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NILPOTENT EXTENSIONS OF ABELIAN p-GROUPS

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This paper arose out of an attempt to solve the following problem due to Suprunenko [**5**, Problem 2.77]. For which pairs of abelian groups A, B is every extension of A by B nilpotent? We obtain complete answers when A and B are p-groups and (a) A has finite exponent or (b) B is divisible or (c) A has infinite exponent, is countable and B is non-divisible. The structure of a basic subgroup of A plays a central role in cases (b) and (c).

1. Introduction. At the outset we must say that the problem is too difficult to solve in complete generality. If $G/A \cong B$, then the nilpotency of G depends solely on properties of the associated homomorphism $\theta:B \to \text{Aut } A$. Thus for instance if A is torsion-free and B finite, G is nilpotent if and only if the extension is a central one, and we would need detailed information on finite subgroups of the group Aut A. Such information is scanty apart from isolated results about free abelian groups of low rank. One could cite the examples of J. de Groot [**2**] of torsion free A for which Aut A is of order 2. Similar comments apply when A is a p-group and B a q-group for distinct primes p, q: nilpotent extensions again have to be central.

Thus we have focused attention on the case where A and B are both p-groups for the same prime p. Here, and generally, extensions constructed using non-nilpotent wreath products provide us with limits for possible general statements. Thus if A has a direct factor isomorphic to a restricted direct power $K^{(I)}$ of a non-trivial group K, where $|I|$ is no less than $|B|$ and B is infinite, then there is a non-nilpotent extension of A by B. Other insights come from looking at wreath products of the form C_{p^m} wr C_{p^n}.

For $p = 2$ everything is easy, due to the existence of the inverter automorphism; it is pretty obvious (Theorem 3.6) that every extension of an abelian group A by C_2, the cyclic group of order 2, is nilpotent if and only if A is a 2-group of finite exponent. Indeed for general p, the case where A has finite exponent is straightforward. When A has infinite exponent, everything is more complex and we are led to consider the three cases where B is divisible, finite, and infinite but not divisible respectively. In each case the conditions obtained for nilpotency are certain finite-rank conditions on the divisible part of A and on the homocyclic components of a basic subgroup of A. In all cases our results extend to general nilpotent

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B, and we are able to give complete answers to the Suprunenko problem in the following cases:

I. A an abelian 2-group, B a nilpotent 2-group (Theorem 3.6);

II. p odd, A an abelian p-group of finite exponent and B a nilpotent p-group (Theorems 3.1 and 3.2);

III. p odd, A an abelian p-group of infinite exponent and B a divisible (abelian) p-group (Theorem 3.5);

IV. p odd, A a countable abelian p-group of infinite exponent and B a non-divisible nilpotent p-group (Theorems 5.3 and 5.4).

The major portion of the paper is needed for cases III and IV. The countability condition in IV arises in the proof of Theorem 5.2, concerning the splitting off from a reduced p-group of "large" factors of a basic subgroup, the underlying reason being that Ulm's theorem is false for uncountable groups in general.

Our methods are a blend of abelian group theory and the commutator calculus of metabelian groups. By far the longest proof is that of Theorem 4.2 (the special case where A is a restricted direct product of cyclic p-groups and B is cyclic of order p). There, it is not the use of deep theorems that is needed but rather, detailed calculations using certain commutator identities repeatedly, together with induction arguments in two directions. The extension of Theorem 4.2 to general abelian groups in Section 5 is usually a relatively routine induction on the Ulm length, but not always (see for instance Step 2 in the proof of Theorem 5.3). Throughout, we have resisted the temptation to consider best possible bounds for the nilpotency class arising; we think that this would complicate matters without good enough cause.

2. Notation and preliminaries. We shall write all groups multiplicatively, even when the concern is exclusively with abelian groups. The cyclic group of order n is denoted by C_n, and the quasicyclic p-groups by C_{p^∞}. For any subset X of a group, $\langle X \rangle$ means the subgroup generated by X; for a p-group G, $\Omega_m(G)$ means

$$\langle x | x \in G \text{ and } x^{p^m} = 1 \rangle.$$

The lower central series of a group G is

$$G = \gamma_1(G) \geqq \gamma_2(G) = G' \geqq \ldots \geqq \gamma_n(G) \geqq \ldots,$$

as usual. The split extension of A by B is written $A \rtimes B$.

We need the following well-known commutator identity in metabelian groups. In it, $[x, ny]$ denotes the iterated commutator $[x, y, y, \ldots, y]$, with n occurrences of y.

(2.1) *For any elements x, y of a metabelian group G and any positive integer n,*

$$[x, y^n] = [x, y]^{\binom{n}{1}}[x, 2y]^{\binom{n}{2}} \dots [x, (n-1)y]^{\binom{n}{n-1}}[x, ny].$$

Also easy is

(2.2) *Let G be a group generated by an abelian normal subgroup A and an element b. Then*

(i) *G is nilpotent if and only if there exists n such that that* $[a, nb] = 1$ *for all a in A*;

(ii) *for every integer r,* $[a^r, b] = [a, b]^r$.

The first of these results follows from the fact that for $t \geqq 2$, $\gamma_{t+1}(G)$ is generated by all the iterated commutators $[a, tb]$, since A is abelian and normal. The second is because $[a, b]$ commutes with a.

Next, an abbreviation for a frequently occurring phrase:

(2.3) *For any groups A, B, the symbol* $\mathcal{N}(A, B)$ *denotes the statement that every extension of A by B is nilpotent.*

At the outset, we show that we can restrict attention to split extensions when A is abelian and B nilpotent. Thus suppose that A is abelian, B is nilpotent of class c and $G/A \cong B$. Then the kernel of the map $G \to \operatorname{Aut} A$ induced by conjugation contains A, and we have an induced map $\theta: B \to \operatorname{Aut} A$ giving rise to the split extension $S = A \rtimes B$. Thus A and B can be viewed as subgroups of S. We let $\bar{B}$ be the image of θ in Aut A and set $\bar{S} = A \rtimes \bar{B}$. With $[A, tG]$, $[A, tB]$ denoting the iterated mutual commutators in the usual way, it is clear that

$$A \geqq \gamma_{c+1}(G) \geqq [A, cG] = [A, cB] = \gamma_{c+1}(S)$$

since A is abelian and B of class c. Continuing, we find that

$$\gamma_i(G) \geqq \gamma_i(S) \geqq \gamma_{i+1}(G) \quad \text{for } i \geqq c + 1.$$

Thus G is nilpotent if and only if S is nilpotent. In general the class of G could be one more than that of S, but for cyclic B the classes are equal. Furthermore $[M, B] = [M, \bar{B}]$ for every subgroup M of A, and it follows that

$$\gamma_i(S) = \gamma_i(\bar{S}) \quad \text{for } i \geqq 2.$$

Thus S is nilpotent if and only if $\bar{S}$ is nilpotent; so in addition to proving that we need only consider split extensions, we also have that B can be viewed as a subgroup of Aut A when it is convenient to do so.

The following facts are in constant use in this article, and they are easily verified in view of the foregoing.

(2.4) *For abelian A* $\mathcal{N}(A, B)$ *if and only if the split extension associated with every homomorphism from B to* Aut *A is nilpotent.*

(2.5) $\mathcal{N}(A, B)$ *implies* $\mathcal{N}(A, C)$ *for every epimorphic image C of B.*

(2.6) *$\mathcal{N}(A_1 \times A_2, B)$ implies $\mathcal{N}(A_1, B)$.*

(2.7) *If C is a characteristic subgroup of A, then $\{\mathcal{N}(C, B)$ and $\mathcal{N}(A/C, B)\} \Rightarrow \mathcal{N}(A, B)$.*

When B is an infinite nilpotent group, it is readily shown that B has a countably infinite image. Thus, since the restricted wreath product K wr H has trivial center [**1**] when $K \neq 1$ and H is infinite, (2.5) and (2.6) allow us to state:

(2.8) *If A has a direct factor isomorphic to the countably infinite restricted power $K^{(N)}$ of a non-trivial group K, and B is an infinite nilpotent group, then $\neg \mathcal{N}(A, B)$.*

3. Extensions of p groups; the case of divisible B. Throughout this section we shall assume that p is prime and A and B are both p-groups, initially not necessarily abelian. Our first result is straightforward, and we omit the proof.

THEOREM 3.1. *If A is a finite p-group and B a nilpotent p-group, then $\mathcal{N}(A, B)$.*

The second is also easy, but not quite so elementary:

THEOREM 3.2. *Let A be an infinite abelian p-group of finite exponent and B a nilpotent p-group. Then $\mathcal{N}(A, B)$ if and only if B is finite.*

Proof. One way this follows is from the well-known theorem of G. Baumslag [**1**] stating that every extension of a nilpotent p-group of finite exponent by a finite p-group is nilpotent.

Conversely, suppose that $\mathcal{N}(A, B)$. As A is infinite abelian of finite exponent, it follows from Prüfer's theorem that A has a direct factor isomorphic with $C_{p^k}^{(n)}$ for some k and so (2.8) says that B must be finite, and the proof is complete.

When A has infinite exponent, the situation is more complicated and results are governed by the structure of a basic subgroup of A. We say that a subgroup L of an abelian p-group A is *basic* (see [**3**]) if

(i) L is a restricted direct product of cyclic groups,

(ii) L is pure in A,

(iii) A/L is divisible.

Every p-group has a basic subgroup, and all basic subgroups of a given group are isomorphic. Additional properties of basic subgroups that we shall need are:

(iv) If

$$L = \prod_{n=1}^{\infty} L_n,$$

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where L_n is the homocyclic component of exponent p^n, then for each n,

$$A = L_1 \times L_2 \times \ldots \times L_n \times A_n^*,$$

where

$$A_n^* = \langle A^{p^n}, L_n^* \rangle \quad \text{and} \quad L_n^* = \prod_{m>n} L_m.$$

(v) If A_1 is a homomorphic image of an abelian p-group A, a basic subgroup of A_1 is a homomorphic image of a basic subgroup of A.

(vi) $A = LA^p$.

We shall need the following definition when we come to investigate $\mathcal{N}(A, B)$ for divisible B.

Definition 3.3. A direct product of cyclic p-groups is said to be *thin* if the ranks of the homocyclic components are all finite.

It turns out that our investigation of the Suprunenko problem for A of infinite exponent requires that we treat the cases of B divisible and B non-divisible separately. Recall that a divisible subgroup of a nilpotent p-group is central, so that a divisible nilpotent p-group is abelian.

A crucial first step is to consider the case where B is the quasicyclic group C_{p^∞}.

THEOREM 3.4. *Let A be an abelian p-group. Then the following statements are equivalent.*

(i) $\mathcal{N}(A, C_{p^\infty})$;

(ii) Aut A *contains no subgroup isomorphic to* C_{p^∞};

(iii) *the divisible part D of A has finite rank and a basic subgroup L of A is thin*;

(iv) *A has no direct factor of the form $K^{(N)}$, where K is non-trivial.*

Proof. That (i) implies (ii) is easy; if $C_{p^\infty} \leqq \text{Aut } A$, the corresponding split extension would be a nilpotent p-group containing a non-central C_{p^∞}, which is impossible. Conversely, if (ii) holds then every homomorphism from C_{p^∞} to Aut A is trivial so that every extension G of A by C_{p^∞} is a central extension and thus nilpotent. In fact it is abelian since G has locally cyclic central factor-group.

The equivalence of (iii) and (iv) is immediate since D and every homocyclic factor of L are direct factors of A.

That (ii) implies (iii) is a direct consequence of (2.8).

The hard part of this theorem is the proof that (iii) implies (ii). We need some lemmas to enable us to do it.

LEMMA 3.4.1. *Let C be a characteristic subgroup of an abelian group A and suppose that A/C is a p-group. If $C_{p^\infty} \leqq$ Aut A, then either*

$$C_{p^\infty} \leqq \text{Aut } C \quad \textit{or} \quad C_{p^\infty} \leqq \text{Aut}(A/C).$$

Proof. Every automorphism of A induces one on C and one on A/C. Furthermore, the obvious map

$$\theta:\text{Aut } A \to \text{Aut } C \times \text{Aut}(A/C)$$

has kernel the stability group of the chain $1 \leqq C \leqq A$, which is isomorphic to $\text{Hom}(A/C, C)$. Since A/C is a p-group it is easily shown that $\text{Hom}(A/C, C)$ has no elements of infinite p-height and so $\ker \theta$ contains no C_{p^∞}. Thus

$$C_{p^\infty} \leqq \text{im } \theta \leqq \text{Aut } C \times \text{Aut}(A/C).$$

But any C_{p^∞} in the direct product $\text{Aut } A \times \text{Aut}(A/C)$ cannot be in the kernels of both projections, so

$$C_{p^\infty} \leqq \text{Aut } A \text{ or } C_{p^\infty} \leqq \text{Aut}(A/C).$$

COROLLARY. *Let A be an abelian p-group with divisible part D, so that $A = D \times R$ where R is reduced. Then $C_{p^\infty} \leqq \text{Aut } A$ if and only if*

$$C_{p^\infty} \leqq \text{Aut } D \quad \textit{or} \quad C_{p^\infty} \leqq \text{Aut } R.$$

LEMMA 3.4.2. *Let A be an abelian p-group, L a basic subgroup of A, and let L_n be the homocyclic component of L of exponent p^n. Then for each $n \geqq 1$, $L_1L_2 \dots L_nA^p$ is characteristic in A.*

Proof. Recall that

$$A = L_1L_2 \dots L_nA_n^*,$$

where

$$A_n^* = A^{p^n}L_n^* \quad \text{and} \quad L_n^* = \langle \cup L_m | m > n \rangle.$$

We shall show that $(L_1L_2 \dots L_n)^\alpha$ is contained in $L_1L_2 \dots L_nA^p$ for each $n \geqq 1$ and each $\alpha \in \text{Aut } A$.

For each $x \in L_1L_2 \dots L_n$,

$$x^\alpha = x_1y_1a^{p^n},$$

where

$$x_1 \in L_1L_2 \dots L_n, \, y_1 \in L_n^* \quad \text{and} \quad a \in A.$$

Since $x^{p^n} = x_1^{p^n} = 1$ it follows that

$$(y_1a^{p^n})^{p^n} = 1.$$

So

$$y_1^{p^n} = a^{-p^{2n}} \quad \text{and} \quad y_1^{p^n} \in L_n^*.$$

But L_n^* is a direct factor of the pure subgroup L and so it is itself pure in A. Thus

$$y_1^{p^n} = y_2^{p^n}, \quad \text{where } y_2 \in L_n^*.$$

Since

$$(y_1y_2^{-p^n})^{p^n} = 1,$$

it follows that $y_1y_2^{-p^n}$ is in $\Omega_n(L_n^*)$, which is contained in L_n^{*p}. Thus $y_1 \in L_n^{*p}$ and so

$$x^\alpha = x_1y_1a^{p^n} = x_1z_1^pa^{p^n}$$

which is in $L_1L_2 \ldots L_nA^p$, as required.

We return now to the proof of (iii) $\Rightarrow$ (ii) in Theorem 3.4. Write $A = D \times R$ as before, and suppose that (ii) is false. Then

$$C_{p^\infty} \leqq \mathrm{Aut}\, A.$$

From the corollary to Lemma 3.4.1, we have either

$$C_{p^\infty} \leqq \mathrm{Aut}\, D \quad \text{or} \quad C_{p^\infty} \leqq \mathrm{Aut}\, R.$$

A theorem of Hartley [**4**] states that the k-th direct power $C_{p^\infty}^{(k)}$ of C_{p^∞} contains an automorphism of order p^d if and only if

$$k \geqq p^{d-1}(p - 1).$$

Thus if $C_{p^\infty} \leqq \mathrm{Aut}\, D$, it follows that D has infinite rank, contradicting (iii). So we may assume that

$$C_{p^\infty} \leqq \mathrm{Aut}\, R.$$

At this point we can assume that A is reduced and that

$$C_{p^\infty} \leqq \mathrm{Aut}\, A.$$

Recall that each homocyclic component L_n of the basic subgroup L of A is finite. We shall derive the contradiction that C_{p^∞} acts trivially on A. From Lemma 3.4.2 it follows that Aut A acts on each factor-group $L_1L_2 \ldots L_nA^p/A^p$, and so C_{p^∞} does. Since each of these factor groups is finite, C_{p^∞} must act trivially on it. Now if x is any element of L, then x is in $L_1L_2 \ldots L_n$ for some n and so

$$x^\alpha \equiv x \bmod A^p.$$

Since $A = LA^p$, it follows that C_{p^∞} acts trivially on A/A^p. Thus in the split extension S of A by our C_{p^∞}, we have

$$[A, C_{p^\infty}] \leqq A^p.$$

This is a start. We consider now the descending series for A defined by $A_0 = A$, $A_{\lambda+1} = A_\lambda^p$ for successor ordinals $\lambda + 1$ and

$$A_\mu = \bigcap_{\lambda<\mu} A_\lambda$$

for limit ordinals μ. We shall establish by transfinite induction that

$$[A, C_{p^\infty}] \leqq A_\lambda \quad \text{for every } \lambda;$$

since A is reduced $A_\lambda = 1$ for some λ, and we will then have shown that $[A, C_{p^\infty}] = 1$, the final contradiction which will complete the proof.

We have seen that $[A, C_{p^\infty}] \leqq A_1$; so suppose that $\mu < 1$ and that

$$[A, C_{p^\infty}] \leqq A_\lambda \quad \text{whenever } \lambda > \mu.$$

If μ is a limit ordinal then

$$A_\mu = \bigcap_{\lambda < \mu} A_\lambda,$$

so by induction

$$[A, C_{p^\infty}] \leqq A_\mu.$$

If $\mu = \lambda + 1$, things are more complicated. Firstly write

$$C_{p^\infty} = \langle \alpha_1, \alpha_2, \dots | \alpha_1^p = 1, \alpha_{i+1}^p = \alpha_i \text{ for } i \geqq 1 \rangle.$$

Given that $[A, \alpha_i] \leqq A_\lambda$ for all i, we must show that

$$[A, \alpha_i] \leqq A_\lambda^p = A_\mu.$$

By the standard commutator identity (2.1), we have for each x in A,

$$[x, \alpha_i] = [x, \alpha_{i+1}^p] = [x, \alpha_{i+1}]^{\binom{p}{1}} [x, 2\alpha_{i+1}]^{\binom{p}{2}} \dots [x, (p-1)\alpha_{i+1}]^{\binom{p}{p-1}} [x, p\alpha_{i+1}].$$

Clearly, the first p factors are in A_λ^p since p divides $\binom{p}{k}$ when $1 \leqq k < p$. But

$$[x, \alpha_{i+1}] \leqq A_\lambda \leqq A_1 = A^p,$$

so

$$[x, 2\alpha_{i+1}] \leqq [A^p, \alpha_{i+1}] = [A, \alpha_{i+1}]^p \leqq A_\lambda^p$$

and so on repeatedly until we see that

$$[x, p\alpha_{i+1}] \in A_\lambda^p.$$

Thus

$$[x, \alpha_i] \in A_\lambda^p = A_\mu$$

and the proof of Theorem 3.4 is now complete.

We can now settle Supunenko's problem for the case where A is an abelian p-group and B is a divisible abelian p-group. Note that we have already done this when A has finite exponent for an arbitrary nilpotent p-group B.

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THEOREM 3.5. *Let A be an abelian p-group of infinite exponent with divisible part D and basic subgroup L, and let B be a non-trivial divisible abelian p-group, Then $\mathcal{N}(A, B)$ if and only if D has finite rank and L is thin.*

Proof. First suppose that $\mathcal{N}(A, B)$. Since B has C_{p^∞} as an image, $\mathcal{N}(A, C_{p^\infty})$ and Theorem 3.4 gives the result.

Conversely, suppose that D has finite rank and L is thin. By Theorem 3.4, Aut A has no C_{p^∞} as a subgroup, and so as B is divisible every homomorphism from B to Aut A is trivial, and so every extension of A by B is nilpotent, of class at most 2.

The case of 2-groups is really easy. Every abelian group A has the inverter automorphism $\alpha:a \to a^{-1}$, and the extension $\langle A, \alpha \rangle$ is nilpotent if and only if A has finite 2-power exponent. This is because every subgroup $\langle a, \alpha \rangle$ has to be nilpotent of bounded class, together with the observation that

$$[a, n\alpha] = a^{\pm 2^n}.$$

Thus we can state:

THEOREM 3.6. *Let A be an abelian 2-group and B a nilpotent 2-group. Then $\mathcal{N}(A, B)$ if and only if one of the following holds:*

(i) *A is finite;*

(ii) *A is an infinite group of finite exponent and B is finite;*

(iii) *A has infinite exponent, its divisible part is of finite rank and its basic subgroup is thin, and B is divisible.*

The proof follows from what has gone before; all we need to say is that if B is not divisible it has C_2 as an image and contemplation of the inverter automorphism of A gives what we want.

The remaining cases are sufficiently long that we go to a new section.

4. Extensions of p-groups: sparse direct products of cyclic groups. In our investigation of p-groups we are now left with the situation where the prime p is odd, A is an abelian p-group of infinite exponent, and B a non-divisible nilpotent p-group. A crucial special case is that where A is a (restricted) direct product of cyclic groups and B is of order p. The reason for its importance is the central role played by the basic subgroup of the bottom group in the general case.

Moreover, direct products of cyclic groups yield all we need in the way of counterexamples. For let W_n stand for the wreath product $C_{p^{n+1}}$ wr C_p of a cyclic group of order p^{n+1} by one of order p. Then as is easy to see, for $t \leqq p$ the factor group $W_n/Z_t(W_n)$ of W_n by the t-th term of its upper central series is an extension of

$$C_{p^{n+1}}^{p-t} \times C_{p^n}^{t-1}$$

by C_p and, by [**6**], it has class $(n + 1)p - n - t$, which is at least $n(p - 1)$. Thus we can state:

Example 4.1. Suppose that A is a restricted direct product of cyclic p-groups and that the ranks r_n of the homocyclic components A_n of the exponent p^n satisfy the inequality

$$r_n + r_{n+1} \geqq p - 1$$

for infinitely many n. Then there is an automorphism b of A of order p such that the split extension $\langle A, b\rangle$ is not nilpotent.

For the conditions give that A has a direct factor X expressible as a direct product $X_1 \times X_2 \dots$, where X_i is of the form

$$C_{p^{n_i+1}}^{p-t_i} \times C_{p^{n_i}}^{t_i-1}$$

for suitable $t_i \leqq p$, and $n_i \to \infty$ as $i \to \infty$. Thus each X_i has an automorphism b_i of order p such that the split extension $\langle X_i, b_i\rangle$ is isomorphic with

$$W_{n_i}/Z_{t_i}(W_{n_i})$$

and so has class going to infinity with i. The b_i extend to an automorphism of order p of A in the obvious way, and the extension $\langle A, b\rangle$ is not nilpotent since it contains subgroups of unbounded class.

Let A be any direct product of cyclic p-groups, A_n the homocyclic component of exponent p^n, and r_n the rank of A_n. We introduced the adjective "thin" in Section 3 in dealing with the case of divisible B; in the case of cyclic B the example just made indicates that we need a more stringent condition.

Definition 4.1. A is said to be *sparse* if

$$r_n + r_{n+1} \leqq p - 2 \quad \text{for all } n \geqq 1;$$

eventually sparse if there exists n_0 such that

$$r_n + r_{n+1} \leqq p - 2 \quad \text{for all } n \geqq n_0.$$

As an aside, we could observe that a sparse 2-group is trivial and that an eventually sparse 2-group is of finite exponent, so that 2-groups fit into the general picture correctly. However we have kept the cases separate, since it was so easy to deal with 2-groups by themselves.

The rest of this section is devoted to the proof of the following result:

THEOREM 4.2. *Let A be a sparse direct product of cyclic p-groups and let r_n be the rank of the homocyclic component of exponent p^n. For every automorphism b of A of order p, the split extension $G = A \rtimes \langle b\rangle$ is nilpotent of class $c \leqq r_1 + 2 \leqq p$.*

Proof. The main thrust will be to prove that $c \leqq p$ and then a few more lines will give $c \leqq r_1 + 2$. Our calculations are based essentially on the following easy result:

LEMMA 4.2.1. *Let G be a p-group generated by an abelian normal subgroup A and an element b of order p^n.*

(i) *If G is nilpotent of class c, then G' has exponent dividing $p^{(c-1)n}$;*

(ii) *If G' has exponent p^e, then G is nilpotent of class at most $ep^n - (e-1)p^{n-1}$;*

(iii) *G' has exponent p^e if and only if b centralizes A^{p^e}.*

Proof. (i) We proceed by induction on c. For $c = 1$ everything is trivial; so we assume that $c > 1$ and that the result is true for groups of class $< c$. Then by induction $G'/\gamma_c(G)$ has exponent at most $p^{(c-2)n}$. But $\gamma_c(G)$ is generated by all commutators of the form

$$d = [x_1, x_2, \dots, x_c] \quad \text{where } x_i \in A \cup \{b\};$$

as A is abelian d can be non-trivial only if

$$x_1 \in A \quad \text{and} \quad x_2 = \dots = x_c = b.$$

Thus as d is central,

$$d^{p^n} = [x_1, b, b, \dots, b]^{p^n} = [x_1, b, \dots, b^{p^n}] = 1,$$

and so $\gamma_c(G)$ is generated by elements of order dividing p^n. But then $\gamma_c(G)$ has exponent p^n and it follows that G' has exponent dividing

$$p^{(c-2)n} \cdot p^n = p^{(c-1)n}.$$

(ii) In this case G' is generated by elements $x = [a, b]$, with $a \in A$. As $b^{p^n} = 1$, we have from (2.1) that

$$1 = [a, b^{p^n}] = [a, b][a, b]^b[a, b]^{b^2} \dots [a, b]^{b^{p^n-1}} = xx^b \dots x^{b^{p^n-1}}.$$

It follows that $\langle x, b \rangle$ is a homomorphic image of the central factor-group of C_{p^e} wr C_{p^n}, so its class is at most

$$ep^n - (e-1)p^{n-1} - 1$$

(see [**6**]). Thus for every $a \in A$,

$$[a, tb] = 1 \quad \text{where } t = ep^n - (e-1)p^{n-1}$$

and (ii) follows since A is abelian.

(iii) is immediate from 2.2 (ii).

We turn now to the proof of Theorem 4.2. We will show that $[A, b]^p = 1$, so that Lemma 4.2.1 gives that G is nilpotent of class at most p. The strategy is to show first that

$$[A, b]^p \leqq A^{p^2},$$

and then inductively that

$$[A, b]^p \leqq A^{p^k} \quad \text{for } k \geqq 1,$$

so that $[A, b]^p = 1$ since

$$\bigcap_{k=1}^{\infty} A^{p^k} = 1.$$

The first step embodies most of the techniques used in the induction step. Observe that when A is sparse, so is A^{p^k} for every k; the sequence of ranks for A^{p^k} is $r_{k+1}, r_{k+2}, \dots$ and the sum of two successive ranks is still at most $p - 2$. This fact will be used repeatedly: after establishing some statement for A we interpret it in A^{p^k} and then by taking p^k-th roots we obtain an additional statement for A. We should point out at this stage that any statement we make will become a triviality should b happen to centralize some A^{p^k} (as we will know must happen, once we have proved the theorem).

Let us repeat here that A is sparse, that r_n is the rank of A_n, the homocyclic component of exponent p^n, and that $b^p = 1$. We will be working inside the split extension $G = \langle A, b \rangle$, and have to make a large number of observations along the way, which we number consecutively. We shall use repeatedly elementary commutator identities, often without comment.

(1) $[A_1, r_1 b] \leqq A^p \cap \Omega_1(A) = \langle A_2^p, A_3^{p^2}, \dots \rangle.$

This is because in A/A^p, the subgroup A_1A^p/A^p is invariant under bA^p, and of order p^{r_1}; so $\langle A_1A^p/A^p, bA^p \rangle$ has class at most r_1. Since (1) holds for all sparse groups it holds for A^p so that

(2) $[A_2^p, r_2 b] \leqq A^{p^2} \cap \Omega_1(A^p) = \langle A_3^{p^2}, A_4^{p^3}, \dots \rangle.$

On taking p-th roots in A, (2) gives

(3) $[A_2, r_2 b] \leqq \langle A_1, A_2^p, A_3^p, A_4^{p^2}, \dots \rangle.$

When (3) is interpreted in A^p, it gives

(4) $[A_3^p, r_3 b] \leqq \langle A_2^p, A_3^{p^2}, A_4^{p^2}, A_5^{p^3}, \dots \rangle.$

Combining (1) and (2) gives

(5) $[A_1, (r_1 + r_2)b] \leqq A^{p^2}.$

When this is interpreted in A^p it gives

(6) $[A_2^p, (r_2 + r_3)b] \leqq A^{p^3}.$

Combining (3) and (1) gives

(7) $[A_2, (r_1 + r_2)b] \leqq \langle A_2^p, A_3^p, A_4^{p^2}, \dots \rangle.$

When this is interpreted in A^p it gives

(8) $[A_3^p, (r_2 + r_3)b] \leqq \langle A_3^{p^2}, A_4^{p^2}, A_5^{p^3}, \dots \rangle.$

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Now interpreting (7) in $A^{p^{s-2}}$ and taking p^{s-2}-th roots,

(9) $$[A_s, (r_{s-1} + r_s)b] \leqq \langle A_1, \ldots, A_{s-2}, A_{s-1}^p, A_s^p, A_{s+1}^p, A^{p^2}\rangle.$$

We can now proceed with the main steps of the proof.

Step I. $[A_2, b]^p \leqq A^{p^2}$.

We know from (2) that

$$[A_2, r_2b]^p \leqq A^{p^2}.$$

If $r_2 = 0$ or 1, we are done. Otherwise $r_2 \geqq 2$. We now turn to the basic identity (2.6). As $b^p = 1$, we have for every $a \in A_2$

$$[a, b]^p[a, 2b]^{\binom{p}{2}} \ldots [a, (p - 1)b]^p[a, pb] = 1.$$

Commutating this with b $r_2 - 2$ times,

$$[a, (r_2 - 1)b]^p[a, r_2b]^{\binom{p}{2}} \ldots [a, (r_2 + p - 3)b]^p$$
$$\times [a, (r_2 + p - 2)b] = 1.$$

Since p divides each binomial coefficient, it follows from this and the fact that

$$[A_2, r_2b]^p \leqq A^{p^2}$$

that all but the first and last terms of this identity are in A^{p^2}. If we can show that the last term is, we will have that

$$[a, (r_2 - 1)b]^p \in A^{p^2},$$

and then we can repeat the argument if $r_2 - 1 \geqq 2$, until we have arrived at a proof that

$$[a, b]^p \in A^{p^2} \quad \text{for all } a \in A_2,$$

which is Step I.

To do this, note that

$$[A_2, r_2b] \leqq \langle A_1, A_2^p, A_3^p, A^{p^2}\rangle;$$

so

$$[A_2, (r_2 + p - 2)b]$$
$$\leqq \langle [A_1, (p - 2)b], [A_2^p, (p - 2)b], [A_3^p, (p - 2)b], A^{p^2}\rangle.$$

But since $r_1 + r_2 \leqq p - 2$, (5) gives that

$$[A_1, (p - 2)b] \leqq A^{p^2};$$

similarly (6) and $r_2 + r_3 \leqq p - 2$ give that

$$[A_2^p, (p - 2)b] \leqq A^{p^2}.$$

Finally (8) with $r_2 + r_3 \leqq p - 2$ implies that

$$[A_3^p, (p - 2)b] \leqq A^{p^2}.$$

We have now done enough to establish Step I.

Step II. If $[A_i, b]^p \leqq A^{p^2}$ for all $i = 1, 2, \ldots, s - 1$, then

$$[A_s, b]^p \leqq A^{p^2}.$$

Starting with the hypothesis that

$$[A_{s-1}, b]^p \leqq A^{p^2},$$

taking p-th roots in A and using the fact that

$$[A_{s-1}, b] \leqq \Omega_{s-1}(A)$$

we obtain

$$[A_{s-1}, b] \leqq \langle A_1, A_2^p, \ldots, A_s^p, A_{s+1}^{p^2}, \ldots \rangle.$$

Interpreting this statement in the sparse group A^p gives

(10) $$[A_s^p, b] \leqq \langle A_2^p, A_3^{p^2}, \ldots, A_{s+1}^{p^2}, A_{s+2}^{p^3}, \ldots \rangle;$$

and taking p-th roots we have

(11) $$[A_s, b] \leqq \langle A_1, A_2, A_3^p, \ldots, A_{s+1}^p, A_{s+2}^{p^2}, \ldots \rangle.$$

We show next that

(12) $$[A_s, r_s b]^p \leqq A^{p^2}.$$

As before we have for all $a \in A_s$,

$$[a, r_s b]^p [a, (r_s + 1)b]^{\binom{p}{2}} \ldots [a, (r_s + p - 2)b]^p$$
$$\times [a, (r_s + p - 1)b] = 1.$$

When we commutate (10) with b r_s times and use Step I, we find that all but the first and last factors of this identity are in A^{p^2}. To show that the last factor is also in A^{p^2}, we commutate (11) $(r_s + p - 2)$ times with b obtaining

(13) $$[A_s, (r_s + p - 1)b] \leqq \langle [A_1, (r_s + p - 2)b], [A_2, (r_s + p - 2)b], [A_3^p, (r_s + p - 2)b], \ldots, [A_{s+1}^p, (r_s + p - 2)b], A^{p^2} \rangle.$$

Now we have four rather different situations to look at. From (5) we have

$$[A_1, (p - 2)b] \leqq A^{p^2},$$

so certainly

$$[A_1, (r_s + p - 2)b] \leqq A^{p^2}.$$

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For the second term in (13), we begin by taking p-th roots in the statement of Step I and obtain

$$[A_2, r_s b] \leqq \langle A_1, A_2^p, A_3^p, A^{p^2} \rangle,$$

so that

$$[A_2, (r_s + p - 2)b] \leqq \langle [A_1, (p - 2)b], [A_2^p, (p - 2)b], [A_3^p, (p - 2)b], A^{p^2} \rangle.$$

Now (5), (6) and (8) together with

$$r_1 + r_2 \leqq p - 2, r_2 + r_3 \leqq p - 2$$

are sufficient to give

$$[A_2, (r_s + p - 2)b] \leqq A^{p^2}.$$

From the hypothesis in Step II, we see that

$$[A_j^p, (r_s + p - 2)b] \leqq A^{p^2} \quad \text{for } 3 \leqq j \leqq s - 1;$$

and it remains for us to consider

$$[A_s^p, (r_s + p - 2)b] \quad \text{and} \quad [A_{s+1}^p, (r_s + p - 2)b]$$

in (13). By (10)

$$[A_s^p, r_s b] \leqq \langle A_2^p, A^{p^2} \rangle$$

from which it follows that

$$[A_s^p, (r_s + p - 2)b] \leqq \langle [A_s^p, (p - 2)b], A^{p^2} \rangle$$

which again by (6) is contained in A^{p^2}.

Turning now to $[A_{s+1}^p, (r_s + p - 2)b]$, we have from (9) with the indices shifted by one that

$$[A_{s+1}, (r_s + r_{s+1})b] \leqq \langle A_1, A_2, \dots, A_{s-1}, A_s^p, A_{s+1}^p, A_{s+2}^p, A^{p^2} \rangle.$$

Thus

$$[A_{s+1}^p, (r_s + r_{s+1})b] \leqq \langle A_2^p, \dots, A_{s+1}^p, A^{p^2} \rangle$$

and so

$$[A_{s+1}^p, (p - 2)b] \leqq \langle A_2^p, \dots, A_{s-1}^p, A^{p^2} \rangle$$

which means that

$$[A_{s+1}^p, (r_s + p - 2)b] \leqq \langle [A_2^p, r_s b], \dots, [A_{s-1}^p, r_s b], A^{p^2} \rangle.$$

The hypotheses of Step II now give that this is in A^{p^2}, and (12) is now established.

The strategy now is to reduce (12) to what we want, viz to

$$[A_s, b]^p \leqq A^{p^2}.$$

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First we take p-th roots in (12), to obtain

(14) $[A_s, r_s b] \leqq \langle A_1, A_2^p, \dots, A_{s+1}^p, A^{p^2}\rangle.$

Interpret this in A^p:

(15) $[A_{s+1}^p, r_{s+1}b] \leqq \langle A_2^p, A_3^{p^2}, \dots, A_{s+2}^{p^2}, A^{p^3}\rangle.$

From this we obtain

$$[A_{s+1}^p, (r_s + r_{s+1})b] \leqq \langle [A_2, r_s b]^p, A^{p^2}\rangle$$

and application of Step I gives

(16) $[A_{s+1}^p, (r_s + r_{s+1})b] \leqq A^{p^2}.$

The last part of the proof that $[A, b]^p \leqq A^{p^2}$ comes from the

LEMMA. *If* $[A_s, kb]^p \leqq A^{p^2}$ *and* $k \geqq 2$, *then*

$$[A_s, (k-1)b]p \leqq A^{p^2}.$$

Proof. For every $a \in A_s$ we have

$$[a, (k-1)b]^p[a, kb]^{\binom{p}{2}} \dots [a, (k+p-3)b]^p$$
$$\times [a, (k+p-2)b] = 1;$$

and as so often before, what we have to do is to show that the last factor in this identity is in A^{p^2}. On taking p-th roots the hypothesis gives

$$[A_s, kb] \leqq \langle A_1, A_2^p, \dots, A_{s+1}^p, A_{s+2}^{p^2}, \dots\rangle.$$

Hence

$$[A_s, (k+p-2)b] \leqq \langle [A_1, (p-2)b], [A_2^p, (p-2)b], \dots, [A_{s+1}^p, (p-2)b], A^{p^2}\rangle.$$

From (5)

$$[A_1, (p-2)b] \leqq A^{p^2},$$

whereas for $2 \leqq j \leqq s-1$ we have that

$$[A_j^p, (p-2)b] \leqq A^{p^2}$$

by our induction hypothesis in Step II. So again we have two terms to consider. From (12) and the fact that $r_s \leqq p-2$ we have

$$[A_s^p, (p-2)b] \leqq A^{p^2}.$$

From (16) and $r_s + r_{s+1} \leqq p-2$, it follows that

$$[A_{s+1}^p, (p-2)b] \leqq A^{p^2}.$$

This establishes the lemma; together with (12) we now have

$$[A_s, b]^p \leqq A^{p^2}.$$

This completes the induction in Step II and we can state

(17) $[A, b]^p \leqq A^{p^2}$.

The last stage in the proof of the theorem is to prove:

Step III. If $[A, b]^p \leqq A^{p^k}$ for some $k \geqq 2$, then $[A, b]^p \leqq p^{k+1}$.

Firstly note that (17) and the hypotheses of Step III give

(18) $[A^{p^2}, b] \leqq A^{p^{k+1}}$ and $[A, 2b]^p \leqq A^{p^{k+1}}$.

We establish Step III by showing that

$$[A_n, b]^p \leqq A^{p^{k+1}} \text{ for all } n.$$

Our induction begins with the statement

(19) $[A_2, b]^p \leqq A^{p^{k+1}}$.

To see this, observe that for a in A_2,

$$[a, b]^p[a, 2b]^{\binom{p}{2}} \dots [a, (p - 1)b]^p[a, pb] = 1.$$

From (18) we can see that all the factors but the first and last are in $A^{p^{k+1}}$. From (7) we have

$$[A_2, (r_1 + r_2)b] \leqq \langle A_2^p, A_3^p, A^{p^2}\rangle$$

so that

$$[A_2, (p - 2)b] \leqq \langle A_2^p, A_3^p, A^{p^2}\rangle.$$

Thus

$$[A_2, pb] \leqq \langle [A_2, 2b]^p, [A_3, 2b]^p, [A, 2b]^{p^2}\rangle.$$

But (18) now gives that

$$[A_2, pb] \leqq A^{p^{k+1}}$$

and so

$$[A_2, b]^p \leqq A^{p^{k+1}}.$$

Next we prove:

If $[A_i, b]^p \leqq A^{p^{k+1}}$ for $1 \leqq i \leqq s - 1$ and $s \geqq 3$, then

(20) $[A_s, b]^p \leqq A^{p^{k+1}}$.

Taking p-th roots in (19) and noting that $k \geqq 2$, we get

$$[A_2, b] \leqq \langle A_1, A_2^p, A_3^{p^2}, \dots, A_{k+1}^{p^k}, A_{k+2}^{p^k}, A_{k+3}^{p^{k+1}}, \dots\rangle.$$

When this is interpreted in $A^{p^{s-2}}$ and p^{s-2}-th roots are taken, it gives

(21) $[A_s, b] \leqq \langle A_1, \dots, A_{s-1}, A_s^p, A_{s+1}^{p^2}, \dots\rangle$.

From this we have immediately

(22) $[A_s^p, b] \leqq \langle A_2^p, \ldots, A_{s-1}^p, A^{p^2}\rangle$.

Next we show that

(23) $[A_s^p, r_s b] \leqq A^{p^{k+1}}$.

As usual we begin with an identity

$$[a, r_s b]^p [a, (r_s + 1)b]^{\binom{p}{2}}$$
$$\ldots [a, (r_s + p - 2)b]^p [a, (r_s + p - 1)b] = 1,$$

valid for $a \in A_s$. By (18), factors other than the first and last are in $A^{p^{k+1}}$ provided that $r_s \geqq 1$; if not there is nothing to prove. From (18) and (21), we see that

$$[A_s, (r_s + p - 1)b] \leqq \langle [A_1, (r_s + p - 2)b],$$
$$\ldots, [A_{s-1}, (r_s + p - 2)b], \ldots [A_s^p, (r_s + p - 2)b], A^{p^{k+1}}\rangle.$$

Going back to (1),

$$[A_1, r_1 b] \leqq \langle A_2^p, A^{p^2}\rangle,$$

so from (18),

$$[A_1, (r_1 + r_2)b] \leqq \langle [A_2, r_2 b]^p, A^{p^{k+1}}\rangle.$$

Applying (19) we get that

$$[A_1, (r_1 + r_2)b] \leqq A^{p^{k+1}}$$

and hence

(24) $[A_1, (p - 2)b] \leqq A^{p^{k+1}}$.

Now taking p-th roots in the hypothesis in (20), we get for $2 \leqq j \leqq s - 1$

$$[A_j, r_s b] \leqq \langle A_1, A_2^p, A_3^{p^2}, \ldots, A_{k+1}^{p^k}, A_{k+2}^{p^k}, \ldots\rangle$$

thus

$$[A_j, (r_s + p - 2)b]$$
$$\leqq \langle [A_1, (p - 2)b], [A_2^p, (p - 2)b], [A^{p^2}, p - 2)b]\rangle.$$

We have just seen that

$$[A_1, (p - 2)b] \leqq A^{p^{k+1}}.$$

That

$$[A_2^p, (p - 2)b] \leqq p^{k+1}$$

follows from (19); and (18) then shows that

$$[A_j, (r_s + p - 2)b] \leqq A^{p^{k+1}} \quad \text{for } j \leqq s - 1.$$

In proving (23) it remains to consider the term $[A_s^p, (r_s + p - 2)b]$. But now, from (22), we get that

$$[A_s^p, (r_s + p - 2)b] \leqq \langle [A_2^p, (p - 2)b], \ldots, [A_{s-1}^p, (p - 2)b], [A^{p^2}, (p - 2)b] \rangle.$$

From our induction hypothesis in (20) together with (18) we see that

$$[A_s^p, (r_s + p - 2)b] \leqq A^{p^{k+1}}.$$

This completes the proof of (23).

Thus to finally complete the proof of (20) and so of the theorem, we only need the following familiar type of lemma:

LEMMA. *If* $[A_s^p, mb] \leqq A^{p^{k+1}}$ *when* $m \geqq 2$, *then*

$$[A_s^p, (m - 1)b] \leqq A^{p^{k+1}}.$$

Proof. For every a in A_s,

$$[a, (m - 1)b]^p [a, mb]^{\binom{p}{2}} \ldots [a, (m + p - 3)b]^p [a, (m + p - 2)b] = 1.$$

As usual the hypotheses of the lemma give that all factors other than the first and last are in $A^{p^{k+1}}$, and we have to show the same for the last. Taking p-th roots in the hypothesis, we get

$$[A_s, mb] \leqq \langle A_1, A_2^p, A_3^{p^2}, \ldots \rangle$$

thus

$$[A_s, (m + p - 2)b] \leqq \langle [A_1, (p - 2)b], [A_2^p, (p - 2)b], [A^{p^2}, (p - 2)b] \rangle.$$

From (24), (19) and (18) we see that

$$[A_s, (m + p - 2)b] \leqq A^{p^{k+1}}.$$

Thus

$$[A_s, (m - 1)b]^p \leqq A^{p^{k+1}}$$

and the lemma is proved.

Combined with (23), this establishes that

$$[A_s, b]^p \leqq A^{p^{k+1}}.$$

This completes the proof that $[A, b]^p = 1$, and we want finally to show that G has class $c \leqq r_1 + 2$. Since $[A, b]$ has exponent p, we have that

$$[A, b] \leqq \langle A_1, A_2^p, A_3^{p^2}, \ldots \rangle;$$

then $[A, 2b] \leqq [A_1, b]$ since A^p is central, and indeed

$$[A, (r_1 + 2)b] \leqq [A_1, (r_1 + 1)b].$$

But $[A_1, r_1 b] \leqq A^p$ from (1), so it is central, and we have that

$$[A, (r_1 + 2)b] \leqq [A_1, (r_1 + 1)b] = 1.$$

So G is of class at most $r_1 + 2$, as claimed. Note that the case $r_1 = 0$ causes no problems, for then $A_1 = 1$ and $[A, b] \leqq A^p$ so that $[A, b]$ is central.

5. Extensions of p-groups: general countable A. Before considering countable p-groups in full generality we show that we can confine attention to reduced groups.

THEOREM 5.1. *Let A be an abelian p-group with divisible part D, let $A = D \times R$ and suppose that B is a non-divisible nilpotent p-group. Then $\mathcal{N}(A, B)$ if and only if the rank r of D is at most $p - 2$ and $\mathcal{N}(R, B)$.*

Proof. If $\mathcal{N}(A, B)$, then $\mathcal{N}(D, B)$ and $\mathcal{N}(R, B)$. Since B is non-divisible, it has C_p as image and so $\mathcal{N}(D, C_p)$. The result of **[4]** used earlier, together with the fact that D is central in every extension of D by C_p shows that $r \leqq p - 2$.

Conversely, suppose that $r \leqq p - 2$ and that $\mathcal{N}(R, B)$. Then D has no p-automorphisms and thus every extension of D by B is central and so nilpotent of class at most 2. Since D is characteristic in A, $A/D \cong R$ and both $\mathcal{N}(D, B)$ and $\mathcal{N}(R, B)$ it follows from 2.7 that $\mathcal{N}(A, B)$ as required.

Next we need a result in abelian group theory that allows us to pull out a large part of a basic subgroup of an abelian group as a direct factor. It is probably known since it is of some interest in abelian group theory, but we can find no reference to it in the literature. Fuchs **[3]** is a good reference for general facts about abelian groups.

THEOREM 5.2. *Let A be a countable reduced abelian p-group of infinite exponent with basic subgroup L. Suppose that $L = X \times Y$, where Y has infinite exponent. Then A has a direct factor isomorphic with X.*

Proof. First we introduce notation for the Ulm sequence of A. We define a transfinitely continued descending series of characteristic subgroups

$$A = A^{(0)} \geqq A^{(1)} \geqq \ldots \geqq A^{(\alpha)} \geqq \ldots$$

of A as follows. Firstly

$$A^{(1)} = \bigcap_{n=1}^{\infty} A^{p^n}.$$

Suppose that all $A^{(\beta)}$ with $\beta < \alpha$ have been defined. If α is a limit ordinal we put

$$A^{(\alpha)} = \bigcap_{\beta<\alpha} A^{(\beta)};$$

if $\alpha = \beta + 1$, then

$$A^{(\alpha)} = (A^{(\beta)})^{(1)}.$$

Since A is reduced there is a least ordinal τ such that $A^{(\tau)} = 1$, called *Ulm type of* A. The factor groups

$$A_\alpha = A^{(\alpha)}/A^{(\alpha+1)} \quad \text{for } \alpha < \tau$$

are the *Ulm factors of* A, while the well-ordered sequence

$$A_0, A_1, A_2, \dots, A_\alpha, \dots \quad (\alpha < \tau)$$

is the *Ulm sequence of* A.

Note that all Ulm factors except possibly a "last" one are of infinite exponent, and being countable abelian p-groups with no elements of infinite height, are direct products of cyclic p-groups. In particular $A/A^{(1)} = A_1$ is such a direct product and we are going to show now that the canonical map $\rho:A \rightarrow A_1$ maps L isomorphically onto a basic subgroup of A_1. This will give in fact $L \cong A_1$ since A_1 is isomorphic to every one of its basic subgroups. The fact that $L\rho$ is basic in A_1 is a routine calculation. But ρ is injective on L since for $b \in L$, $b\rho = 1$ gives

$$b \in \bigcap_{k=1}^{\infty} A^{p^k}.$$

But L is pure and so

$$b \in \bigcap_{k=1}^{\infty} L^{p^k} = 1,$$

thus completing the proof that $L \cong A_1$.

The hypotheses give that $A_1 \cong X \times Y$. Consider now the sequence of groups

$$Y, A_2, A_3, \dots$$

in which the first group in the Ulm sequence of A is replaced by Y. By Zippin's theorem **[3]** there exists a group C with this new sequence as its Ulm sequence. It follows that $X \times C$ has

$$X \times Y, A_2, A_3, \dots$$

as its Ulm sequence, that is, it has the same Ulm sequence as A. By Ulm's theorem, $A \cong X \times C$, and the theorem is established.

Note that we have not proved (nor can we prove) that X is itself a direct factor of A. Fortunately, we do not need to. Moreoever, we do not know if the result is true for uncountable A.

We now have the main theorem for countable A and finite B. The proof is long and not quite routine.

THEOREM 5.3. *Let A be a countable abelian p-group of infinite exponent with divisible part D of rank r, and basic subgroup L; let B be a non-trivial finite p-group. Then $\mathcal{N}(A, B)$ if and only if $r \leqq p - 2$ and L is eventually sparse.*

Proof. Suppose that $\mathcal{N}(A, B)$. Writing $A = D \times R$, with R reduced, we have seen in Theorem 5.1 that $r \leqq p - 2$ and $\mathcal{N}(R, B)$. But a basic subgroup of R is a basic subgroup of A; and we suppose that L is not eventually sparse. Then clearly L can be split as a direct product $X \times Y$ where Y is of infinite exponent and X is a non-sparse direct product of cyclic groups. By Theorem 5.2, R and hence A has a direct factor isomorphic to X. Now Example 4.1 shows that there is a non-nilpotent extension of X by C_p and thus $\neg\mathcal{N}(A, C_p)$. But B is finite so that $\mathcal{N}(A, B)$ implies $\mathcal{N}(A, C_p)$, a contradiction.

Conversely suppose that $r \leqq p - 2$ and that L is eventually sparse. Theorem 5.1 tells us that we may assume that A is reduced. We have already seen in Theorem 4.2 that if A is a sparse direct product of cyclic groups and $B = C_p$, every extension of A by B is nilpotent of class $c \leqq p$, indeed that $c \leqq r_1 + 2$ where r_1 is the rank of the homocyclic factor of exponent p. To get full generality in the theorem we simply strengthen the proof step by step.

Step 1. If A is a sparse direct product of cyclic p-groups and b is an automorphism of A of order p^n, then the split extension $\langle A, b\rangle$ is of class at most $p + p^2 + \ldots + p^n$ and b acts as an automorphism of order dividing p^{n-1} on A^p.

Proof. The case $n = 1$ has been established in Theorem 4.2. Induct on n. Suppose that the result is true when $n = k$ and let b be an automorphism of order p^{k+1} on A. Then b^p has order p^k on A and so by induction order at most p^{k-1} on A^p. Thus b has order at most p^k on A^p, which is part of what we want. Now $\langle A/A^p, bA^p\rangle$ is nilpotent of class at most p^{k+1} since A/A^p has exponent p and bA^p has order at most p^{k+1}. Thus

$$[A, p^{k+1}b] \leqq A^p.$$

But A^p is sparse and b has order at most p^k on A^p, so by induction we have

$$[A^p, (p + p^2 + \ldots + p^k)b] = 1.$$

Combining these two results we have

$$[A, (p + p^2 + \dots + p^{k+1})b] = 1,$$

as required.

Step 2. *If A is a countable reduced abelian p-group with sparse basic subgroup L, and b is an automorphism of A of order* p^n, *then* $\langle A, b\rangle$ *is nilpotent of class* $c \leqq p + p^2 + \dots + p^n$ *and b acts as an automorphism of order dividing* p^{n-1} *on* A^p.

Proof. The hard part here is to do the case $n = 1$. The rest then proceeds by an easy induction. So let $n = 1$. Recall the notation for the Ulm sequence in the proof of Theorem 5.2. We know that L is isomorphic with the first Ulm factor $A/A^{(1)}$. We induct on the Ulm type τ of A. When $\tau = 1$ we have $A \cong L$ so that A is a sparse direct product of cyclic groups and the result is given by Step 1. Suppose the result holds for groups of Ulm type less than τ.

Since $L \cong A/A^{(1)}$, the hypotheses carry over to the groups $A/A^{(\alpha)}$. Thus for all $\alpha < \tau$, $\langle A/A^{(\alpha)}, b\rangle$ has class $c \leqq p$ and b acts trivially on $A^p/A^{(\alpha)}$. Thus

$$[A, b]^p \leqq A^{(\alpha)} \quad \text{for all } \alpha < \tau.$$

If τ is a limit ordinal we have

$$1 = A^{(\tau)} = \bigcap_{\alpha<\tau} A^{(\alpha)}$$

so that $[A, b]^p = 1$ and Lemma 4.2.1 gives what we want. If on the other hand $\tau = \lambda + 1$, the situation is more complicated and we show first $\langle A, b\rangle$ has class at most $p + 1$.

Suppose first that $A^{(\lambda)}$ has finite exponent, p^e say. Then $\langle A^{(\lambda)}, b\rangle$ is nilpotent, so that as

$$[A, pb] \leqq A^{(\lambda)},$$

$\langle A, b\rangle$ is nilpotent. By Lemma 4.2.1 some power A^{p^s} of A is centralized by b and thus so is $A^{(\lambda)}$. Thus

$$[A, (p + 1)b] = 1$$

and $\langle A, b\rangle$ is of class at most $p + 1$, as claimed. In general $A^{(\lambda)}$ is a direct product of cyclic groups, so that

$$\bigcap_{n=1}^{\infty} A^{(\lambda)p^n} = 1.$$

However what we just showed gives that

$$\langle A/A^{(\lambda)p^n}, bA^{(\lambda)p^n}\rangle$$

is of class at most $p + 1$ for all n, so that

$$\gamma_{p+2}(\langle A, b\rangle) \leqq \bigcap_{n=1}^{\infty} A^{(\lambda)p^n} = 1$$

and $\langle A, b\rangle$ is of class at most $p + 1$ in all cases.

However, we have to reduce this $p + 1$ to p, and we proceed as follows. First we show that b centralizes A^{p^2}. For all a in A we have

$$1 = [A, pb]^p[a, (p + 1)b]^{\binom{p}{2}} \dots [a, (2p - 1)b],$$

so that $[a, pb]^p = 1$. However this gives that

$$(*) \quad 1 = [a, b]^{p^2}[a, 2b]^{p\binom{p}{2}} \dots [a, (p - 1)b]^{p\binom{p}{p-1}}$$

so that commutation $(p - 2)$ times with b gives

$$[a, (p - 1)b]^{p^2} = 1,$$

and the last term of (*) drops out. Repeating this commutation process we successively eliminate the last factor of (*) until we reach the conclusion that

$$[a, b]^{p^2} = 1 \quad \text{for all } a.$$

In other words,

$$[A^{p^2}, b] = 1.$$

Since $L \cong A/A^{(1)}$ we know from Theorem 4.2 that $\langle A/A^{(1)}, b\rangle$ has class at most $r_1 + 2$ where r_1 is the rank of the homocyclic component L_1 of exponent p. Thus

$$[A, (r_1 + 2)b] \leqq A^{(1)} \leqq A^{p^2}$$

so that

$$[A, (r_1 + 3)b] = 1.$$

If $r_1 + 3 \leqq p$ we are done. Otherwise $r_1 + 2 \leqq p$ so that, as L is sparse, $r_1 + r_2 \leqq p - 2$; so $r_1 = p - 2$ and $r_2 = 0$. Thus $L_2 = 0$. From properties of basic subgroups there is a decomposition of A as

$$A = L_1 \times L_2 \times K.$$

where a basic subgroup of K begins with a homocyclic component of exponent at least p^3. Since $L_2 = 0$ here, this decomposition gives

$$A/A^{(1)} = L_1A^{(1)}/A^{(1)} \times KA^{(1)}/A^{(1)};$$

the fact that the two factors intersect trivially follows since L_1 has no elements of infinite height. Since $L \cong A/A^{(1)}$ is sparse with $KA^{(1)}/A^{(1)}$ having its homocyclic components of exponent at least p^3 and b acts as an automorphism on $A/A^{(1)}$, it follows from Step 1 that $[A/A^{(1)}, bA^{(1)}]$ is of exponent p, and hence that

$$[A/A^{(1)}, bA^{(1)}] \leqq L_1A^{(1)}/A^{(1)} \times K^{p^2}A^{(1)}/A^{(1)},$$

that is,

$$[A, b] \leqq L_1A^{p^2}.$$

Thus $[A, 2b] \leqq [L_1, b]$ since b centralizes A^{p^2}. But $[L_1, b]$ is elementary abelian and has rank at most r_1. Thus so is $[A, 2b]$ and so, since it is b-invariant,

$$[A, (r_1 + 2)b] = 1.$$

So $[A, pb] = 1$ and Step 2 is complete in the case $n = 1$; and as we said at the beginning, the rest of the proof is straightforward.

Step 3. *Let A be a countable reduced abelian p-group with eventually sparse basic subgroup L, and let b be an automorphism of order p^n on A. Then $\langle A, b\rangle$ is nilpotent of class*

$$c \leqq p + p^2 + \dots + p^n + mp^n - (m - 1)p^{n-1},$$

where m is such that L is sparse after exponent p^m.

Proof. Write $L = Y \times K$ where $Y = L_1 \times \dots \times L_m$, so that K is sparse. Further, set $U = \Omega_m(A)$. Then $A = Y \times A_1$ and A_1 has K as basic subgroup, and

$$A/U = A_1/\Omega_m(A_1).$$

But a basic subgroup of A maps epimorphically to one of A/U, so that A/U has sparse basic subgroup (it is isomorphic with $K/\Omega_m(K)$). Also A is reduced and U has finite exponent, so A/U is reduced. By Step 2 we can conclude that $\langle A/U , b\rangle$ has class at most $p + p^2 + \dots + p^n$. So

$$[A, (p + p^2 + \dots + p^n)b] \leqq \Omega_m(A).$$

But $\langle \Omega_m(A), b\rangle$ has class at most $mp^n - (m - 1)p^{n-1}$, and the result follows.

Step 4. We can now complete the proof of the full theorem. We have that A is a countable, reduced abelian p-group with eventually sparse basic subgroup L and B is a finite p-group. Let G be a split extension of A by B; we must show that G is nilpotent. Suppose that L is sparse after exponent p^m and that B has exponent p^n. From Step 3, each subgroup $\langle A, b\rangle$ with b in B is nilpotent of class depending only on p, m, n. Hence $[A, b]$ has finite exponent depending only on p, m, n (Lemma 4.2.1), and $[A, B]$ has finite exponent. However, B is nilpotent and $G/A \cong B$, so that $\gamma_k(G) \leqq A$ for some k and further $\gamma_{k+1}(G) \leqq [A, B]$ since A is abelian. Thus $\gamma_{k+1}(G)$ has finite exponent and G acts on $\gamma_{k+1}(G)$ as a finite p-group, so that G is nilpotent and the proof of Theorem 5.3 is complete.

Finally, we deal with the case where A is countable and B is infinite and non-divisible. Set $B_{ab} = B/B'$.

THEOREM 5.4. *Let A be a countable abelian p-group of infinite exponent and B an infinite non-divisible nilpotent p-group. Suppose $A = D \times R$ where D is divisible of rank r and R reduced and let L be a basic subgroup of A.*

(i) *Let R be finite. Then $\mathcal{N}(A, B)$ if and only if $r \leqq p - 2$.*

(ii) *Let R be infinite. Then $\mathcal{N}(A, B)$ if and only if $r \leqq p - 2$, L is thin and eventually sparse and $B_{ab} = E \times M$ where E has finite exponent and M is divisible.*

Proof. Part (i) is an immediate consequence of Theorems 5.1 and 3.1. and so we turn to (ii). If $\mathcal{N}(A, B)$ we see from Theorem 5.1 that $r \leqq p - 2$ and $\mathcal{N}(R, B)$. Moreover, L can be taken as a basic subgroup of R and since $\mathcal{N}(R, B)$ it follows from Theorem 2.8 that L must be thin. Otherwise R would have as direct factor an infinite homocyclic group, namely some L_n. Note that R must have infinite exponent here and so L does also. Next B is not divisible, so has C_p as an image. Thus $\mathcal{N}(R, C_p)$ and we see from Theorem 5.3 that L is eventually sparse. Now let $B_{ab} = E \times M$ where E is reduced and M divisible. We claim that E has finite exponent. Assume not. Since E maps onto its basic subgroup, it follows that B has a homomorphic image K isomorphic with the restricted direct product

$$C_p \times C_{p^2} \times \ldots \times C_{p^n} \times \ldots .$$

Further $\mathcal{N}(R, K)$. Since L has infinite exponent, we can write $L = X \times Y$, where both X and Y are of infinite exponent. By Theorem 5.2, $R \cong X \times S$ for some S and we see that $\mathcal{N}(X, K)$. However, $\mathcal{N}(X, K)$ is false in this situation, for the following reason. For any $n > 1$ the group

$$V_n = \langle x, y | x^{p^n} = y^{p^{n-1}} = 1, x^y = x^{1+p} \rangle$$

has class n, and obviously it is possible to make an extension of X by K containing V_n for infinitely many n. This completes the proof of (ii) in one direction.

We begin the proof of the converse by considering the case where B is reduced.

LEMMA. *If B is an infinite reduced nilpotent p-group and $B_{ab} = E \times M$ where E has finite exponent and M is divisible, then B has finite exponent (and therefore $M = 1$).*

Proof. We have $E = F/B'$ and $M = N/B'$ for suitable F, N. From a result of Zaleskii [7], $F' = B'$ and F is a basic subgroup of B (in his terminology). Since F/F' has finite exponent, so does F. Zaleskii also showed in [7] that $B = FZ$ where Z is the center of B. With bars denoting images mod B', we have $B_{ab} = \bar{F}\bar{Z}$ and thus $\bar{Z}$ is a direct product of a group of finite exponent and a divisible group. But a basic subgroup K of Z maps to one of $\bar{Z}$, so that K has finite exponent, and also so does B'. Thus Z has finite exponent and finally B does.

Assume now that B is reduced in the statement of Theorem 5.4. Because of Theorem 5.1, we may assume that A is reduced.

Case 1. A is a direct product of cyclic groups.

Then $A \cong L$ and so A is thin and eventually sparse. Let the exponent of B be p^e. Then from Theorem 5.3 we see that the split extension $\langle A, b\rangle$ has class depending only on A and p^e, for each b in B. Thus by Lemma 4.2.1, there exists a positive integer k such that $[A, b]^{p^k} = 1$ for all b in B. Thus

$$[A, B] \leqq \Omega_k(A).$$

With A_i denoting the i-th homocyclic factor of A, we have

$$\Omega_k(A) \leqq \langle A_1, A, \dots, A_{2k-1}, A^{p^k}\rangle;$$

that is, $[A, B] \leqq HA^{p^k}$ for a finite group H. With $G = \langle A, B\rangle$, the split extension, we have

$$\gamma_l(G) \leqq [A, B]$$

for some l. So the image of $\gamma_l(G)$ in G/A^{p^k} is a finite p-group, which means that

$$[\gamma_l(G), mB] \leqq A^{p^k} \quad \text{for some } m \geqq 1.$$

Thus $\gamma_{l+m+1}(G) = 1$ since B fixes the elements of A^{p^k}.

Case 2. A is not a direct product of cyclic groups.

Then $A/A^{(1)}$ is such a product, and from Case 1 we have that

$$\bar{G} = \langle A/A^{(1)}, B\rangle$$

is nilpotent. Thus

$$\gamma_c(G) \leqq A^{(1)} \quad \text{for some } c.$$

Theorem 5.3 now gives that there exists a $k \geqq 0$ such that

$$[A, b]^{p^k} = 1 \quad \text{for all } b \text{ in } B.$$

So

$$\gamma_c(G) \leqq A^{(1)} \leqq A^{p^k}$$

and as B fixes all elements of A^{p^k},

$$\gamma_{c+1}(G) = 1.$$

If B is not reduced, its divisible part C is central and it is easy to see that

B/C is reduced. Furthermore $(B/C)_{ab}$ is still a direct product of a group of finite exponent and a divisible group. Since L is thin, C acts trivially on A by Theorem 3.4. Hence every homomorphism $B \to \text{Aut } A$ factors through a map $B/C \to \text{Aut } A$. But $\mathcal{N}(A, B/C)$ from above, and so $\mathcal{N}(A, B)$. This completes the proof of Theorem 5.4.

Remarks. For these concluding remarks A remains an abelian p-group and B a nilpotent p-group. We have now classified all pairs A, B with B divisible for which $\mathcal{N}(A, B)$. Further, when B is non-divisible we have classified all A, B with A countable or of finite exponent for which $\mathcal{N}(A, B)$. There remains the case where A is uncountable, of infinite exponent and B is non-divisible. From our previous work there are many obvious necessary conditions for $\mathcal{N}(A, B)$, for example, restrictions on the structure of countable direct factors of A. Again because of Theorem 5.1 the divisible part of A must have rank at most $p - 2$ and one can restrict attention to the case where A is reduced. Let L be a basic subgroup of such an A and let B be infinite and non-divisible. Again from (2.8) we see that L must be thin and since, as is well-known, the cardinality $|A|$ of A is at most $|L|^{\aleph_0}$ we conclude that $|A| \leqq 2^{\aleph_0}$. In other words, if B is infinite, non-divisible and $|A|$ exceeds $2^{\aleph_0}$, then $\neg \mathcal{N}(A, B)$. For the case where B is a finite, non-trivial p-group recall that $\mathcal{N}(A, B)$ implies $\mathcal{N}(A, C_p)$. Consider the special case where A is the torsion subgroup of an unrestricted product of cyclic groups. Here a basic subgroup of A is the restricted direct product of those same cyclic groups and again it must be eventually sparse. Otherwise the automorphism constructed in Example 4.1 could clearly be extended to A, thereby giving rise to a non-nilpotent extension of A by C_p. Whether eventual sparseness is sufficient for $\mathcal{N}(A, C_p)$ in this special case we do not know.

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