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SOME RESULTS ON COINCIDENCE POINTS

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In this paper we prove some coincidence point theorems for nonself single-valued and multivalued maps satisfying a nonexpansive condition. These extend fixed point theorems for multivalued maps of a number of authors.

1. INTRODUCTION

Let (X, d) be a complete metric space, M a nonempty subset of X, and for $S = X$ or $S = M$ let $CB(S)$ (respectively $K(S)$) denote the family of all nonempty *closed bounded* (respectively *compact*) subsets of S endowed with the Hausdorff metric H. A multivalued map T of M into $CB(X)$ is called a *contraction* if there exists a constant $h \in (0,1)$ such that $H\big(T(x), T(y)\big) \leqslant h\, d(x, y)$, for all $x, y \in M$. If we have the Lipschitz constant $h = 1$, then T is called a *nonexpansive* mapping. A point x in M is said to be a fixed point of T if $x \in T(x)$. Nadler [**15**] and Markin [**12**] initiated such a geometric approach to multivalued maps. In [**15**] Nadler proved a fixed point result for multivalued contraction maps of a complete metric space, which is a generalisation of the Banach Contraction Principle. Since then various well-known results for single-valued self contraction and nonexpansive mappings have been extended to multivalued analogues. For example, see [**4, 5, 9, 11, 17**].

On the other hand Kaneko [**8**] has introduced a notion of multivalued f-contraction map as follows. Let f be a single-valued continuous map of M into X. Then a multivalued map T of M into $CB(X)$ is called an *f-contraction* if there exists a constant $h \in (0,1)$ such that $H\big(T(x), T(y)\big) \leqslant h\, d\big(f(x), f(y)\big)$ for all $x, y \in M$. If we have the Lipschitz constant $h = 1$, then T is called a *f-nonexpansive* mapping. A point x in M is said to be a *coincidence point* of f and T if $f(x) \in T(x)$. We denote by $C(f \cap T)$ the set of coincidence points of f and T. In [**8**] Kaneko has proved coincidence and common fixed point results for self f-contraction maps, extending results of Jungck [**7**], Nadler [**15**] and others. Recently Daffer and Kaneko [**2**] have studied multivalued f-nonexpansive maps and extended results of Smithson [**19**] and Kaneko [**8**] for such maps of connected metric spaces, using the concept of an f-orbit of the multifunction as a major tool.

Geometric fixed point theory in Functional Analysis for such multivalued maps has been extensively developed. One of its developments has led to substantial weakenings

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in the assumption that the values of the mapping be subsets of its domain. For example, see [**1, 3, 6, 13, 18, 20, 21, 22**].

In this note we continue the geometric approach and obtain coincidence point results for nonself f-contraction and f-nonexpansive mappings without commutativity assumptions. In particular, we prove in Section 2 a coincidence point result (Theorem 2.1) for f-contraction maps in a complete metrically convex space. At the same time we also obtain a coincidence point result (Theorem 2.2) for such maps satisfying the weakly inward condition in a Banach space, which contains results of Reich [**18**] and Martinez-Yanez [**14**] as special cases. Applying these results for f-contraction maps in Section 3, we prove some more general results on coincidence points for f-nonexpansive maps, which in turn generalise results due to Assad and Kirk [**1**], Itoh and Takahashi [**6**], Yanagi [**20**], Zhang [**22**], and many others.

First we recall the following definitions. A metric space X is said to be *metrically convex* [**1**], if for each $x, y \in M$ with $x \neq y$, there exists $z \in X$, $x \neq z \neq y$, such that

$$d(x,z) + d(z,y) = d(x,y).$$

A Banach space X is said to be an *Opial space* [**16**] if for each sequence $\{x_n\}$ in X which converges weakly to x and for all $y \neq x$ we have

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|.$$

Hilbert spaces and Banach spaces having weakly continuous duality mappings are Opial spaces [**16**]. On the other hand it is well-known that L^p spaces ($p \neq 2$) are not Opial spaces [**9**], [**16**]. A multivalued map T of $M \subseteq X$ into 2^X (the family of nonempty subsets of X) is said to be: (i) *demiclosed* if for every sequence $\{x_n\} \subset M$ and any $y_n \in T(x_n)$, $n = 1, 2, \ldots$, such that $x_n \xrightarrow{w} x$ and $y_n \to y$, we have $x \in M$ and $y \in T(x)$. Here and throughout the paper $\to$ and $\xrightarrow{w}$ denote strong and weak convergence respectively; (ii) *weakly inward* if $T(x) \subset cl I_M(x)$ for closed M and $x \in M$, where $I_M(x) = \{z \in X : z = x + \lambda(y - x) \text{ for some } y \in M,\ \lambda \geqslant 1\}$. The set $I_M(x)$ has been called the *inward set* at x.
A subset M is said to be *star-shaped* with respect to $q \in M$ if $\{(1-\lambda)x + \lambda q : 0 < \lambda < 1\} \subset M$ for each $x \in M$. The point q is known as a *star-centre* of M. Clearly the star-shaped subsets include the convex subsets as a proper subclass.

2. Coincidence points for f-contraction maps

We start with a coincidence point result for complete metrically convex spaces.

Theorem 2.1. *Let M be a nonempty subset of a complete metrically convex space X. Let $f : M \to X$ be any map with its range G closed and $T : M \to CB(X)$ an f-contraction map such that $T(x) \subset G$ for all $f(x) \in \partial G$. Then $C(f \cap T) \neq \emptyset$.*

Proof: Define $J : G \to CB(X)$ by $J(z) = Tf^{-1}(z)$ for all $z \in G$. Note that for each $z \in G$ and any $x, y \in f^{-1}(z)$, the f-contractiveness of T implies

$$H\big(T(x), T(y)\big) \leqslant hd\big(f(x), f(y)\big) = 0$$

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and hence $J(z) = T(p)$ for all $p \in f^{-1}(z)$. Now we show that J is a contraction. For any $w, z \in G$, we have $H\big(J(w), J(z)\big) = H\big(T(x), T(y)\big)$ for any $x \in f^{-1}(w)$ and $y \in f^{-1}(z)$. But T is an f-contraction so there exists $h \in (0,1)$ such that

$$H\big(J(w), J(z)\big) \leqslant hd\big(f(x), f(y)\big) = hd(w, z),$$

which implies that J is a contraction map. Also note that $J(z) \subset G$ for every $z \in \partial G$. Thus by [**1**, Theorem 1], there is a point $z_0 \in G$ such that $z_0 \in J(z_0)$. Since $J(z_0) = T(x_0)$ for any $x_0 \in f^{-1}(z_0)$, so $f(x_0) \in T(x_0)$. ∎

If we take M and N to be subsets of a Banach space, then according to [**1**] the boundary of a closed set N relative to M is defined by

$$\partial_M(N) = \Big\{a \in N : B(a, r) \cap (M \backslash N) \neq \emptyset \text{ for each } r > 0\Big\},$$

where $B(a, r) = \Big\{x \in X : \|x - a\| < r\Big\}$.

Corollary 2.1. *Let M be a nonempty closed convex subset of a Banach space, N a subset of M. Let $f : N \to M$ be any map with its range G closed and let $T : N \to CB(M)$ be an f-contraction map such that $T(x) \subset G$ for all $f(x) \in \partial_M G$. Then $C(f \cap T) \neq \emptyset$.*

Proof: Since in this case M is a complete metrically convex space, the result follows if we replace M by N and X by M in the above theorem. ∎

For a more general boundary condition we have the following coincidence point result for general Banach spaces.

Theorem 2.2. *Let M be a nonempty subset of a Banach space X. Let $f : M \to X$ be any map with its range G closed and $T : M \to K(X)$ an f-contraction map such that $T(x) \subset clI_G(z)$ for all $x \in f^{-1}(z)$. Then $C(f \cap T) \neq \emptyset$.*

Proof: As in the proof of the above theorem, define

$$J(z) = Tf^{-1}(z) \text{ for all } z \in G.$$

Then $J(z) = T(p)$ for all $p \in f^{-1}(z)$ and J is a multivalued contraction map from G into $K(X)$. Also note that $J(z) \subset clI_G(z)$ for any $z \in G$; that is, J is weakly inward. Thus by [**21**, Theorem 2.1] there exists $z_0 \in G$ such that $z_0 \in J(z_0)$ and hence there exists $x_0 \in M$ such that $f(x_0) \in T(x_0)$. ∎

If $f = I$, the identity on M, and T is a single-valued map then we have the following fixed point result of Martinez-Yanez [**14**].

Corollary 2.2. *Let M be a nonempty closed subset of a Banach space X. Let $T : M \to X$ be a weakly inward contraction map. Then T has a unique fixed point.*

3. Coincidence points for f-nonexpansive maps

First, for the sake of completeness we give the proof of the following useful lemma [**10**].

Lemma 3.1. *Let M be a nonempty weakly compact subset of an Opial space X. Let $f : M \to X$ be a weakly continuous map and $T : M \to K(X)$ be an f-nonexpansive multivalued map. Then $f - T$ is demiclosed.*

Proof: Let $\{x_n\} \subset M$ and $y_n \in (f-T)x_n$ be such that $x_n \xrightarrow{w} x$ and $y_n \to y$. It is obvious that $x \in M$ and $f(x_n) \xrightarrow{w} f(x)$. Since $y_n \in f(x_n) - T(x_n)$, we get

$$y_n = f(x_n) - u_n, \qquad \text{for some } \; u_n \in T(x_n). \tag{3.1.1}$$

Since $T(x)$ is a compact set, there is a $v_n \in T(x)$ such that

$$\|u_n - v_n\| \leqslant H\big(T(x_n), T(x)\big) \leqslant \big\|f(x_n) - f(x)\big\|. \tag{3.1.2}$$

From (3.1.1) and (3.1.2), passing to the limit with respect to n, we obtain

$$\liminf_{n\to\infty}\big\|f(x_n) - f(x)\big\| \geqslant \liminf_{n\to\infty}\big\|u_n - v_n\big\| = \liminf_{n\to\infty}\|f(x_n) - y_n - v_n\|. \tag{3.1.3}$$

$T(x)$ being compact, for a convenient subsequence still denoted by $\{v_n\}$, we have $v_n \to v \in T(x)$. Then (3.1.3) yields

$$\liminf_{n\to\infty}\big\|f(x_n) - f(x)\big\| \geqslant \liminf_{n\to\infty}\big\|f(x_n) - y - v\big\|.$$

Since X is an Opial space and $f(x_n) \xrightarrow{w} f(x)$, this yields $f(x) = y + v$. Thus $y = f(x) - v \in f(x) - T(x)$, which proves that $f - T$ is demiclosed. ∎

The following result contains Theorem 2 of Assad and Kirk [**1**], which in turn improved a result of Lami Dozo [**9**].

Theorem 3.1. *Let M be a nonempty closed convex subset of an Opial space X and N a nonempty weakly compact subset of M. Let $f : N \to M$ be a weakly continuous map with its range G star-shaped and let $T : N \to K(M)$ be an f-nonexpansive map such that $T(x) \subset G$ for $f(x) \in \partial_M G$. Then $C(f \cap T) \neq \emptyset$.*

Proof: Let q be a star-centre of G; then for any $z \in G$ and any λ $(0 < \lambda < 1)$, $(1-\lambda)z + \lambda q \in G$. Also note that G is closed and bounded. Now, for each n, define

$$T_n(x) = (1 - h_n)T(x) + h_n q,$$

where $x \in N$ and $\{h_n\}$ is any sequence with $h_n \to 0$ $(n \to \infty)$ and $0 < h_n < 1$. Clearly, for each n, T_n maps N into $K(M)$. Now, if $z \in \partial_M(G)$, then $T(x) \subset G$ for any $x \in f^{-1}(z)$. Since G is star-shaped with respect to q, so $T_n(x) \subset G$ for any $x \in f^{-1}(z)$. Furthermore, we have

$$H\big(T_n(x), T_n(y)\big) \leqslant (1 - h_n)\big\|f(x) - f(y)\big\|,$$

for each n and any $x, y \in N$. By Corollary 2.1 there exists $x_n \in N$ such that

$$f(x_n) \in T_n(x_n) = (1 - h_n)T(x_n) + h_n q,$$

so there is some $u_n \in T(x_n)$ such that

$$f(x_n) = (1 - h_n)u_n + h_n q.$$

Thus,

$$\left\|f(x_n) - u_n\right\| = \frac{h_n}{1-h_n}\left\|q - f(x_n)\right\| \to 0 \text{ as } n \to \infty.$$

Since N is weakly compact, for a convenient subsequence still denoted by $\{x_n\}$, we have $x_n \xrightarrow{w} x_0 \in N$. Now as $f(x_n) - u_n \in (f - T)(x_n)$ and by Lemma 3.1, $f - T$ is demiclosed, we conclude that $0 \in (f - T)(x_0)$ and hence $f(x_0) \in T(x_0)$. This completes the proof. $\square$

Applying our Theorem 2.2, we have the following coincidence point results for general Banach spaces.

Theorem 3.2. *Let M be a nonempty subset of a Banach space X and let $f : M \to X$ with its range G closed, bounded and star-shaped. Let $T : M \to K(X)$ be an f-nonexpansive map which satisfies the following conditions:*

(i) $T(x) \subset clI_G(z)$ for all $x \in f^{-1}(z)$

(ii) $(f - T)M$ is closed.

Then $C(f \cap T) \neq \emptyset$.

Proof: Let q be a star-centre of G; then $I_G(z)$ is also star-shaped with respect to q for each $z \in G$ [**22**]. For each n, define $T_n : M \to K(X)$ by

$$T_n(x) = (1 - h_n)T(x) + h_n q,$$

where $x \in M$ and $\{h_n\}$ is any sequence with $h_n \to 0$ $(n \to \infty)$ and $0 < h_n < 1$. Then it is easy to see that for each n, T_n is an f-contraction map and $T_n(x) \subset clI_G(z)$ for all $x \in f^{-1}(z)$. By Theorem 2.2, there exists $x_n \in M$ such that $f(x_n) \in T_n(x_n)$ and hence, as in the proof of Theorem 3.1, $f(x_n) - u_n \to 0$ as $n \to \infty$ for some $u_n \in T(x_n)$. Since $(f - T)M$ is closed and $f(x_n) - u_n \in (f - T)M$, we get $0 \in (f - T)M$. Hence there is a point $x_0 \in M$ such that $f(x_0) \in T(x_0)$. $\square$

Theorem 3.3. *Let M be a nonempty weakly compact subset of a Banach space X and $f : M \to X$ a weakly continuous map with its range G star-shaped. Let $T : M \to K(X)$ be an f-nonexpansive map which satisfies the following conditions:*

(i) $T(x) \subset clI_G(z)$ for all $x \in f^{-1}(z)$

(ii) $f - T$ is demiclosed.

Then $C(f \cap T) \neq \emptyset$.

Proof: Note that G is weakly compact and hence it is a closed subset of X. Let q be a star-centre of G; then $I_G(z)$ is also star-shaped with respect to q. Now, following the proof of the above theorem we get a sequence $\{x_n\}$ in M and $u_n \in T(x_n)$ such that $f(x_n) - u_n \to 0$ as $n \to \infty$. Since M is weakly compact, for a convenient subsequence still denoted by $\{x_n\}$, we have $x_n \xrightarrow{w} x_0 \in M$. Hence by using demiclosedness of $f - T$, we obtain $0 \in (f - T)(x_0)$, that is, $f(x_0) \in T(x_0)$. $\square$

By virtue of Lemma 3.1, we have the following result for Opial spaces.

Corollary 3.1. *Let M be a nonempty weakly compact subset of an Opial space X and $f : M \to X$ a weakly continuous map with its range G star-shaped. Let $T : M \to K(X)$ be an f-nonexpansive map such that $T(x) \subset clI_G(z)$ for all $x \in f^{-1}(z)$. Then $C(f \cap T) \neq \emptyset$.*

If $f = I$, the identity on M, then Theorem 3.3 reduces to the following main fixed point result of Zhang [**22**], which in turn generalised a result of Yanagi [**20**].

COROLLARY 3.2. *Let M be a nonempty weakly compact star-shaped subset of a Banach space X. Let $T : M \to K(X)$ be a weakly inward nonexpansive map such that $I - T$ is demiclosed. Then T has a fixed point.*

The following result extends a Theorem of Itoh and Takahashi [**6**].

COROLLARY 3.3. *Let M be a nonempty weakly compact subset of an Opial space X and $f : M \to X$ a weakly continuous map with its range G star-shaped. Let $T : M \to K(X)$ be an f-nonexpansive map such that for each $z \in \partial G$, $T(x) \subset G$ for all $x \in f^{-1}(z)$. Then $C(f \cap T) \neq \emptyset$.*

PROOF: Since for all $z \in G$, $G \subset I_G(z)$ and $I_G(z) = X$ if z is an interior point of G [**22**], thus $T(x) \subset clI_G(z)$ for all $x \in f^{-1}(z)$ and hence the result follows by Corollary 3.1. ∎

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