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ON NONCOMMUTING SETS AND CENTRALISERS IN INFINITE GROUPS

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Abstract

A subset X of a group G is a set of pairwise noncommuting elements if $ab \neq ba$ for any two distinct elements a and b in X. If $|X| \geq |Y|$ for any other set of pairwise noncommuting elements Y in G, then X is called a maximal subset of pairwise noncommuting elements and the cardinality of such a subset (if it exists) is denoted by $\omega(G)$. In this paper, among other things, we prove that, for each positive integer n, there are only finitely many groups G, up to isoclinism, with $\omega(G) = n$, and we obtain similar results for groups with exactly n centralisers.

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1. Introduction and results

Let G be a nonabelian group. We call a subset X of G a set of pairwise noncommuting elements if $ab \neq ba$ for any two distinct elements a and b in X. If $|X| \geq |Y|$ for any other set of pairwise noncommuting elements Y in G, then X is called a maximal subset of pairwise noncommuting elements, and the cardinality of such a subset (if it exists) is called the clique number of G, denoted by $\omega(G)$. By a famous result of Neumann [11] answering a question of Erdős, we know that the finiteness of $\omega(G)$ is equivalent to the finiteness of the factor group $G/Z(G)$, where $Z(G)$ is the centre of G. Pyber [12] showed that the size of $\omega(G)$ is related to the index of the centre of G: there is a constant c such that $[G : Z(G)] \leq c^{\omega(G)}$. The clique numbers of groups have been investigated by many authors (see, for example, [2, 6, 8]).

It is easy to see that if H is an arbitrary abelian group and G is a group with $\omega(G) = n$ then $\omega(G \times H) = n$. Therefore, there can be infinitely many groups K with $\omega(K) = n$. In this paper, we first show that the clique numbers of any two isoclinic groups [10] are the same (Lemma 2.1). By using this result, we show that for each positive integer n there are only finitely many groups G, up to isoclinism, with $\omega(G) = n$. We state our main results.

THEOREM 1.1. *Let n be a positive integer and G be an arbitrary group with $\omega(G) = n$.*

(1) *There are only finitely many groups H, up to isoclinism, with $\omega(H) = n$.*
(2) *There exists a finite group K such that K is isoclinic to G and $\omega(K) = n$.*

From this result, we deduce a sufficient condition for the solubility of a group in terms of its clique number.

THEOREM 1.2. *A group G with $\omega(G) \leq 20$ is soluble and this estimate is sharp.*

For any group G, let $C(G)$ denote the set of centralisers of G. We say that a group G has *n centralisers* (G is a C_n-group) if $|C(G)| = n$. Finally, we obtain similar results for groups with a finite number n of centralisers (Lemma 3.2 and Theorems 3.3–3.5).

2. Pairwise noncommuting elements

The groups G and H are said to be isoclinic if there are two isomorphisms $\varphi : G/Z(G) \to H/Z(H)$ and $\phi : G' \to H'$ such that if

$$\varphi(g_1 Z(G)) = h_1 Z(H) \quad \text{and} \quad \varphi(g_2 Z(G)) = h_2 Z(H),$$

with $g_1, g_2 \in G, h_1, h_2 \in H$, then

$$\phi([g_1, g_2]) = [h_1, h_2].$$

Isoclinism is an equivalence relation weaker than isomorphism and was introduced by Hall [10] to help classify groups. A stem group is defined as a group whose centre is contained inside its derived subgroup. It is known that every group is isoclinic to a stem group and if we restrict to finite groups, a stem group has the minimum order among all groups isoclinic to it (see [10] for more details).

To prove our main results, we need the following lemma.

LEMMA 2.1. *For every two isoclinic groups G and H, we have $\omega(G) = \omega(H)$.*

PROOF. Suppose that G and H are two isoclinic groups. From Hall [10], there exist commutator maps

$$\alpha : G/Z(G) \times G/Z(G) \longrightarrow G', \quad (xZ(G), yZ(G)) \mapsto ([x, y])$$

and

$$\alpha' : H/Z(H) \times H/Z(H) \longrightarrow H', \quad (xZ(H), yZ(H)) \mapsto ([x, y])$$

and isomorphisms $\beta : G/Z(G) \longrightarrow H/Z(H)$ and $\gamma : G' \longrightarrow H'$ such that

$$\alpha'(\beta \times \beta) = \gamma(\alpha)$$

where

$$\beta \times \beta : G/Z(G) \times G/Z(G) \longrightarrow H/Z(H) \times H/Z(H).$$

Now assume that the set $X = \{x_1, x_2, \ldots, x_n\}$ is a maximal subset of pairwise noncommuting elements of G. It follows that $x_i Z(G) \neq x_j Z(G)$ for all $1 \leq i < j \leq n$.

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Therefore, there exist n elements $y_i \in H \backslash Z(H)$ such that $\beta(x_i Z(G)) = y_i Z(H)$. To complete the proof it is enough to show that the set $Y = \{y_1, y_2, \ldots, y_n\}$ is a subset of pairwise noncommuting elements of H. Suppose, on the contrary, that there exist $y_i, y_j \in H$ for some $1 \leq i \neq j \leq n$, such that $[y_i, y_j] = 1$. As mentioned above,

$$\alpha'(\beta \times \beta)((x_i Z(G), x_j Z(G))) = \gamma(\alpha)(x_i Z(G), x_j Z(G))$$

and so $\alpha'(y_i Z(H), y_j Z(H)) = \gamma([x_i, x_j])$ and $1 = [y_i, y_j] = \gamma([x_i, x_j])$. It follows that $[x_i, x_j] = 1$, a contradiction. Thus $\omega(G) = |X| = |Y| \leq \omega(H)$ and so $\omega(G) \leq \omega(H)$. Similarly, $\omega(H) \leq \omega(G)$, and this completes the proof. □

Proof of Theorem 1.1. (1) Assume that G is a group with $\omega(G) = n$. From Pyber [12], there is a constant c such that $[G : Z(G)] \leq c^{\omega(G)} \leq f(n)$. Therefore, by Schur's theorem, the derived subgroup G' is finite (in fact, $|G'| \leq f(n)^{2f(n)^3}$) and the number of isomorphism types of $G/Z(G)$ and G' is bounded above by a function of n. For every choice of $G/Z(G)$ and G' there are only finitely many commutator maps from $G/Z(G) \times G/Z(G)$ to G'. It follows, in view of Lemma 2.1, that G is determined by finitely many isoclinism types.

(2) Since $\omega(G) = n$, by Pyber [12], G is a centre-by-finite group. On the other hand, according to the main theorem of Hall [10, page 135], there exists a group K such that G is isoclinic to K and $Z(K) \subseteq [K, K] = K'$. Since G is isoclinic to K, it follows that K is centre-by-finite and so, according to Schur's theorem, K' is finite. Therefore $Z(K)$ and $K/Z(K)$ are finite, so K is finite, and so Lemma 2.1 completes the proof. □

Proof of Theorem 1.2. Assume that G is a group with $\omega(G) \leq 20$. According to Theorem 1.1, there exists a finite group K such that G is isoclinic to K and $\omega(G) = \omega(K)$. Thus, replacing G by the factor group $G/Z(G)$, it can be assumed without loss of generality that G is a finite group with $\omega(G) \leq 20$. But in this case the result follows from the main result of [9]. Note that the alternating group of degree five, A_5, is a group with $\omega(A_5) = 21$ and so the estimate is sharp. □

3. Groups with a finite number of centralisers

As mentioned in the introduction, there are interesting relations between centralisers and pairwise noncommuting elements. So we now consider groups with a finite number n of centralisers (C_n-groups). From the result of Neumann [11], the finiteness of $\omega(G)$ in G is equivalent to the finiteness of the factor group $G/Z(G)$. Centralisers are subgroups containing the centre of the group, so from the finiteness of the factor group $G/Z(G)$ it follows that G has a finite number of centralisers. Also, if G has a finite number of centralisers, then it is easy to see that $\omega(G)$ is finite. These remarks give the following theorem.

Theorem 3.1. *For any group G, the following statements are equivalent.*

(1) *G has finitely many centralisers.*
(2) *G is a centre-by-finite group.*
(3) *G has finitely many pairwise noncommuting elements.*

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It is clear that a group is a C_1-group if and only if it is abelian. The class of C_n-groups was introduced by Belcastro and Sherman in [7] and investigated by many authors (see, for example, [1, 3, 4, 13, 14, 16]).

Since every group G with a finite number of centralisers is centre-by-finite, by an argument similar to the one in the proof of Lemma 2.1, we have the following result.

Lemma 3.2. *For every two isoclinic groups G and H, $|C(G)| = |C(H)|$.*

Proof. Let β be the isomorphism $\beta : G/Z(G) \longrightarrow H/Z(H)$ and let x be an element of G. There exists a subgroup K of H such that $\beta(C_G(x)/Z(G)) = K/H$. By an argument similar to the one in the proof of Lemma 2.1, there exists an element $y \in K$ such that $K = C_H(y)$ and $yZ(H) = \beta(xZ(G))$. The isomorphism β induces a bijection between the subgroups of G containing $Z(G)$ and the subgroups of H containing $Z(H)$, and the result follows. □

By an argument similar to the one in the proof of Theorem 1.1, we obtain the following result.

Theorem 3.3. *Let n be a positive integer and let G be an arbitrary C_n-group.*

(1) *There are only finitely many groups H, up to isoclinism, with $|C(H)| = n$.*
(2) *There exists a finite group K such that K is isoclinic to G and $|C(G)| = |C(K)|$.*

For any group G, it is easy to see that if $x, y \in G$ and $xy \neq yx$, then $C_G(x) \neq C_G(y)$. From this, it follows easily that $1 + \omega(G) \leq |C(G)|$ (note that $C_G(e) = G$, where e is the identity of G). Thus, by using Theorem 1.2, we generalise [15, Theorem A].

Theorem 3.4. *A group G with $|C(G)| \leq 20$ is soluble and this estimate is sharp.*

Finally, by using case (2) of Theorem 3.3, we generalise the main results of [1, 4, 5, 7] for infinite groups.

Theorem 3.5. *Let G be an arbitrary C_n-group.*

(1) $G/Z(G) \cong C_2 \times C_2$ *if and only if* $n = 4$.
(2) $G/Z(G) \cong C_3 \times C_3$ *or* S_3 *if and only if* $n = 5$.
(3) $G/Z(G) \cong D_8,\ A_4, C_2 \times C_2 \times C_2$ *or* $C_2 \times C_2 \times C_2 \times C_2$ *whenever* $n = 6$.
(4) $G/Z(G) \cong C_5 \times C_5, D_{10}$ *or* $\langle x, y | x^5 = y^4 = 1, x^y = x^3\rangle$ *if and only if* $n = 7$.
(5) $G/Z(G) \cong C_2 \times C_2 \times C_2, A_4$ *or* D_{12} *whenever* $n = 8$.

Proof. It is enough to note that there exists a finite C_n-group K such that K is isoclinic to G and hence $G/Z(G) \cong K/Z(K)$. So the statements in the theorem follow from the main results in [1, 4, 5, 7]. □

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