phi(\mathbf{z} \cdot \mathbf{F}(\mathbf{u}))\, \mathrm{d}\mathbf{u}\, \mathrm{d}\mathbf{z} \\ &\quad + O(NP^{ns-1+nm(d-1)\Delta} \mathrm{meas}(\mathfrak{M}_\Gamma)) \\ &= S(\Gamma) P^{n(s-md)} \int_{|\mathbf{t}| \ll P^{nm(d-1)\Delta}} \int_{\mathcal{B}} \Phi(\mathbf{t} \cdot \mathbf{F}(\mathbf{y}))\, \mathrm{d}\mathbf{y}\, \mathrm{d}\mathbf{t} \\ &\quad + O(P^{n((s-md)+m(nm+2)(d-1)\Delta)-1}) \\ &= S(\Gamma)\mathfrak{J}(\Delta)P^{n(s-md)} + O(P^{n(s-md)+nm(nm+2)(d-1)\Delta-1}). \end{aligned} \tag{4.7}$$

Substituting (4.7) into (4.4) gives

$$\int_{\mathfrak{M}(nm(d-1)\Delta)} S(\mathbf{r})\, \mathrm{d}\mathbf{r} = \mathfrak{S}(\Delta)\mathfrak{J}(\Delta)P^{n(s-md)} + O(P^{n(s-md)-\eta}),$$

by Lemma 5(i) provided Δ is sufficiently small. □

Now, we make a more detailed investigation of the term $\mathfrak{S}(\Delta)\mathfrak{J}(\Delta)$. Put

$$\mathfrak{S}(\infty) = \sum_{N=1}^{\infty} \sum_{\substack{\Gamma \\ \mathrm{Nm}(\mathfrak{a}_\Gamma)=N}} S(\Gamma),$$

and

$$\mathfrak{J}(\infty) = \int_{V^m} \int_{\mathcal{B}} \Phi(\mathbf{t} \cdot \mathbf{F}(\mathbf{y}))\, \mathrm{d}\mathbf{y}\, \mathrm{d}\mathbf{t}.$$

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LEMMA 8. *If G and κ are as in Lemma* 6, *then*

(i) $\mathfrak{S}(\infty)$ *converges absolutely, and*
(ii) $\mathfrak{S}(\Delta) - \mathfrak{S}(\infty) \ll P^{-\zeta}$ *for some positive* $\zeta = \zeta(\Delta)$.

Proof. Put $N = \mathrm{Nm}(\mathfrak{a}_\Gamma)$. Choose $\Delta = 1/nm(d-1) - \varepsilon'$. If alternative (i) of Lemma 2 holds for $P = N$, then

$$S(\Gamma) = N^{-sn} \sum_{\mathbf{x}(N)} \Phi(\Gamma \cdot \mathbf{G}(\mathbf{x}))$$

$$\ll N^{-sn} N^{sn-(m+1)-(\delta-\varepsilon)\Delta} \ll N^{-(m+1)-\eta}, \tag{4.8}$$

for some positive η if ε is sufficiently small. Alternative (ii) gives an α with $|\alpha| \ll N^{m(d-1)\Delta}$ such that

$$|\alpha\gamma_i - \nu_i| \ll N^{-d+m(d-1)\Delta}.$$

Let $0 \neq \beta \in \mathfrak{a}_\Gamma$ such that $|\beta| \ll N^{1/n}$. Then

$$|\beta\alpha\gamma_i - \beta\nu_i| \ll N^{-d+m(d-1)\Delta+1/n}.$$

Since $m(d-1)\Delta + 1/n < 2 \leqslant d$ for small ε, it follows that for large N we must have

$$\gamma_i = \frac{\nu_i}{\alpha} \quad \text{for all } i.$$

But $N \ll |\mathrm{Nm}(\alpha)| \ll |\alpha|^n \ll N^{nm(d-1)\Delta}$ and $nm(d-1)\Delta < 1$, giving a contradiction for N large. Thus, we may assume that (4.8) holds.

(i) By (4.8) and Lemma 5(i)

$$\mathfrak{S}(\infty) \ll \sum_{N=1}^{\infty} N^{m+\varepsilon} N^{-m-1-\eta} \ll \sum_{N=1}^{\infty} N^{-1-(\eta-\varepsilon)} < \infty,$$

since ε may be taken smaller than η.

(ii) Similarly

$$\mathfrak{S}(\Delta) - \mathfrak{S}(\infty) \ll \sum_{N \gg P^{nm(d-1)\Delta}} N^{-1-(\eta-\varepsilon)} \ll P^{-\zeta}.$$

□

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LEMMA 9. *With* **F** *and* κ *as above,*

(i) $\mathfrak{J}(\infty)$ *exists, and*
(ii) $\mathfrak{J}(\Delta) - \mathfrak{J}(\infty) \ll P^{-nm(d-1)\Delta}$.

Proof. Thinking of the system **F** as the rational system $\mathbf{F}^*$, the lemma follows from Lemma 8.1 and Section 3 of [S2]. □

We can now tie everything together to deduce the main part of our Theorem, namely the formula for $N(\mathbf{F}, \mathcal{B}, P)$. Put

$$\mu = \mu(\mathbf{G}, \mathcal{B}) = \mathfrak{S}(\infty)\mathfrak{J}(\infty).$$

PROPOSITION 1. *Suppose* **G** *is a system of polynomials satisfying the hypotheses of the Theorem. Then for any box* $\mathcal{B}$

$$N(P) = \mu P^{n(s-md)} + o(P^{n(s-md)}).$$

Proof. Choose Δ small. By Lemmas 6 and 7

$$N(P) = \mathfrak{S}(\Delta)\mathfrak{J}(\Delta)P^{n(s-md)} + o(P^{n(s-md)}).$$

By Lemmas 8 and 9

$$\begin{aligned}\mathfrak{S}(\Delta)\mathfrak{J}(\Delta) &= \mathfrak{S}(\infty)\mathfrak{J}(\infty) + O(P^{-\zeta}\mathfrak{J}(\infty)\\ &\quad + P^{-nm(d-1)\Delta}\mathfrak{S}(\infty) + P^{-nm(d-1)\Delta-\zeta}).\end{aligned}$$

It follows that

$$N(P) = \mu P^{n(s-md)} + o(P^{n(s-md)}).$$ □

It remains to check that $\mu > 0$ under the conditions stated in the Theorem.

PROPOSITION 2. *Suppose* **G** *satisfies the hypotheses of the Theorem. Then*

(i) $\mathfrak{S}(\infty) > 0$, *if* **G** *has a non-singular solution in* $\mathfrak{n}_v^s$ *for each finite place* v, *and*
(ii) $\mathfrak{J}(\infty) > 0$ *for some* $\mathcal{B}$, *if* **F** *has a non-singular solution in* K_v^s *for each infinite place* v *and if* $\dim(V(\mathbf{0})) = n - m$. *In particular, this holds for any* $\mathcal{B}$ *centered at a point* $\oplus\mathbf{z}_i$, *where each* $\mathbf{z}_i$ *is a non-singular solution over* K_i *of* **F**.

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Proof. Both of these results are standard. For a prime ideal $\mathfrak{p}$, put

$$\mu(\mathfrak{p}) = \sum_{j=1}^{\infty} \sum_{\substack{\Gamma \\ \mathfrak{a}_\Gamma = \mathfrak{p}^j}} S(\Gamma).$$

Then we have

$$\mathfrak{S}(\infty) = \prod_{\mathfrak{p}} \mu(\mathfrak{p}).$$

Let $\mathfrak{n}_\mathfrak{p}$ be the ideal in $\mathfrak{o}_\mathfrak{p}$ generated by $\mathfrak{n}$. If **G** has a non-singular solution in $\mathfrak{n}_\mathfrak{p}^s$, then standard arguments (e.g. [DL, Lemma 10]) are easily generalized to our setting, and they show that $\mu(\mathfrak{p}) > 0$ and (i) is true. Also, again thinking of **F** as $\mathbf{F}^*$, it is a straight-forward exercise to show that a non-singular solution of **F** in V^s corresponds to a non-singular solution of $\mathbf{F}^*$ in $\mathbb{R}^{sn}$, and *vice versa*. It is clear that $\dim(V^*(\mathbf{0})) = n \dim(V(\mathbf{0}))$ and $\dim(V^*_{\text{sing}}) = n \dim(V_{\text{sing}})$, whence (ii) follows from [B1, Section 6]. □

Proof of Theorem. Combine Propositions 1 and 2. □

5. Proof of corollary 1

In this section we deduce Corollary 1 and make some related comments.

Proof of Corollary 1. Let X and $\mathbf{F} = (f_1, \ldots, f_m)$ be as in the statement of the corollary. We assume that $X(K_v) \neq \emptyset$ for all places v. To prove that weak approximation holds for X, it suffices to prove that the following holds for X. For any $\varepsilon > 0$, any finite set of places S, and any set of points $\{\mathbf{x}_v = (x_{v1} : \cdots : x_{vs}) \in X(K_v) : v \in S\}$ there exists a point $\mathbf{x} = (x_1 : \cdots : x_s) \in X(K)$ such that

$$|x_i - x_{vi}|_v < \varepsilon,$$

for every i and every $v \in S$.

Let ε, S, and $\{\mathbf{x}_v\}_{v \in S}$ be given. By possibly adjusting ε, we may assume that $\mathrm{ord}_v(x_{vi}) \geqslant 0$ for every i and every $v \in S$. Write $S = S_\infty \cup S_f$, where S_∞ consists of infinite places, and S_f consists of finite places. By the Chinese Remainder Theorem, we can find $\mathbf{a} = (a_1, \ldots, a_s) \in \mathfrak{o}^s$ such that $|a_i - x_{vi}|_v < \varepsilon$ for every i and every $v \in S_f$. Let

$$r_v = \min_i \mathrm{ord}_v(a_i - x_{vi}),$$

and let $\mathfrak{p}_v$ be the prime ideal corresponding to v. Put

$$\mathfrak{n} = \mathfrak{n}_S = \prod_{v \in S_f} \mathfrak{p}_v^{r_v}.$$

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Let

$$\mathbf{G} = (f_1(\mathbf{y}+\mathbf{a}), \ldots, f_m(\mathbf{y}+\mathbf{a})).$$

Then $\mathbf{G}$ is a system of polynomials satisfying the hypotheses of the Theorem and whose associated homogeneous system is $\mathbf{F}$.

Let t be a positive integer. Let $C \equiv 1\ (\mathfrak{n}^t)$ be a positive integer such that

$$C > \frac{2c}{\varepsilon},$$

where $c = c(K, \omega)$ is a positive constant such that for any $r \in V$,

$$|r^{(i)}| \leqslant c|r|.$$

For any infinite place v, let

$$\mathbf{r}_v = \begin{cases} C\mathbf{x}_v, & \text{if } v \in S_\infty, \\ \text{any point of } X(K_v), & \text{otherwise.} \end{cases}$$

Since $X(K_v)$ is non-empty by assumption, such an $\mathbf{r}_v$ is always possible. Put

$$\mathbf{r}_0 = \bigoplus_{i=1}^{n_1+n_2} \mathbf{r}_{v_i},$$

where v_i is the infinite place corresponding to the embedding σ_i. Finally, let $\mathcal{B} = \mathcal{B}(\mathbf{r}_0)$ be the box centered at the point $\mathbf{r}_0$.

Applying the Theorem to $\mathbf{G}$, with $\mathfrak{n}$ and $\mathcal{B}$ chosen as above, we have that

$$N(\mathbf{G}, \mathcal{B}, P) = \mu(\mathbf{G}, \mathcal{B}) P^{n(s-md)} + o(P^{n(s-md)}),$$

where $N(\mathbf{G}, \mathcal{B}, P)$ counts zeros of $\mathbf{G}$ in $\mathfrak{n}^s \cap P\mathcal{B}$. For a finite place $v \notin S$, $\mathfrak{n}_v = \mathfrak{o}_v$, whence any point in $X(K_v)$ gives rise to a non-singular zero of $\mathbf{G}$ in $\mathfrak{n}_v$. For a finite $v \in S$, the point $\mathbf{x}_v - \mathbf{a}$ is a non-singular zero of $\mathbf{G}$, and certainly, $\mathbf{x}_v - \mathbf{a} \in \mathfrak{n}_v^s$. Thus, all the conditions needed for $\mu(\mathbf{G}, \mathcal{B})$ to be positive are satisified. It follows that for any sufficiently large integer $P \equiv 1\ (\mathfrak{n}^t)$, there exists a point $\mathbf{y} \in (\mathfrak{n}^s \cap P\mathcal{B})$ such that $\mathbf{x} = \mathbf{y} + \mathbf{a}$ is a zero of $\mathbf{F}$. In fact, for P large enough, the number of such solutions is greater than one, so we may assume that $\mathbf{y} \neq -\mathbf{a}$, so $\mathbf{x} \neq \mathbf{0}$.

For each $v \in S_\infty$ and each i, we have that

$$|CPx_{vi} - y_i| \leqslant cP,$$

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whence

$$\begin{aligned}\left|x_{vi} - \frac{1}{CP}x_i\right|_v &\leqslant \left|x_{v_i} - \frac{1}{CP}y_i\right|_v + \frac{|a_i|_v}{CP}\\ &\leqslant \frac{c}{C} + \frac{c|a_i|}{CP}\\ &< \varepsilon, \end{aligned} \tag{5.1}$$

for CP sufficiently large.

For each $v \in S_f$ and each i, we have that

$$\begin{aligned}\left|x_{vi} - \frac{1}{CP}x_i\right|_v &\leqslant |x_{vi} - x_i|_v + 2\\ &\leqslant |(x_i - a_i) - (x_{vi} - a_i)|_v + 2\\ &< 2\varepsilon, \end{aligned} \tag{5.2}$$

for $CP \equiv 1\ (\mathfrak{n}^t)$ for sufficiently large t.

It follows from (5.1) and (5.2) that weak approximation holds for X. □

As a particular case of Corollary 1, weak approximation holds for any smooth cubic hypersurface of dimension at least 15. One could, in fact do better in this case. For example, essentially following the arguments of Pleasants [P], we could significantly weaken the hypothesis of non-singularity. In light of the result of [Sk], one should also be able to show that weak approximation holds for any smooth cubic hypersurface of dimension at least 11.

A final remark. In [ERS] the asymptotic formulae that result from applying the Circle Method to non-singular quadratic forms are used to give a new proof of Siegel's Mass Formulae over $\mathbb{Q}$. One would hope to do the same for arbitrary number fields. Indeed, for smooth quadratic hypersurfaces of dimension at least 3, the desired asymptotics follow from our Theorem. Unfortunately, the inductive arguments of [ERS] require that they also hold in dimension 2. This appears to be out of reach of the methods of this paper. Over $\mathbb{Q}$, one uses the Kloosterman variant of the Circle Method, which makes great use of the Farey dissection of the unit interval, of which there is no satisfactory generalization. This was a limiting factor in [Sk] as well.

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