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REMARKS ON SOME FUNDAMENTAL RESULTS ABOUT HIGHER-RANK GRAPHS AND THEIR C^*-ALGEBRAS

ROBERT HAZLEWOOD[1], IAIN RAEBURN[2], AIDAN SIMS[3]
AND SAMUEL B. G. WEBSTER[3]

[1] *School of Mathematics and Statistics, University of New South Wales, Sydney, NSW 2052, Australia* (robbiehazlewood@gmail.com)
[2] *Department of Mathematics and Statistics, University of Otago, PO Box 56, Dunedin 9054, New Zealand* (iraeburn@maths.otago.ac.nz)
[3] *School of Mathematics and Applied Statistics, University of Wollongong, NSW 2522, Australia* (asims@uow.edu.au; sbgwebster@gmail.com)

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Abstract Results of Fowler and Sims show that every k-graph is completely determined by its k-coloured skeleton and collection of commuting squares. Here we give an explicit description of the k-graph associated with a given skeleton and collection of squares and show that two k-graphs are isomorphic if and only if there is an isomorphism of their skeletons which preserves commuting squares. We use this to prove directly that each k-graph Λ is isomorphic to the quotient of the path category of its skeleton by the equivalence relation determined by the commuting squares, and show that this extends to a homeomorphism of infinite-path spaces when the k-graph is row finite with no sources. We conclude with a short direct proof of the characterization, originally due to Robertson and Sims, of simplicity of the C^*-algebra of a row-finite k-graph with no sources.

Keywords: higher-rank graph; C^*-algebra; Cuntz–Krieger algebra

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1. Introduction

A k-graph is a combinatorial object akin to a directed graph, in which each path λ has a k-dimensional shape $d(\lambda) \in \mathbb{N}^k$, called its degree, instead of a one-dimensional length. C^*-algebras associated to graphs and k-graphs have attracted significant attention recently because they at once encompass a great many interesting examples [**2**, **6**, **11**, **17**], and are remarkably tractable [**3**–**5**, **7**, **12**, **13**, **21**]. Indeed, Spielberg [**25**] showed how to construct every Kirchberg algebra from combinations of graph C^*-algebras and 2-graph C^*-algebras. However, k-graphs themselves are, from a combinatorial point of view, substantially more complicated than their one-dimensional counterparts, and one of the keys to using them effectively is a good visual description.

A crucial feature of k-graphs is the factorization property, which says that, given any path λ and any decomposition $d(\lambda) = m + n$, there is a unique factorization $\lambda = \mu\nu$ such that $d(\mu) = m$ and $d(\nu) = n$. In particular, writing $e_1, \dots, e_k$ for the generators of $\mathbb{N}^k$, if ef is a path with $d(e) = e_i$ and $d(f) = e_j$, then $d(ef) = e_j + e_i$, so there is a unique expression $ef = f'e'$ where $d(f') = e_j$ and $d(e') = e_i$. This is called a *square* of Λ. We can regard the list $\mathcal{C}_\Lambda$ of all such squares as data associated with the *skeleton* of Λ, which is the k-coloured directed graph E_Λ with the same vertices as Λ and with edges $\bigcup_{i=1}^k d^{-1}(e_k)$, where edges of different degree are coloured with different colours.

Theorem 2.2 of [**9**] characterizes exactly which coloured graphs E and collections $\mathcal{C}$ of squares arise from k-graphs; [**9**, Theorem 2.1] implies that for each such pair $(E, \mathcal{C})$ there is a unique k-graph up to isomorphism whose skeleton is E and whose commuting squares are those in $\mathcal{C}$. The latter theorem is an existence result; it does not explicitly describe the k-graph $\Lambda_{E,\mathcal{C}}$. It is more or less folklore (and can be dug out of the proof of [**9**, Theorem 2.1]) that $\Lambda_{E,\mathcal{C}}$ can be described along the lines outlined for $k = 2$ in [**14**, § 6]: paths in $\Lambda_{E,\mathcal{C}}$ are described as paths in E in which the colours occur in a fixed preferred order. But this is unsatisfactory because it is difficult to recognize a path when it is written as a concatenation of sub-paths, or to decide when one path is a sub-path of another; to do so requires tedious calculations using the collection $\mathcal{C}$ of squares.

In § 4 we provide a concrete description of the k-graph $\Lambda_{E,\mathcal{C}}$. Inspired by the construction of 2-graphs from two-dimensional shift spaces in [**18**], we show that the paths in Λ can be regarded as coloured-graph morphisms from a collection of model k-coloured graphs into E. An advantage of this construction is that, under this presentation, each path explicitly encodes all of its subpaths. In § 5 we use this to provide an explicit proof that Λ is the quotient of the path category E_Λ^* of E_Λ by the equivalence relation $\sim$ determined by $\mathcal{C}$. We then show that if Λ is row finite and has no sources in the sense of [**14**], then the topology on the infinite-path space of Λ coincides with the quotient topology on $E^\infty/{\sim}$. We also present an example showing that the corresponding statement is false for boundary paths in non-row-finite k-graphs. Our final section gives a direct and elementary proof that if Λ is a row-finite k-graph with no sources, then $C^*(\Lambda)$ is simple if and only if Λ is both aperiodic and cofinal (see § 6 for details). This result first appeared in [**21**], but the proof there was indirect, proceeding via reference to the results of [**14**], which were proved using groupoid technology. Since aperiodicity and cofinality have been characterized in a number of different ways in the literature, we use the presentations which are best suited to the description of $\Lambda_{E,\mathcal{C}}$ from § 4: specifically, the description of aperiodicity introduced in [**21**], and the cofinality condition of [**15**]. The key graph-theoretic component of our proof, Lemma 6.2, has already found applications elsewhere; it was precisely the statement needed to establish the Cuntz–Krieger Uniqueness Theorem [**1**, Theorem 4.7] for the Kumjian–Pask algebras introduced there.

2. Background

A *directed graph* $E = (E^0, E^1, r, s)$ consists of countable sets E^0, E^1 and functions $r, s \colon E^1 \to E^0$. Since all the graphs in this paper are directed, we will drop the adjective.

We call elements of E^0 *vertices*, and elements of E^1 *edges*. For an edge $e \in E^1$, we call $s(e)$ the *source* of e and $r(e)$ the *range* of e. A *path of length* n is a sequence $\mu = \mu_1\mu_2\cdots\mu_n$ of edges such that $s(\mu_i) = r(\mu_{i+1})$ for $1 \leqslant i \leqslant n-1$. We write $|\mu|$ for the length n of μ. For $v \in E^0$, we define $|v| = 0$. We denote by E^n the set of all paths of length n, and define $E^* := \bigcup_{n\in\mathbb{N}} E^n$. We extend r and s to E^* by setting $r(\mu) = r(\mu_1)$ and $s(\mu) = s(\mu_n)$. By an infinite path in E, we mean a sequence $x = \nu_1\nu_2\cdots$ where $r(\nu_{i+1}) = s(\nu_i)$ for all i, and we write $r(x) = r(\nu_1)$. We write E^∞ for the set of all infinite paths and call $W_E := E^* \cup E^\infty$ the *path space* of E.

For $k \in \mathbb{N}$, a *k-graph* is a pair (Λ, d), where Λ is a countable category and d is a functor from Λ to $\mathbb{N}^k$ which satisfies the *factorization property*: for every $\lambda \in \mathrm{Mor}(\Lambda)$ and $m, n \in \mathbb{N}^k$ with $d(\lambda) = m + n$, there are unique elements $\mu, \nu \in \mathrm{Mor}(\Lambda)$ such that $\lambda = \mu\nu$, $d(\mu) = m$ and $d(\nu) = n$ (see [**14**, Definition 1.1]). Elements $\lambda \in \mathrm{Mor}(\Lambda)$ are called *paths*, and by convention we write $\lambda \in \Lambda$ to mean $\lambda \in \mathrm{Mor}(\Lambda)$. The functor d is called the degree map. We write r, s for the usual maps from Λ to its identity morphisms: formally, $r(\lambda) = \mathrm{id}_{\mathrm{cod}(\lambda)}$ and $s(\lambda) = \mathrm{id}_{\mathrm{dom}(\lambda)}$.

For $m \in \mathbb{N}^k$ and $v \in \mathrm{Obj}(\Lambda)$, we define $\Lambda^m := \{\lambda \in \Lambda\colon d(\lambda) = m\}$ and $v\Lambda^m := \{\lambda \in \Lambda^m\colon r(\lambda) = v\}$. More generally, given $\lambda \in \Lambda$ and $F, G \subseteq \Lambda$, we define $\lambda G = \{\lambda\nu\colon \nu \in G,\ r(\nu) = s(\lambda)\}$ and $F\lambda = \{\mu\lambda\colon \mu \in F,\ s(\mu) = r(\lambda)\}$, and then

$$F\lambda G = \bigcup_{\mu\in F} \mu\lambda G = \bigcup_{\nu\in G} F\lambda\nu.$$

A morphism between k-graphs (Λ_1, d_1) and (Λ_2, d_2) is a functor $f\colon \Lambda_1 \to \Lambda_2$ which respects the degree maps. The factorization property implies that $v \mapsto \mathrm{id}_v$ is a bijection between $\mathrm{Obj}(\Lambda)$ and Λ^0, allowing us to identify $\mathrm{Obj}(\Lambda)$ with Λ^0. In particular, we will henceforth regard r and s as maps from Λ to Λ^0.

3. Coloured graphs and coloured-graph morphisms

Consider the free semigroup $\mathbb{F}_k$ on k-generators $\{c_1, \dots, c_k\}$. A *k-coloured graph* is a graph E together with a map $c\colon E^1 \to \{c_1, \dots, c_k\}$, which we extend to a functor $c\colon E^* \to \mathbb{F}_k^+$. We write q for the canonical quotient map $q\colon \mathbb{F}_k^+ \to \mathbb{N}^k$ determined by $q(c_i) = e_i$ for all i. So each path $x \in E^*$ has both a colouring $c(x) \in \mathbb{F}_k^+$ and a shape $q(c(x)) \in \mathbb{N}^k$. If there are multiple k-coloured graphs around, we write c_E for the colour map associated with the graph E. In this paper, we will draw edges of colour c_1 as solid lines, edges of colour c_2 as dashed lines, and edges of colour c_3 as dotted lines.

A *graph morphism* ψ from a graph E to a graph F consists of functions $\psi^0\colon E^0 \to F^0$ and $\psi^1\colon E^1 \to F^1$ such that $r_F(\psi^1(e)) = \psi^0(r_E(e))$ and $s_F(\psi^1(e)) = \psi^0(s_E(e))$ for all $e \in E^1$. Given graph morphisms $\psi\colon E \to F$ and $\phi\colon F \to G$, we write $\phi \circ \psi$ for the graph morphism from E to G given by $(\phi\circ\psi)^i = \phi^i \circ \psi^i$ for $i = 0, 1$. A *coloured-graph morphism* is a graph morphism ψ such that $c_E(e) = c_F(\psi(e))$ for every $e \in E^1$.

The following example describes the model k-coloured graphs which will underly the construction used in our main theorem in § 4. In the example, $n + v_i$ is a formal symbol intended to suggest an edge of colour c_i pointing from the integer grid point $n + e_i$ to the integer grid point n.

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Example 3.1. For $m \in (\mathbb{N} \cup \{\infty\})^k$, we define a coloured graph $E_{k,m}$ by

$$E^0_{k,m} = \{n \in \mathbb{N}^k \colon 0 \leqslant n \leqslant m\}, \quad E^1_{k,m} = \{n + v_i \colon n, n + e_i \in E^0_{k,m}\},$$
$$r(n + v_i) = n, \quad s(n + v_i) = n + e_i \quad \text{and} \quad c(n + v_i) = c_i.$$

Fix $n+v_i \in E^1_{k,m}$ and $m \in \mathbb{N}^k$. We define $(n+v_i)+m := (n+m)+v_i$, and as a notational convenience, if E is a coloured graph and $x \in E^1$ with $c(x) = c_i$, we sometimes write $n + v_{c(x)}$ for the edge $n + v_i$. Given a coloured-graph morphism $\lambda\colon E_{k,m} \to E$ we say λ has degree m and write $d(\lambda) = m$, and define $r(\lambda) := \lambda(0)$ and $s(\lambda) := \lambda(m)$.

Given a k-coloured graph E and distinct $i, j \in \{1, \dots, k\}$, an $\{i, j\}$-*square* (or just a *square*) in E is a coloured-graph morphism $\phi\colon E_{k,e_i+e_j} \to E$. If $\lambda\colon E_{k,m} \to E$ is a coloured-graph morphism and ϕ is a square in E, then ϕ *occurs in* λ if there exists $n \in \mathbb{N}^k$ such that $\phi(x) = \lambda(x + n)$ for all $x \in E_{k,e_i+e_j}$.

Let E be a k-coloured graph. A *complete collection of squares* is a collection $\mathcal{C}$ of squares in E such that, for each $x \in E^*$ with $c(x) = c_i c_j$ and $i \neq j$, there exists a unique $\phi \in \mathcal{C}$ such that $x = \phi(v_i)\phi(e_i + v_j)$. We write $\phi(v_i)\phi(e_i + v_j) \sim_{\mathcal{C}} \phi(v_j)\phi(e_j + v_i)$, so for each $c_i c_j$-coloured path $x \in E^*$, there is a unique $c_j c_i$-coloured path y such that $x \sim_{\mathcal{C}} y$. If $\mathcal{C}$ is clear from the context, we just write $x \sim y$. A coloured-graph morphism $\lambda\colon E_{k,m} \to E$ is $\mathcal{C}$-*compatible* if every square occurring in λ belongs to $\mathcal{C}$.

For $p, q \in \mathbb{N}^k$ with $p \leqslant q$, define $E_{k,[p,q]}$ to be the subgraph of $E_{k,q}$ such that

$$E^0_{k,[p,q]} = \{n \in \mathbb{N}^k \colon p \leqslant n \leqslant q\},$$
$$E^1_{k,[p,q]} = \{x \in E^1_{k,q} \colon s(x), r(x) \in E^0_{k,[p,q]}\}.$$

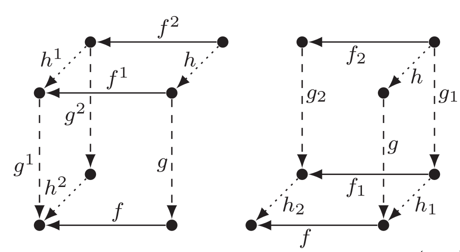
Given a coloured-graph morphism $\lambda\colon E_{k,m} \to E$ and $p, q \in \mathbb{N}^k$ such that $p \leqslant q \leqslant m$, define $\lambda|^*_{E_{k,[p,q]}} \colon E_{k,q-p} \to E$ by

$$\lambda|^*_{E_{k,[p,q]}}(a) = \lambda(p + a). \tag{3.1}$$

The '$*$' is to remind us that this non-standard restriction involves a translation.

We say a complete collection of squares $\mathcal{C}$ in a k-coloured graph E is *associative* if for every path fgh in E such that f, g, h are edges of distinct colour, the edges f_1, f_2, g_1, g_2, h_1, h_2 and f^1, f^2, g^1, g^2, h^1, h^2 determined by

$$fg \sim g^1 f^1, \quad f^1 h \sim h^1 f^2 \quad \text{and} \quad g^1 h^1 \sim h^2 g^2$$
$$gh \sim h_1 g_1, \quad f h_1 \sim h_2 f_1 \quad \text{and} \quad f_1 g_1 \sim g_2 f_2$$



(3.2)

satisfy $f^2 = f_2$, $g^2 = g_2$ and $h^2 = h_2$.

Let E be a k-coloured graph, and let $m \in \mathbb{N}^k \setminus \{0\}$. Let $|m| := \sum_{i=1}^k m_i$. Fix $x \in E^*$ and a coloured-graph morphism $\lambda\colon E_{k,m} \to E$. Recall that q is the quotient map from

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$\mathbb{F}_k^+$ to $\mathbb{N}^k$. We say x *traverses* λ if $q(c(x)) = d(\lambda)$ and $\lambda(q(c(x_1 \dots x_{l-1})) + v_{c(x_l)}) = x_l$ for all $0 < l \leqslant |m|$. By definition, $d(\lambda) = m$ and $|x| = |d(\lambda)|$. If $m = 0$, then $x \in E^0$ and $\mathrm{dom}(\lambda) = \{0\}$, and we say x traverses λ if $x = \lambda(0)$. Observe that for any coloured-graph morphism λ and any decomposition $d(\lambda) = e_{i_1} + e_{i_2} + \cdots + e_{i_m}$ there is a corresponding path $x := \lambda(0 + v_{i_1})\lambda(e_{i_1} + v_{i_2}) \cdots \lambda((d(\lambda) - e_{i_m}) + v_{i_m})$ which traverses λ; in particular, for every finite coloured-graph morphism λ there is a path which traverses λ.

We can also make sense of infinite coloured paths which traverse infinite coloured-graph morphisms. If $x \in E^\infty$ and $\lambda\colon E_{k,p} \to E$ is a coloured-graph morphism of non-finite degree (so $p \in (\mathbb{N} \cup \{\infty\})^k \setminus \mathbb{N}^k$), then we say that x traverses λ if $x_1 \cdots x_n$ traverses $\lambda|_{E_{k,d(x_1 \cdots x_n)}}$ for every $n \in \mathbb{N}$.

Remark 3.2. Let E be a k-coloured graph and let $\lambda\colon E_{k,m} \to E$ be a coloured-graph morphism where $m \in \mathbb{N}^k$. Fix $p \leqslant m$. If $x \in E^*$ traverses $\lambda|_{[0,p]}$ and $y \in E^*$ traverses $\lambda|^*_{[p,m]}$, then $d(\lambda) = m = p + (m - p) = q(c(x)) + q(c(y))$, and for $l \leqslant |xy| = |x| + |y|$ we have

$$\begin{aligned} \lambda(d((xy)_1 \cdots (xy)_{l-1}) + v_{c((xy)_l)}) &= \begin{cases} \lambda|_{[0,p]}(q(c(x_1 \cdots x_{l-1})) + v_{c(x)_l}) & \text{if } l \leqslant |x|, \\ \lambda|^*_{[p,m]}(q(c(y_1 \cdots y_{l-p-1})) + v_{c(y_{l-p})}) & \text{otherwise,} \end{cases} \\ &= \begin{cases} x_l & \text{if } l \leqslant |x|, \\ y_{l-p} & \text{otherwise,} \end{cases} \end{aligned}$$

so xy traverses λ.

4. From k-coloured graphs to k-graphs

In this section we present an explicit description of the unique k-graph associated to a k-coloured graph E and a complete collection $\mathcal{C}$ of squares in E that is associative (see Theorem 4.4).

We begin by showing how a k-graph defines a skeleton and a collection of squares. If Λ is a k-graph, $\lambda \in \Lambda$ and $m \leqslant n \leqslant d(\lambda)$, then we write $\lambda(m, n)$ for the unique element of Λ^{m-n} such that $\lambda = \lambda'\lambda(m,n)\lambda''$ with $d(\lambda') = m$ and $d(\lambda'') = d(\lambda) - n$. We write $\lambda(n)$ for $s(\lambda(0,n)) \in \Lambda^0$.

Definition 4.1. Let Λ be a k-graph. We define a coloured graph E_Λ and a collection $\mathcal{C}_\Lambda$ of squares associated to Λ as follows. Let E_Λ be the k-coloured graph with $E_\Lambda^0 = \{\bar{v}\colon v \in \Lambda^0\}$, $E_\Lambda^1 = \bigcup_{i=1}^k \{\bar{f}\colon f \in \Lambda^{e_i}\}$ and $c(\bar{f}) = c_i \iff d(f) = e_i$. Define $\pi\colon E_\Lambda^0 \to \Lambda$ by $\pi(\bar{v}) = v$ and $\pi\colon E_\Lambda^1 \to \Lambda$ by $\pi(\bar{f}) = f$, and extend this to a map $\pi\colon E_\Lambda^* \to \Lambda$ by $\pi(\overline{f_1} \cdots \overline{f_n}) = f_1 \cdots f_n$. For distinct $i, j \leqslant k$ and $\lambda \in \Lambda^{e_i+e_j}$ define a coloured-graph morphism $\phi_\lambda\colon E_{k,e_i+e_j} \to E_\Lambda$ by

$$\phi^0_\lambda(n) = \overline{\lambda(n)} \quad \text{and} \quad \phi^1_\lambda(n + v_i) := \overline{\lambda(n, n + e_i)}. \tag{4.1}$$

Let $\mathcal{C}_\Lambda := \bigcup_{i<j\leqslant k} \{\phi_\lambda\colon \lambda \in \Lambda^{e_i+e_j}\}$. We call E_Λ the *skeleton* of Λ.

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Lemma 4.2. *Let Λ be a k-graph. Fix distinct $i, j \leqslant k$ and $\lambda \in \Lambda^{e_i+e_j}$. Then ϕ_λ is the unique coloured-graph morphism from $E_{k,e_i+e_j} \to E_\Lambda$ such that*

$$\pi(\phi_\lambda(0+v_i)\phi_\lambda(e_i+v_j)) = \lambda = \pi(\phi_\lambda(0+v_j)\phi_\lambda(e_j+v_i)). \tag{4.2}$$

Moreover, $\mathcal{C}_\Lambda$ is a complete collection of squares in E_Λ that is associative.

Proof. Fix distinct $i, j \leqslant k$ and $\lambda \in \Lambda^{e_i+e_j}$. Then

$$\pi(\phi_\lambda(0+v_i)\phi_\lambda(e_i+v_j)) = \pi(\overline{\lambda(0,e_i)}\,\overline{\lambda(e_i,e_i+e_j)}) = \lambda(0,e_i)\lambda(e_i,e_i+e_j) = \lambda.$$

The symmetric calculation shows that $\pi(\phi_\lambda(0+v_j)\phi_\lambda(e_j+v_i)) = \lambda$. Hence ϕ_λ satisfies (4.2). To see that it is the unique such coloured-graph morphism, suppose that $f \in c^{-1}(i)$ and $g \in c^{-1}(j)$ and $\pi(fg) = \lambda$. Then the factorization property forces $\pi(f) = \lambda(0,e_i)$ and $\pi(g) = \lambda(e_i, e_i+e_j)$. Since π is injective on E^1_Λ, it follows that $f = \overline{\lambda(0,e_i)} = \phi_\lambda(0+v_i)$ and $g = \overline{\lambda(e_i,e_i+e_j)} = \phi_\lambda(e_i+v_j)$. A symmetric argument applies with i and j interchanged, and this proves the first statement of the lemma.

To see that the collection $\mathcal{C}_\Lambda$ is complete, fix $f, g \in E^1_\Lambda$ with $s(f) = r(g)$ and $c(f) \neq c(g)$, say $c(f) = c_i$ and $c(g) = c_j$. Then $\pi(f) \in \Lambda^{e_i}$ and $\pi(g) \in \Lambda^{e_j}$, so $\pi(fg) \in \Lambda^{e_i+e_j}$, and the factorization property ensures that fg traverses $\phi_{\pi(fg)}$. Moreover, if $\lambda \in \Lambda^{e_i+e_j}$ is another path such that fg traverses ϕ_λ, then

$$\lambda = \lambda(0,e_i)\lambda(e_i,e_i+e_j) = \pi(\phi_\lambda(0+v_i)\phi_\lambda(e_i+v_j)) = \pi(fg),$$

so $\phi_{\pi(fg)}$ is the unique element of $\mathcal{C}_\Lambda$ such that fg traverses $\phi_{\pi(fg)}$. For the associativity condition, suppose we have f, g, h, f^i, g^i, h^i and f_i, g_i, h_i as in (3.2). By associativity of composition in Λ, we have

$$\pi(h_2g_2f_2) = \pi(fgh) = \pi(h^2g^2f^2),$$

so the factorization property in Λ forces $\pi(h_2) = \pi(h^2)$, $\pi(g_2) = \pi(g^2)$ and $\pi(f_2) = \pi(f^2)$. Since π is injective on E^1_Λ, it follows that $h_2 = h^2$, $g_2 = g^2$ and $f_2 = f^2$, as required. □

Notation 4.3. Let E be a k-coloured graph and let $\mathcal{C}$ be a complete collection of squares in E that is associative. For each $m \in \mathbb{N}^k$ we write $\Lambda^m_{(E,\mathcal{C})}$ for the set of all $\mathcal{C}$-compatible coloured-graph morphisms $E_{k,m} \to E$. Let

$$\Lambda_{(E,\mathcal{C})} := \bigcup_{m\in\mathbb{N}^k} \Lambda^m_{(E,\mathcal{C})}.$$

Let $d\colon \Lambda_{(E,\mathcal{C})} \to \mathbb{N}^k$ and $r, s\colon \Lambda_{(E,\mathcal{C})} \to \Lambda^0_E$ be as defined in Example 3.1. For $v \in E^0$ we define $\lambda_v\colon E_{k,0} \to E$ by $\lambda_v(0) = v$, and for $1 \leqslant i \leqslant k$ and $f \in E^1$ with $c(f) = c_i$ we define $\lambda_f\colon E_{k,e_i} \to E$ by $\lambda_f(0) = r(f)$, $\lambda_f(e_i) = s(f)$ and $\lambda_f(0+v_i) = f$.

Our first main theorem shows that the notation above describes a k-graph whose skeleton is isomorphic to E under an isomorphism which carries the commuting squares of Λ to the elements of $\mathcal{C}$.

Theorem 4.4. *Fix a k-coloured graph E and a complete collection of squares $\mathcal{C}$ in E that is associative. If $\mu\colon E_{k,m} \to E$ and $\nu\colon E_{k,n} \to E$ are $\mathcal{C}$-compatible coloured-graph morphisms such that $s(\mu) = r(\nu)$, then there exists a unique $\mathcal{C}$-compatible coloured-graph morphism $\mu\nu\colon E_{k,m+n} \to E$ such that $(\mu\nu)|_{E_{k,m}} = \mu$ and $(\mu\nu)|^*_{E_{k,[m,m+n]}} = \nu$. Under this composition map, the set $\Lambda = \Lambda_{(E,\mathcal{C})}$ of Notation 4.3, endowed with the structure maps defined there, is a k-graph. There is an isomorphism $\rho\colon E \to E_\Lambda$ such that $\rho^0(v) = \overline{\lambda_v}$ for all $v \in E^0$ and $\rho^1(f) = \overline{\lambda_f}$ for all $f \in E^1$; this ρ satisfies $\rho \circ \phi \in \mathcal{C}_\Lambda$ for all $\phi \in \mathcal{C}$.*

Our second main theorem says that the k-graph $\Lambda_{(E,\mathcal{C})}$ is uniquely determined, up to isomorphism, by the isomorphism class of $(E, \mathcal{C})$.

Theorem 4.5. *Fix a k-graph Γ, a k-coloured graph E and a complete collection $\mathcal{C}$ of squares in E that is associative. Suppose that $\psi\colon E_\Gamma \to E$ is a coloured-graph isomorphism such that $\psi \circ \phi \in \mathcal{C}$ for all $\phi \in \mathcal{C}_\Gamma$. Then for each $\gamma \in \Gamma$ there is a $\mathcal{C}$-compatible coloured-graph morphism $\theta_\gamma\colon E_{k,d(\gamma)} \to E$ such that*

$$\theta^0_\gamma(m) = \psi^0(\overline{\gamma(m)}) \qquad \text{for } m \in E^0_{k,d(\gamma)}, \tag{4.3}$$

$$\theta^1_\gamma(m + v_i) = \psi^1(\overline{\gamma(m, m + e_i)}) \quad \text{for } m + v_i \in E^1_{k,d(\gamma)}. \tag{4.4}$$

Moreover, the map $\theta\colon \gamma \mapsto \theta_\gamma$ is an isomorphism $\Gamma \cong \Lambda_{(E,\mathcal{C})}$.

The key technical result which we need to prove Theorems 4.4 and 4.5 says that every path in the coloured graph E determines a unique element of Λ. We first use the associativity condition to prove this in the special case of a tri-coloured path of length 3, and then deal with arbitrary paths using an inductive argument.

Lemma 4.6. *Let E be a k-coloured graph and let $\mathcal{C}$ be a complete collection of squares in E that is associative. If $f, g, h \in E^1$ are of distinct colour and fgh is a path in E, then there is a unique $\mathcal{C}$-compatible coloured-graph morphism $\lambda\colon E_{k,d(fgh)} \to E$ such that fgh traverses λ.*

Proof. The completeness of $\mathcal{C}$ implies that there exist paths f^i, g^i, h^i and f_i, g_i, h_i satisfying the equations in (3.2). Let λ be the coloured-graph morphism such that each of fgh, fh_1g_1, $h_2f_1g_1$, $h_2g_2f_2$, g^1f^1h and $g^1h^1f^2$ traverses λ. Associativity of $\mathcal{C}$ ensures that λ is $\mathcal{C}$-compatible. Since the values of the f^i, g^i, h^i and f_i, g_i, h_i are determined by f, g, h and $\mathcal{C}$, if fgh also traverses μ, then $\mu = \lambda$. □

Proposition 4.7. *Let E be a k-coloured graph and let $\mathcal{C}$ be a complete collection of squares in E that is associative. For every $x \in E^*$ there is a unique $\mathcal{C}$-compatible coloured-graph morphism $\lambda_x\colon E_{k,d(x)} \to E$ such that x traverses λ_x.*

Remark 4.8. The notation of Proposition 4.7 is consistent with that of Notation 4.3, since λ_v and λ_f (see Notation 4.3) are the unique morphisms such that v traverses λ_v and f traverses λ_f.

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Proof of Proposition 4.7. We prove this by induction on $|x|$. If $|x| = 0$, then the result is trivial.

Now suppose as an inductive hypothesis that, for every $y \in E^*$ with $|y| \leqslant n$, the path y traverses a unique coloured-graph morphism $\lambda_y \colon E_{k,d(y)} \to E$. Fix a path $x \in E^*$ with $|x| = n+1$, and write $x = yf$, where $f \in E^1$, with $c(f) = c_i$, say.

Let $m := q(c(y))$. By the inductive hypothesis, y traverses a unique $\mathcal{C}$-compatible coloured-graph morphism λ_y. We complete the proof by consideration of three cases: $|\{j \neq i \colon m_j > 0\}| = 0$, $|\{j \neq i \colon m_j > 0\}| = 1$ and $|\{j \neq i \colon m_j > 0\}| \geqslant 2$.

Suppose first that $|\{j \neq i \colon m_j > 0\}| = 0$. Then $E_{k,d(x)} = E_{k,m} \cup E_{k,[m,m+e_i]}$, so the formulae

$$\lambda_x|_{E_{k,m}} = \lambda_y, \quad \lambda_x(m+v_i) = f \quad \text{and} \quad \lambda_x(m+e_i) = s(f) \tag{4.5}$$

completely specify a coloured-graph morphism λ_x such that x traverses λ_x. Furthermore, λ_x is the unique such coloured-graph morphism: if x also traverses μ, then μ satisfies the formulae (4.5). This completes the proof when $|\{j \neq i \colon m_j > 0\}| = 0$.

Suppose for the rest of the proof that $|\{j \neq i \colon m_j > 0\}| \geqslant 1$ (we will consider separately later the cases $|\{j \neq i \colon m_j > 0\}| = 1$ and $|\{j \neq i \colon m_j > 0\}| \geqslant 2$). Then

$$E_{k,m+e_i} = E_{k,m} \cup \Bigl(\bigcup_{j \neq i, m_j > 0} E_{k,m+e_i-e_j} \Bigr) \cup \Bigl(\bigcup_{j \neq i, m_j > 0} E_{k,[m-e_j,m+e_i]} \Bigr) \tag{4.6}$$

(the union here is taken inside the enveloping graph $E_{k,m+e_i}$ and is not a disjoint union; for example, $E_{k,m+e_i-e_j} \cap E_{k,m+e_i-e_l} = E_{k,m+e_i-e_j-e_l}$). For each $j \neq i$ such that $m_j > 0$, fix, for the remainder of the proof, a path z^j which traverses $\lambda_y|_{E_{k,m-e_j}}$.

Claim 4.9. *Suppose that $j \neq i$ satisfies $m_j > 0$. Let ϕ^j be the unique square in $\mathcal{C}$ traversed by $\lambda_y((m-e_j)+v_j)f$. Let $g^j = \phi^j(0+v_i)$ and $h^j = \phi^j(e_i+v_j)$, so $g^j h^j \sim \lambda_y((m-e_j)+v_j)f$. Then there is a unique coloured-graph morphism $\lambda^j \colon E_{k,m-e_j+e_i} \to E$ such that $\lambda^j|_{E_{k,m-e_j}} = \lambda_y|_{E_{k,m-e_j}}$ and $\lambda^j((m-e_j)+v_i) = g^j$.*

To prove Claim 4.9, observe that $|z^j g^j| = n$, so the inductive hypothesis implies that $z^j g^j$ traverses a unique $\mathcal{C}$-compatible coloured-graph morphism λ^j. Since z^j traverses both $\lambda^j|_{E_{k,m-e_j}}$ and $\lambda_y|_{E_{k,m-e_j}}$, the inductive hypothesis implies that the two are equal. This proves Claim 4.9.

Suppose now that $|\{j \neq i \colon m_j > 0\}| = 1$; let j be the unique element of this set. Then Claim 4.9 and (4.6) imply that there is a well-defined function $\lambda_x \colon E_{k,m+e_i} \to E$ such that

$$\lambda_x|_{E_{k,m}} = \lambda_y, \quad \lambda_x|_{E_{k,m+e_i-e_j}} = \lambda^j \quad \text{and} \quad \lambda_x|^*_{E_{k,[m-e_j,m+e_i]}} = \phi^j. \tag{4.7}$$

This λ_x is $\mathcal{C}$-compatible by construction, and x traverses λ_x. For uniqueness, fix a $\mathcal{C}$-compatible coloured-graph morphism μ traversed by x. Then $z^j g^j h^j$ traverses μ. Hence y traverses $\mu|_{E_{k,m}}$ and $z^j g^j$ traverses $\mu|_{E_{k,m-e_j+e_i}}$. The inductive hypothesis forces $\mu|_{E_{k,m}} = \lambda_y$ and $\mu|_{E_{k,m-e_j+e_i}} = \lambda^j$. That μ is $\mathcal{C}$-compatible implies that

$$\mu|^*_{E_{k,[m-e_j,m+e_i]}} = \phi^j.$$

So $\mu = \lambda_x$. This proves the lemma when there is a unique $j \neq i$ such that $m_j \neq 0$, as claimed.

We now consider the last remaining case: suppose that there are at least two distinct $j, l \neq i$ such that $m_j, m_l > 0$.

Claim 4.10. *For distinct $j, l \neq i$ with $m_j, m_l \neq 0$, we have*

$$\lambda^j|_{E_{k,m+e_i-e_j-e_l}} = \lambda^l|_{E_{k,m+e_i-e_j-e_l}}.$$

To establish Claim 4.10, observe that since i, j, l are all different, Lemma 4.6 implies that $\lambda_y((m - e_j - e_l) + v_l)\lambda_y((m - e_j) + v_j)f$ traverses a unique $\mathcal{C}$-compatible graph morphism $\psi^{j,l}$. We show that

$$\lambda^j|^*_{E_{k,[m-e_l-e_j,m+e_i-e_j]}} = \psi^{j,l}|_{E_{k,e_i+e_l}} \quad \text{and} \quad \lambda^l|^*_{E_{k,[m-e_l-e_j,m+e_i-e_l]}} = \psi^{j,l}|_{E_{k,e_i+e_j}}. \tag{4.8}$$

By symmetry, it suffices to establish that

$$\lambda^j|^*_{E_{k,[m-e_l-e_j,m+e_i-e_j]}} = \psi^{j,l}|_{E_{k,e_i+e_l}}.$$

Since $\lambda_y((m - e_j) + v_j)f \sim g^j h^j$, and since $\psi^{j,l}$ is a $\mathcal{C}$-compatible coloured-graph morphism,

$$\lambda_y((m - e_j - e_l) + v_l)g^j h^j = \lambda^j((m - e_j - e_l) + v_l)\lambda^j((m - e_j) + v_i)h^j$$

traverses $\psi^{j,l}$. Since $\mathcal{C}$ is a complete collection of squares,

$$\lambda^j|^*_{E_{k,[m-e_l-e_j,m+e_i-e_j]}} = \psi^{j,l}|_{E_{k,e_i+e_l}}.$$

This proves (4.8).

To complete the proof of Claim 4.10, note that

$$\lambda^j|_{E_{k,m-e_j-e_l}} = \lambda_y|_{E_{k,m-e_j-e_l}} = \lambda^l|_{E_{k,m-e_j-e_l}}.$$

Suppose that z traverses this morphism. Equation (4.8) implies that $z\psi^{j,l}(0+v_i)$ traverses each of $\lambda^j|_{E_{k,m+e_i-e_j-e_l}}$ and $\lambda^l|_{E_{k,m+e_i-e_j-e_l}}$. The inductive hypothesis now establishes Claim 4.10.

For $j \neq i$ such that $m_j > 0$, let ϕ^j and λ^j be as in Claim 4.9. Then Claim 4.10 implies that the formulae

$$\lambda_x|_{E_{k,m}} = \lambda_y|_{E_{k,m}}, \quad \lambda_x|_{E_{k,m+e_i-e_j}} = \lambda^j \quad \text{and} \quad \lambda_x|^*_{E_{k,[m-e_j,m+e_i]}} = \phi^j$$

determine a well-defined coloured-graph morphism $\lambda_x \colon E_{k,m+e_i} \to E$. Moreover, λ_x is $\mathcal{C}$-compatible because each square occurring in λ_x occurs in λ_y, in one of the λ^j or in one of the ϕ^j.

To see that λ_x is the unique $\mathcal{C}$-compatible coloured-graph morphism which x traverses, fix a $\mathcal{C}$-compatible coloured-graph morphism μ traversed by x. Then y traverses $\mu|_{E_{k,m}}$, so the inductive hypothesis implies that $\mu|_{E_{k,m}} = \lambda_y$. Fix $j \neq i$ such that $m_j > 0$. That $\mathcal{C}$

is a complete collection of squares and that $\lambda_y((m-e_j)+v_j)f$ traverses $\mu|^*_{E_{k,[m-e_j,m+e_i]}}$ implies that

$$\mu|^*_{E_{k,[m-e_j,m+e_i]}} = \phi^j.$$

In particular, $\mu((m-e_j)+v_i) = g^j$, and hence $z^j g^j$ traverses $\mu|_{E_{k,m-e_j+e_i}}$. The inductive hypothesis forces $\mu|_{E_{k,m+e_i-e_j}} = \lambda^j$. It now follows from (4.6) that $\mu = \lambda_x$. □

Corollary 4.11. *Let E be a k-coloured graph and let $\mathcal{C}$ be a complete collection of squares in E that is associative. If $\mu\colon E_{k,m} \to E$ and $\nu\colon E_{k,n} \to E$ are $\mathcal{C}$-compatible coloured-graph morphisms such that $s(\mu) = r(\nu)$, then there exists a unique $\mathcal{C}$-compatible coloured-graph morphism $\mu\nu\colon E_{k,m+n} \to E$ called the composition of μ and ν such that $(\mu\nu)|_{E_{k,m}} = \mu$ and $(\mu\nu)|^*_{E_{k,[m,m+n]}} = \nu$.*

Proof. Fix $x, y \in E^*$ such that x traverses μ and y traverses ν. Proposition 4.7 implies that xy traverses a unique $\mathcal{C}$-compatible coloured-graph morphism $\mu\nu$. Then x traverses $(\mu\nu)|_{E_{k,m}}$ and y traverses $(\mu\nu)|^*_{E_{k,[m,m+n]}}$, so Proposition 4.7 implies that $(\mu\nu)|_{E_{k,m}} = \mu$ and $(\mu\nu)|^*_{E_{k,[m,m+n]}} = \nu$.

Moreover, if λ is any other coloured-graph morphism such that $\lambda|_{E_{k,m}} = \mu$ and $\lambda|^*_{E_{k,[m,m+n]}} = \nu$, then Remark 3.2 shows that xy traverses λ and uniqueness in Proposition 4.7 forces $\lambda = \mu\nu$. □

Remark 4.12. Let E be a k-coloured graph and let $\mathcal{C}$ be a complete collection of squares in E that is associative. Fix $m \leqslant n$ in $\mathbb{N}^k$ and suppose that $\lambda\colon E_{k,n} \to E$ is a $\mathcal{C}$-compatible coloured-graph morphism. Corollary 4.11 implies that $\mu := \lambda|_{E_{k,m}}$ and $\nu := \lambda|^*_{E_{k,[m,n]}}$ satisfy $\mu\nu = \lambda$. Suppose that $\mu'\colon E_{k,m} \to E$ and $\nu'\colon E_{k,n-m} \to E$ are another two $\mathcal{C}$-compatible coloured-graph morphisms such that $\mu'\nu' = \lambda$. Then $\mu' = \lambda|_{E_{k,m}} = \mu$ and $\nu' = \lambda|^*_{E_{k,[m,n]}} = \nu$. So μ and ν are the unique coloured-graph morphisms with $d(\mu) = m$, $d(\nu) = n$ and $\lambda = \mu\nu$.

Corollary 4.13. *Let E be a k-coloured graph and let $\mathcal{C}$ be a complete collection of squares in E that is associative. If $\lambda\colon E_{k,l} \to E$, $\mu\colon E_{k,m} \to E$ and $\nu\colon E_{k,n} \to E$ are $\mathcal{C}$-compatible coloured-graph morphisms such that $s(\lambda) = r(\mu)$ and $s(\mu) = r(\nu)$, then $\lambda(\mu\nu) = (\lambda\mu)\nu$.*

Proof. Fix $x_\lambda, x_\mu, x_\nu \in E^*$ such that x_λ traverses λ, x_μ traverses μ and x_ν traverses ν. Repeated applications of Remark 3.2 show that $x_\lambda x_\mu$ traverses $\lambda\mu$. Hence $x_\lambda x_\mu x_\nu = (x_\lambda x_\mu)x_\nu$ traverses $(\lambda\mu)\nu$. Similarly, $x_\lambda x_\mu x_\nu = x_\lambda(x_\mu x_\nu)$ traverses $\lambda(\mu\nu)$. So Proposition 4.7 implies that $(\lambda\mu)\nu = \lambda(\mu\nu)$. □

Proof of Theorem 4.4. The first statement of the theorem is precisely Corollary 4.11. We must check that Λ is a category. For composable μ, ν we have

$$s(\mu\nu) = (\mu\nu)(d(\mu\nu)) = (\mu\nu)|^*_{E_{k,[d(\mu),d(\mu)+d(\nu)]}}(d(\nu)) = \nu(d(\nu)) = s(\nu),$$

and similarly $r(\mu\nu) = r(\mu)$. Associativity of composition follows from Corollary 4.13. For $v \in E^0$, we have $r(\lambda_v) = \lambda_v(0) = v$ and $s(\lambda_v) = \lambda_v(d(\lambda_v)) = \lambda_v(0) = v$. Moreover, if

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$r(\mu) = \lambda_v$ and $s(\nu) = \lambda_v$, then Remark 4.12 implies that $\mu = \lambda_v \mu$ and $\nu = \nu\lambda_v$. Hence Λ is a category.

Since $\mathbb{N}^k$ as a category has only one object, d trivially respects r and s. It follows immediately from the definition of composition (see Corollary 4.11) that d respects composition. So d is a functor. Remark 4.12 shows that d satisfies the factorization property. So (Λ, d) is a k-graph.

It remains to show that ρ defines an isomorphism of E with E_Λ and that $\rho \circ \phi \in \mathcal{C}_\Lambda$ for each $\phi \in \mathcal{C}$. The map $v \mapsto \overline{\lambda_v}$ is a bijection. We established above that $f \mapsto \lambda_f$ is a range- and source-preserving bijection between $c^{-1}(c_i) \subset E^1$ and Λ^{e_i}. We defined $E^1_\Lambda = \{\bar{f} \colon f \in \bigcup_{i=1}^k \Lambda^{e_i}\}$ (see Definition 4.1). For each $f \in E^1$, λ_f is the unique coloured-graph morphism traversed by f, and $\overline{\lambda_f} \in E^1_\Lambda$ satisfies $c_{E_\Lambda}(\overline{\lambda_f}) = c_i = c(f)$, $r(\overline{\lambda_f}) = \overline{\lambda_f(0)} = \overline{\lambda_{r(f)}}$ and $s(\overline{\lambda_f}) = \overline{\lambda_f(e_i)} = \overline{\lambda_{s(f)}}$. Since ρ^1 is bijective, the pair $(\rho^0, \rho^1) \colon E \to E_\Lambda$ is an isomorphism of coloured graphs. To see that ρ preserves squares, fix $\psi \in \mathcal{C}$. Then $\rho \circ \psi$ is the square ϕ_ψ of (4.1) and hence belongs to $\mathcal{C}_{\Lambda_{(E,\mathcal{C})}}$ as required. □

Proof of Theorem 4.5. For $\gamma \in \Gamma$ define $\theta_\gamma \colon E_{k,m} \to E$ as in (4.3) and (4.4). Then $r(\theta^1_\gamma(m + v_i)) = r(\psi^1(\overline{\gamma(m, m + e_i)})) = \psi^0(\gamma(m)) = \theta^0_\gamma(m)$ and similarly at the source, so θ_γ is a graph morphism. Since ψ^1 preserves colour, we have

$$c_E(\theta^1_\gamma(m + v_i)) = c_E(\psi^1(\overline{\gamma(m, m + e_i)})) = c_{E_\Gamma}(\overline{\gamma(m, m + e_i)}) = c_i = c_{E_{k,d(\gamma)}}(m + v_i),$$

so θ_γ is a coloured-graph morphism.

To see that θ_γ is $\mathcal{C}$-compatible, fix a square α occurring in θ_γ. Then there exist $m \in \mathbb{N}^k$ and $i, j \leqslant k$ such that $\alpha(x) = \theta_\gamma(x + m)$ for all $x \in E_{k,e_i+e_j}$. Let $\lambda := \gamma(m, m + e_i + e_j)$. Then $\alpha^0(n) = \theta^0_\gamma(m + n) = \psi^0(\lambda(n))$ for $0 \leqslant n \leqslant e_i + e_j$, and $\alpha^1(n + v_l) = \theta^1_\gamma(m + n + v_l) = \psi^1(\overline{\lambda(n, n + e_l)})$ whenever $n, n + e_l \leqslant e_i + e_j$. That is, $\alpha = \psi \circ \phi_\lambda$, where $\phi_\lambda \in \mathcal{C}_\Gamma$ is as in Definition 4.1. By hypothesis, that $\phi_\lambda \in \mathcal{C}_\Gamma$ implies that $\alpha \in \mathcal{C}$, and hence θ_γ is $\mathcal{C}$-compatible. Hence $\theta_\gamma \in \Lambda^{d(\gamma)}_{(E,\mathcal{C})}$.

The assignment $\gamma \mapsto \theta_\gamma$ is a degree-, range- and source-preserving map $\theta \colon \Gamma \to \Lambda_{(E,\mathcal{C})}$. To see that θ is injective, fix $\gamma, \gamma' \in \Gamma$ and suppose that $\theta_\gamma = \theta_{\gamma'}$. Write $\gamma = \gamma_1 \cdots \gamma_n$ where each $d(\gamma_i) = e_{j_i}$, and $\gamma' = \gamma'_1 \cdots \gamma'_n$ where each $d(\gamma'_i) = d(\gamma_i)$. For $i \leqslant n$ define $p_i := \sum_{j=1}^i d(\gamma_j)$. Then, for each $i \leqslant n$,

$$\psi^1(\overline{\gamma_i}) = \theta^1_\gamma(p_{i-1} + v_{j_i}) = \theta^1_{\gamma'}(p_{i-1} + v_{j_i}) = \psi^1(\overline{\gamma'_i}).$$

Since ψ^1 is injective, it follows that $\overline{\gamma_i} = \overline{\gamma'_i}$ and hence $\gamma_i = \gamma'_i$. So θ is injective.

To see that θ preserves composition, fix $\gamma, \gamma' \in \Gamma$ with $s(\gamma) = r(\gamma')$, and fix paths $\psi^1(\overline{\gamma_1}) \cdots \psi^1(\overline{\gamma_m})$ and $\psi^1(\overline{\gamma'_1}) \cdots \psi^1(\overline{\gamma'_n})$ which traverse θ_γ and $\theta_{\gamma'}$. Then

$$\psi^1(\overline{\gamma_1}) \cdots \psi^1(\overline{\gamma_m})\psi^1(\overline{\gamma'_1}) \cdots \psi^1(\overline{\gamma'_n})$$

traverses both $\theta_{\gamma\gamma'}$ and $\theta_\gamma\theta_{\gamma'}$. So $\theta_{\gamma\gamma'} = \theta_\gamma\theta_{\gamma'}$ by Proposition 4.7. So θ is a functor.

To see that θ is surjective, fix $\lambda \in \Lambda_{(E,\mathcal{C})}$ and a path $f_1 \cdots f_m$ which traverses λ. Then each $f_i \in E^1$, and since ψ^1 is surjective, each $f_i = \psi^1(g_i)$ for some $g_i \in E^1_\Gamma$. Each $g_i = \overline{\gamma_i}$ for some $\gamma_i \in \Gamma$. Let $\gamma := \gamma_1 \cdots \gamma_n$. Then $f_1 \cdots f_m = \psi^1(\overline{\gamma_1}) \cdots \psi^1(\overline{\gamma_m})$ traverses both λ and θ_γ. So Proposition 4.7 implies that $\theta_\gamma = \lambda$ and hence θ is surjective. Thus θ is an isomorphism $\Gamma \cong \Lambda_{(E,\mathcal{C})}$. □

5. Topology of path spaces

In [**16**, Proposition 4.3] the authors appeal to general category-theoretic results [**23**] to see that, given a k-coloured graph E and a complete collection of squares $\mathcal{C}$ in E that is associative, the corresponding k-graph $\Lambda_{(E,\mathcal{C})}$ is isomorphic to the quotient of the category E^* under the equivalence relation $\sim$ generated by

$$\bigcup_{n\geqslant 2}\{(x,y)\in E^n\times E^n\colon \text{there exists } i<n \text{ such that } x_j=y_j \text{ whenever } j\notin\{i,i+1\} \text{ and } x_ix_{i+1}\sim_{\mathcal{C}} y_iy_{i+1}\}. \tag{5.1}$$

We start this section with a direct proof of this assertion by showing that each equivalence class for $\sim$ is the set of paths which traverse some $\lambda\in\Lambda_{(E,\mathcal{C})}$. We show that the quotient map extends to a surjection from the space of all paths in E to the space of all paths in Λ.

We then restrict attention to k-graphs which are row finite with no sources in the sense that $0<|v\Lambda^{e_i}|<\infty$ for all $v\in\Lambda^0$ and $i\leqslant k$ (see [**14**]). In this context, the space Λ^∞ of infinite paths in Λ (see Remark 5.3 for a precise definition)—under the topology with basic open sets $\mathcal{Z}(\mu):=\{x\in\Lambda^\infty\colon x(0,d(\mu))=\mu\}$ indexed by $\mu\in\Lambda$—is a locally compact Hausdorff space. Furthermore, it is the unit space of the groupoid $\mathcal{G}_\Lambda$ used to define $C^*(\Lambda)$ in [**14**].

We show that Λ^∞ is the topological quotient of the space

$$\partial^c E:=\{x\in E^\infty\colon |\{i\colon c(x_i)=c_j\}|=\infty \text{ for each } j\leqslant k\}. \tag{5.2}$$

We also show that $\partial^c E$ is a closed subspace of E^∞. Lastly, we present an example which shows that these results do not necessarily hold if Λ is not row finite.

The following elementary lemma can be deduced from more general results in the literature (see, for example, [**10**, Theorem 3.9]), but we provide a straightforward proof for completeness. Recall that q denotes the quotient map from $\mathbb{F}_k^+$ to $\mathbb{N}^k$.

Lemma 5.1. *Fix $w,w'\in\mathbb{F}_k^+$ and suppose that $q(w)=q(w')$. Then there is a finite sequence $(w^i)_{i=1}^m$ in $\mathbb{F}_k^+$ such that $w^1=w$, $w^m=w'$, and for each $i<m$ there exists $j_i<|w|$ such that $w^i_l=w^{i+1}_l$ for $l\notin\{j_i,j_i+1\}$, $w^i_{j_i}=w^{i+1}_{j_i+1}$ and $w^i_{j_i+1}=w^{i+1}_{j_i}$.*

Proof. The result is trivial if $|w|=0$. Suppose $|w|\geqslant 1$ and the result holds for words of length $|w|-1$. Since $q(w)=q(w')$ there exists j such that $w_j=w'_1$. Let

$$\begin{aligned} w^2 &= w_1\cdots w_{j-2}w_jw_{j-1}w_{j+1}\cdots w_{|w|},\\ w^3 &= w_1\cdots w_jw_{j-2}w_{j-1}w_{j+1}\cdots w_{|w|},\\ &\ \ \vdots\\ w^j &= w_jw_1\cdots w_{j-2}w_{j-1}w_{j+1}\cdots w_{|w|}. \end{aligned}$$

Let $x=w_1\cdots w_{j-2}w_{j-1}w_{j+1}\cdots w_{|w|}$ and $x'=w'_2\cdots w'_{|w|}$. Then $w^j=w'_1x$, $w'=w'_1x'$, $q(x)=q(x')$ and $|x|=|w|-1$. Apply the inductive hypothesis to x and x' to obtain $x^1,\dots,x^n$. The sequence $w^1,\dots w^j,w'_1x^2,\dots,w'_1x^n$ has the desired properties. □

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Proposition 5.2. *Let E be a k-coloured graph and let $\mathcal{C}$ be a complete collection of squares in E that is associative. Let $\sim$ be the equivalence relation on E^* generated by (5.1). For $x, y \in E^*$, we have $x \sim y$ if and only if x and y traverse the same $\mathcal{C}$-compatible graph morphism λ. The structure maps $s([x]) := s(x)$, $r([x]) := r(x)$, $d([x]) := q(c(x))$ and $[x][y] := [xy]$ are well defined on $E^*/{\sim}$, and under these operations $E^*/{\sim}$ is a k-graph which is isomorphic to $\Lambda_{(E,\mathcal{C})}$.*

Proof. For a pair (x, y) as in (5.1), we have $r(x) = r(y)$, $s(x) = s(y)$ and $q(c(x)) = q(c(y))$, so the formulae $s([x]) := s(x)$, $r([x]) := r(x)$ and $d([x]) := q(c(x))$ are well defined.

If $x \sim y$, then there is a finite sequence of pairs (x^l, x^{l+1}), $1 \leqslant l \leqslant m-1$, each of the form described in (5.1), such that $x^1 = x$ and $x^m = y$. So it suffices to fix (x, y) as in (5.1) and show that x and y traverse the same $\mathcal{C}$-compatible coloured-graph morphism. For this, let ϕ be the square in $\mathcal{C}$ traversed by $x_i x_{i+1}$ and hence also by $y_i y_{i+1}$. By Proposition 4.7, $x_1 \cdots x_{i-1} = y_1 \cdots y_{i-1}$ traverses a unique $\mathcal{C}$-compatible morphism μ and $x_{i+2} \cdots x_n = y_{i+2} \cdots y_n$ traverses a unique $\mathcal{C}$-compatible morphism ν. By Corollary 4.11, there is a unique $\mathcal{C}$-compatible $\lambda = \mu\phi\nu$ which agrees, upon restriction, with μ, ϕ and ν. Each of x and y traverses this λ.

Now suppose that x and y traverse a common $\mathcal{C}$-compatible morphism λ. Then, in particular, $q(c(x)) = q(c(y))$. By Lemma 5.1, there is a finite sequence $(w^i)_{i=1}^m$ in $\mathbb{F}_k^+$ such that $w^1 = c(x)$, $w^m = c(y)$, and for each $i \leqslant m-1$ there exists $j_i < |x|$ such that $w^i_l = w^{i+1}_l$ for $l \notin \{j_i, j_i+1\}$, and $w^i_{j_i} = w^{i+1}_{j_i+1}$ and $w^i_{j_i+1} = w^{i+1}_{j_i}$. For each i, let z^i be the path which traverses λ such that $c(x^i) = w^i$, and for each $i \leqslant m$ and $l \leqslant |x|$, let $p^i_l := q(c(x^i_1 \cdots x^i_l))$. Then for each $i \leqslant m$, both $x^i_1 \cdots x^i_{j_i-1}$ and $x^{i+1}_1 \cdots x^{i+1}_{j_i-1}$ traverse $\lambda(0, p^i_{j_i-1})$, so they are equal, and likewise

$$x^i_{j_i+2} \cdots x^i_{|x|} = x^{i+1}_{j_i+2} \cdots x^{i+1}_{|x|}.$$

Moreover, both $x^i_{j_i} x^i_{j_i+1}$ and $x^{i+1}_{j_i} x^{i+1}_{j_i+1}$ traverse $\lambda|^*_{[p^i_{j_i-1}, p^i_{j_i+1}]}$, which, since λ is $\mathcal{C}$-compatible, belongs to $\mathcal{C}$. Thus the pair (x^i, x^{i+1}) is a pair of paths as in (5.1), and it follows that $x \sim y$ as required.

By the preceding two paragraphs, the assignment $\rho\colon \lambda \mapsto [x]$ for any x which traverses λ is a well-defined bijection from $\Lambda_{(E,\mathcal{C})}$ to $E^*/{\sim}$ which preserves range, source and degree. By definition of composition in $\Lambda_{(E,\mathcal{C})}$, if x traverses μ and y traverses μ, then xy traverses $\lambda\mu$. So if $[x] = [x']$ and $[y] = [y']$, then x and x' both traverse μ, and y and y' both traverse ν, so xy and $x'y'$ both traverse $\mu\nu$. Thus

$$[xy] = \rho(\mu\nu) = [x'y'],$$

showing that the composition on $E^*/{\sim}$ is well defined. So ρ is a degree-preserving bijective functor, and hence an isomorphism of k-graphs. $\square$

We recall the k-graphs $\Omega_{k,m}$ described in [**19**, Examples 2.2]. For $m \in (\mathbb{N} \cup \{\infty\})^k$, define $\Omega_{k,m}$ to be the category with $\mathrm{Obj}(\Omega_{k,m}) = \{n \in \mathbb{N}^k : n \leqslant m\}$, $\mathrm{Mor}(\Omega_{k,m}) = \{(p, q) \in \mathbb{N}^k \times \mathbb{N}^k : p \leqslant q \leqslant m\}$, $s(p, q) = q$, $r(p, q) = p$ and $(p, q)(q, r) = (p, r)$. Then, with

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$d(p,q) = q-p$, the pair $(\Omega_{k,m}, d)$ is a row-finite k-graph. By convention, $\Omega_k = \Omega_{k,(\infty,\ldots,\infty)}$. Note that there is only one possible complete collection of squares $\mathcal{C}$ in the k-coloured graph $E_{k,m}$; this collection is also associative, and the k-graph $\Lambda_{E_{k,m},\mathcal{C}}$ of Theorem 4.4 is isomorphic to $\Omega_{k,m}$.

Remark 5.3. Let Λ be a k-graph. For $m \in \mathbb{N}^k$, the factorization property gives a bijection $\lambda \mapsto x_\lambda$ between Λ^m and the set of graph morphisms from $\Omega_{k,m}$ to Λ: for $\lambda \in \Lambda$ and $p \leqslant q \leqslant d(\lambda)$, $x_\lambda(p,q)$ is the unique element of Λ^{q-p} such that $\lambda = \lambda' x_\lambda(p,q)\lambda''$ for some λ', λ''. By analogy, for $m \in (\mathbb{N} \cup \{\infty\})^k$, we call a k-graph morphism $x\colon \Omega_{k,m} \to \Lambda$ a *path of degree* m in Λ, and we write $d(x)$ for m and $r(x)$ for $x(0)$. We continue to denote the collection of all such paths by Λ^m. It is conventional to identify λ with x_λ, and in particular to denote $x_\lambda(p,q)$ by $\lambda(p,q)$, so $\lambda = \lambda(0,p)\lambda(p,q)\lambda(q,d(\lambda))$ whenever $0 \leqslant p \leqslant q \leqslant d(\lambda)$.

We shall write W_Λ for the *path space* $W_\Lambda := \bigcup_{m \in (\mathbb{N}\cup\{\infty\})^k} \Lambda^m$ of Λ.

Proposition 5.4 (cf. [14, Remarks 2.2]). *Let E be a k-coloured graph and let $\mathcal{C}$ be a complete collection of squares in E that is associative. The map $x \mapsto \lambda_x$ from E^* to $\Lambda = \Lambda_{(E,\mathcal{C})}$ of Proposition 4.7 extends uniquely to a degree-preserving map $\pi\colon W_E \to W_\Lambda$ such that, for $x \in W_E$ and $i \in \mathbb{N}$ with $i \leqslant |x|$, $\pi(x)(0, d(x_1 \cdots x_i)) = \lambda_{x_1 \cdots x_i}$. Moreover, π is surjective.*

Remark 5.5. We have used the same symbol π both for the map from W_E to W_Λ of Proposition 5.4, and for the map from E_Λ to Λ of Definition 4.1. This notation is consistent because Theorem 4.4 yields a coloured-graph isomorphism $E \cong E_\Lambda$ which carries elements of $\mathcal{C}$ to elements of $\mathcal{C}_\Lambda$.

Proof of Proposition 5.4. For $x \in W_E$ and $m \leqslant n \leqslant d(x)$, let j be the least element of $\mathbb{N}$ such that $d(x_1 \cdots x_j) \geqslant n$, and define $\pi(x)(m,n) := \lambda_{x_1 \cdots x_j}|^*_{E_{k,[m,n]}}$. Proposition 4.7 implies that, for $j \leqslant l$, we have

$$\lambda_{x_1 \cdots x_l}|_{E_{k,d(x_1 \cdots x_j)}} = \lambda_{x_1 \cdots x_j}.$$

Hence $\pi(x)(0, d(x_1 \cdots x_j)) = \lambda_{x_1 \cdots x_j}$ for all $j \leqslant |x|$. The factorization property in Λ implies that $\pi(x)$ is a k-graph morphism from $\Omega_{k,d(x)}$ to Λ. For uniqueness of π, observe that, by uniqueness of factorizations in Λ, any $y \in W_\Lambda$ such that $y(x_1 \cdots x_i) = \lambda_{x_1 \cdots x_i}$ for all $i \leqslant d(x)$ must satisfy $y(m,n) = \lambda_{x_1 \cdots x_j}|^*_{E_{k,[m,n]}}$ whenever $d(x_1 \cdots x_j) \geqslant n$.

To see that π is surjective first note that if $\lambda \in \Lambda$ then any path x which traverses λ satisfies $\pi(x) = \lambda$. So fix $y \in W_\Lambda \setminus \Lambda$. Fix a sequence $(m_j)_{j=0}^\infty$ such that $m_0 = 0$, $m_{j+1} - m_j \in \{e_1, \ldots, e_k\}$ for all j and $\bigvee_{j \in \mathbb{N}} m_j = d(y)$. For each $j \in \mathbb{N}$, define $x_j := y(m_{j-1}, m_j) \in E^1$. Then $x = x_1 x_2 \cdots \in W_E$, and $\pi(x)(m,n) = y(m,n)$ for all m, n by uniqueness of factorizations in Λ, so $\pi(x) = y$. □

If $\pi\colon W_E \to W_\Lambda$ is the surjection of Proposition 5.4, then Proposition 4.11 implies that $\pi(x)\pi(y) = \pi(xy)$ when x and y are finite with $r(y) = s(x)$.

Now let Λ be a row-finite k-graph with no sources. Recall that Λ^∞ is the collection of k-graph morphisms from Ω_k to Λ, and $\partial^c E$ is the collection of infinite paths in E which contains infinitely many edges of each colour.

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Remark 5.6. Let E be a k-coloured graph and let $\mathcal{C}$ be a complete collection of squares in E that is associative. Let $\Lambda = \Lambda_{(E,\mathcal{C})}$ be the corresponding k-graph as in Theorem 4.4. Identify Λ^0 with E^0. Then for each $v \in \Lambda^0$ and $i \leqslant k$, we have $|v\Lambda^{e_i}| = |\{e \in E^1 : r(e) = v \text{ and } c(e) = c_i\}|$. Hence Λ is row finite and has no sources if and only if $0 < |\{e \in E^1 : r(e) = v \text{ and } c(e) = c_i\}| < \infty$ for all $v \in E^0$ and $i \leqslant k$.

Recall that if Λ is a row-finite k-graph, then the topology on Λ^∞ has basic open sets $\mathcal{Z}(\mu) = \{x \in \Lambda^\infty : x(0, d(\mu)) = \nu\}$ indexed by $\mu \in \Lambda$ and is a locally compact Hausdorff topology. If E is a k-coloured graph and $\mathcal{C}$ a complete collection of squares in E such that $\Lambda_{(E,\mathcal{C})}$ is row finite with no sources, then E is also row-finite and has no sources. So the sets $\mathcal{Z}(y)$, where $y \in E^*$, form a basis for a locally compact Hausdorff topology on E^∞, and we endow $\partial^c E$ with the subspace topology.

Proposition 5.7. *Let E be a k-coloured graph and let $\mathcal{C}$ be a complete collection of squares in E that is associative. Let $\Lambda = \Lambda_{(E,\mathcal{C})}$ as in Theorem 4.4. Suppose that Λ is row finite and has no sources. Then the surjection of W_E onto W_Λ of Proposition 5.4 restricts to a surjection $\pi : \partial^c E \to \Lambda^\infty$. Moreover, $U \subseteq \Lambda^\infty$ is open if and only if $\pi^{-1}(U) \subseteq \partial^c E$ is open.*

Proof. That $\partial^c E$ is precisely $\pi^{-1}(\Lambda^\infty)$ follows from the definitions of the two sets and of π. Since $\pi : W_E \to W_\Lambda$ is surjective, it follows that its restriction to $\partial^c E$ is surjective onto Λ^∞.

Suppose that U is open in Λ^∞, and fix $x \in \pi^{-1}(U)$. We seek a basic open set B_x in $\partial^c E$ such that $x \in B_x \subset \pi^{-1}(U)$. Since U is open, there exists $\mu \in \Lambda$ such that $\pi(x) \in \mathcal{Z}(\mu) \subset U$. Fix $n \in \mathbb{N}$ such that $q(c(x_1 \cdots x_n)) > d(\mu)$. Then $\pi(x_1 \cdots x_n) \in \mathcal{Z}(\mu)$. Let $y_x = x_1 \cdots x_n$. Then $x \in \mathcal{Z}(y_x)$. To see that $\mathcal{Z}(y_x) \subset \pi^{-1}(U)$ fix $y \in \mathcal{Z}(y_x)$, say $y = y_x y'$. Then $\pi(y) = \pi(x_1 \cdots x_n y') = \pi(x_1 \cdots x_n)\pi(y') \in \mathcal{Z}(\mu) \subset U$, so $y \in \pi^{-1}(U)$ as required.

For the reverse implication, suppose that $\pi^{-1}(U)$ is open in $\partial^c E$ and fix $\lambda \in U$. We seek a basic open set B_λ such that $\lambda \in B_\lambda \subset U$. Fix $x \in E^\infty$ which traverses λ. Then $x \in \partial^c E$, and $x \in \pi^{-1}(U)$ which is open. Hence there exists a basic open set $B_x \in \partial^c E$ such that $x \in B_x \subset \pi^{-1}(U)$. So $B_x = \mathcal{Z}(y_x)$ for some $y_x \in E^*$, and

$$\lambda = \pi(x) = \pi(y_x x') = \pi(y_x)\pi(x') \in \mathcal{Z}(\pi(y_x)).$$

To see that $\mathcal{Z}(\pi(y_x)) \subset U$, let $\mu \in \mathcal{Z}(\pi(y_x))$. Write $\mu = \pi(y_x)\mu'$, and let $x_{\mu'}$ be a path in E^* which traverses μ'. Then $y_x x_{\mu'} \in \mathcal{Z}(y_x) \subset \pi^{-1}(U)$, which implies that $\mu = \pi(y_x x_{\mu'}) \in U$. □

Proposition 5.7 implies that, when $\Lambda_{E,\mathcal{C}}$ is row finite with no sources, the topology on Λ^∞ is the quotient topology inherited from $\partial^c E$ under π. In particular, π is continuous.

The role in the proof of Proposition 5.7 of the hypothesis that Λ is row finite and has no sources is not readily apparent. Indeed the proposition is valid for arbitrary k-graphs Λ when Λ^∞ is endowed with the topology with basis $\{\mathcal{Z}(\mu) : \mu \in \Lambda\}$. We have included the hypothesis because, from the point of view of C^*-algebras associated to higher-rank graphs, the result is only interesting for row-finite k-graphs with no sources. We close the

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section by discussing why this is, what would be the corresponding result for arbitrary k-graphs and why it does not hold.

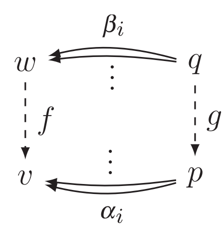
When Λ is row finite with no sources, Λ^∞ is homeomorphic to the unit space of the groupoid $\mathcal{G}_\Lambda$ of [**14**]; it is also homeomorphic to the spectrum of the commutative subalgebra of $C^*(\Lambda)$ spanned by the projections $s_\lambda s_\lambda^*$ (see [**26**] and the opening of § 6). If Λ is not row finite, this is no longer the case: Λ^∞ need not even be locally compact. To see this, suppose that Λ is the 1-graph with one vertex and infinitely many edges $\{f_i \colon i \in \mathbb{N}\}$. Given any $x \in \Lambda^\infty$, any neighbourhood of x contains $\mathcal{Z}(x_1 \cdots x_n)$ for some n, and the cover $\mathcal{Z}(x_1 \cdots x_n) = \bigcup_{i=1}^\infty \mathcal{Z}(x_1 \cdots x_n f_i)$ has no finite subcover.

Instead, given a finitely aligned k-graph, let

$$\begin{aligned} \Lambda^{\leqslant\infty} := \{x \in W_\Lambda \colon\ & \text{there exists } n \leqslant d(x) \text{ such that} \\ & (n \leqslant p \leqslant d(x) \text{ and } p_i = d(x)_i) \text{ implies } x(p)\Lambda^{e_i} = \emptyset\} \end{aligned} \tag{5.3}$$

as in [**20**]. Endow W_Λ with the topology with basic open sets $\mathcal{Z}(\mu \setminus G) := \mathcal{Z}(\mu) \setminus (\bigcup_{\lambda \in G} \mathcal{Z}(\mu\lambda))$, where μ ranges over Λ and G ranges over all finite subsets of $s(\mu)\Lambda$. Then the unit space $\partial\Lambda$ of the groupoid $\mathcal{G}_\Lambda$ constructed in [**8**] is the closure of $\Lambda^{\leqslant\infty}$ in W_Λ; this is also homeomorphic to the spectrum of $\overline{\operatorname{span}}\{s_\lambda s_\lambda^* \colon \lambda \in \Lambda\}$ [**26**]. So the natural question to ask for finitely aligned k-graphs is whether this topology on $\partial\Lambda$ coincides with the quotient topology determined by the surjection $\pi \colon \pi^{-1}(\partial\Lambda) \to \partial\Lambda$, where $\pi^{-1}(\partial\Lambda)$ is given the relative topology coming from W_E.

Example 5.8. Let E be the 2-coloured graph:



Let $\mathcal{C}$ be the collection of graph morphisms $\lambda_i \colon E_{2,(1,1)} \to E$ such that $\alpha_i g$ and $f\beta_i$ both traverse λ_i for each i. This is a complete collection of squares in E. Since E has only two colours, $\mathcal{C}$ is associative. Let Λ be the 2-graph constructed from $(E, \mathcal{C})$ as in Theorem 4.4, and let $\pi \colon W_E \to W_\Lambda$ be the surjection of Proposition 5.4. Then $\pi(v)\Lambda^{\leqslant\infty} = \Lambda^{(1,1)} = \{\lambda_i \colon i \in \mathbb{N}\}$ and $\pi(\alpha_i g) = \lambda_i = \pi(f\beta_i)$ for all i.

We claim that $\alpha_i g \to v$ in W_E but that $\lambda_i \to \pi(f)$ in W_Λ. To see that $\alpha_i g \to v$ in W_E, fix a basic open set $\mathcal{Z}(y \setminus F) \subset E^*$ containing v. Then $y = v$. Since F is finite, there are only finitely many i such that either α_i or $\alpha_i g$ belongs to F. Let $N_0 := \max\{i \colon \alpha_i \in F \text{ or } \alpha_i g \in F\}$. Then $\alpha_n g \in \mathcal{Z}(v \setminus F)$ for all $n \geqslant N_0$, whence $\alpha_i g \to v$ as $i \to \infty$.

To see that $\lambda_i \to \pi(f)$ in W_Λ, fix a basic open set $\mathcal{Z}(\mu \setminus G) \subset \Lambda$ containing $\pi(f)$. Then either $\mu = \pi(f)$ or $\mu = \pi(v)$. We show that $\lambda_i \in \mathcal{Z}(\mu \setminus G)$ for large i. First suppose that $\mu = \pi(f)$. Then G is a finite collection of paths of the form $\pi(\beta_i)$. Let $N_1 = \max\{i \colon \pi(\beta_i) \in G\}$. Then $\lambda_n = \pi(f\beta_n) \in \mathcal{Z}(\pi(f) \setminus G)$ for all $n \geqslant N_1$. Now suppose that $\mu = \pi(v)$. Since G does not contain $\pi(f)$, it is a finite subset of $\{\pi(\alpha_i), \lambda_i \colon i \in \mathbb{N}\}$.

Let $N_2 = \max\{i \colon \pi(\alpha_i) \in G \text{ or } \lambda_i \in G\}$. Then $\lambda_n \in \mathcal{Z}(\pi(v) \setminus G)$ for all $n \geqslant N_2$. Hence $\lambda_i \to \pi(f)$ as $i \to \infty$.

We now have $\pi(\lim \alpha_i g) = \pi(v) \neq \pi(f) = \lim \lambda_i = \lim \pi(\alpha_i g)$, so π is not continuous.

6. Simplicity of C^*-algebras of higher-rank graphs

Suppose that Λ is a k-graph which is row finite and has no sources. For such Λ, a *Cuntz–Krieger Λ-family* in a C^*-algebra B consists of partial isometries $\{t_\lambda \colon \lambda \in \Lambda\}$ satisfying the *Cuntz–Krieger relations* [**14**]:

(CK1) $\{t_v \colon v \in \Lambda^0\}$ are mutually orthogonal projections;

(CK2) $t_{\lambda\mu} = t_\lambda t_\mu$ for all $\lambda, \mu \in \Lambda$ with $s(\lambda) = r(\mu)$;

(CK3) $t_\lambda^* t_\lambda = t_{s(\lambda)}$ for all $\lambda \in \Lambda$; and

(CK4) $t_v = \sum_{\lambda \in v\Lambda^m} t_\lambda t_\lambda^*$ for all $v \in \Lambda^0$ and $m \in \mathbb{N}^k$.

The graph C^*-algebra $C^*(\Lambda)$ is the C^*-algebra generated by a universal Cuntz–Krieger Λ-family $\{s_\lambda \colon \lambda \in \Lambda\}$; it follows from [**14**, Proposition 2.11] that each vertex projection s_v is non-zero.

As in [**21**], we say that Λ is *aperiodic* if for every vertex $v \in \Lambda^0$ and each pair $m \neq n \in \mathbb{N}^k$ there is a path $\lambda \in v\Lambda$ such that $d(\lambda) \geqslant m \vee n$ and

$$\lambda(m, m + d(\lambda) - (m \vee n)) \neq \lambda(n, n + d(\lambda) - (m \vee n)).$$

Lemma 3.2 of [**21**] implies that this formulation of aperiodicity in terms of finite paths is equivalent to the aperiodicity condition used in [**14**]. So the next theorem follows from [**14**, Theorem 4.6].

Theorem 6.1 (Cuntz–Krieger Uniqueness Theorem). *Let Λ be a row-finite, aperiodic k-graph with no sources. Suppose that $\{t_\lambda \colon \lambda \in \Lambda\}$ is a Cuntz–Krieger Λ-family, and let π be the homomorphism of $C^*(\Lambda)$ such that $\pi(s_\lambda) = t_\lambda$ for all $\lambda \in \Lambda$. If each t_v is non-zero, then π is faithful.*

The proof of this theorem in [**14**] uses a groupoid model for $C^*(\Lambda)$. Here we outline a direct proof that flows from the finite-path formulation of aperiodicity via the following lemma.

Lemma 6.2. *Let (Λ, d) be an aperiodic k-graph with no sources. Suppose that $v \in \Lambda^0$ and $l \in \mathbb{N}^k$. Then there exists $\lambda \in \Lambda$ such that $r(\lambda) = v$, $d(\lambda) \geqslant l$ and*

$$\alpha, \beta \in \Lambda v, \quad d(\alpha), d(\beta) \leqslant l \quad \text{and} \quad \alpha \neq \beta \Longrightarrow (\alpha\lambda)(0, d(\lambda)) \neq (\beta\lambda)(0, d(\lambda)). \tag{6.1}$$

Proof. We list pairs (m, n) of distinct elements of $\mathbb{N}^k$ with $0 \leqslant m, n \leqslant l$ as $\{(m^{(i)}, n^{(i)}) \colon 1 \leqslant i \leqslant p\}$. Then an induction on i shows that there exist μ_i and $l^{(i)} \in \mathbb{N}^k$ such that $r(\mu_1) = v$, $r(\mu_i) = s(\mu_{i-1})$ for $i \geqslant 1$, $d(\mu_i) = (m^{(i)} \vee n^{(i)}) + l^{(i)}$ and $\mu_i(m^{(i)}, m^{(i)} + l^{(i)}) \neq \mu_i(n^{(i)}, n^{(i)} + l^{(i)})$. We now choose an arbitrary path λ' with $d(\lambda') \geqslant l$ and $r(\lambda') = s(\mu_p)$, and claim that $\lambda := \mu_1\mu_2 \cdots \mu_p\lambda'$ has the required properties. We trivially have $d(\lambda) \geqslant l$.

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Suppose that α and β are distinct paths with source v and $d(\alpha) \vee d(\beta) \leqslant l$. If $d(\alpha) = d(\beta) = d$, say, then the initial segments $\alpha = (\alpha\lambda)(0, d)$ and $\beta = (\beta\lambda)(0, d)$ are not equal, and $\alpha\lambda \neq \beta\lambda$. So suppose that $d(\alpha) \neq d(\beta)$, say $(d(\alpha), d(\beta)) = (m^{(i)}, n^{(i)})$. Let

$$d := \sum_{j=1}^{i-1} d(\mu_j).$$

Then

$$(\alpha\lambda)(d(\alpha) + d + n^{(i)}, d(\alpha) + d + n^{(i)} + l^{(i)}) = \mu_i(n^{(i)}, n^{(i)} + l^{(i)})$$

is not the same as

$$\mu_i(m^{(i)}, m^{(i)} + l^{(i)}) = (\beta\lambda)(d(\beta) + d + m^{(i)}, d(\beta)d + m^{(i)} + l^{(i)}).$$

Since

$$d(\beta) + d + m^{(i)} = d + m^{(i)} + n^{(i)} = d(\alpha) + d + n^{(i)},$$

it follows that

$$(\alpha\lambda)(d + m^{(i)} + n^{(i)}, d + m^{(i)} + n^{(i)} + l^{(i)}) \neq (\beta\lambda)(d + m^{(i)} + n^{(i)}, d + m^{(i)} + n^{(i)} + l^{(i)}).$$

The presence of the factor λ' forces $d(\lambda) \geqslant d + m^{(i)} + n^{(i)} + l^{(i)}$, so $(\alpha\lambda)(0, d(\lambda)) \neq (\beta\lambda)(0, d(\lambda))$, as required. $\square$

Remark 6.3. The technical condition established in Lemma 6.2 is actually equivalent to aperiodicity. There is no doubt a direct combinatorial proof of this, but to see it quickly, observe that our proof of Theorem 6.1 shows that the conclusion of Lemma 6.2 implies that every ideal of $C^*(\Lambda)$ contains a vertex projection, and paragraphs two to four of the proof of Theorem 6.6 show that if every ideal of $C^*(\Lambda)$ contains a vertex projection, then Λ is aperiodic.

Proposition 6.4. *Suppose that Λ is a row-finite aperiodic k-graph with no sources, and let $\{t_\lambda : \lambda \in \Lambda\}$ be a Cuntz–Krieger Λ-family in a C^*-algebra B such that $t_v \neq 0$ for all $v \in \Lambda^0$. Let F be a finite subset of Λ and let $a\colon (\mu, \nu) \mapsto a_{\mu,\nu}$ be a $\mathbb{C}$-valued function on $F \times F$ such that $s(\mu) = s(\nu)$ whenever $a_{\mu,\nu} \neq 0$. Then*

$$\Big\| \sum_{\mu,\nu \in F} a_{\mu,\nu} t_\mu t_\nu^* \Big\| \geqslant \Big\| \sum_{\mu,\nu \in F, d(\mu)=d(\nu)} a_{\mu,\nu} t_\mu t_\nu^* \Big\|.$$

Proof. Let $a := \sum_{\mu,\nu \in F} a_{\mu,\nu} t_\mu t_\nu^*$ and let

$$a_0 := \sum_{\mu,\nu \in F, d(\mu)=d(\nu)} a_{\mu,\nu} t_\mu t_\nu^*.$$

Define $n := \bigvee_{\mu \in F} d(\mu)$, and let $G := \bigcup_{\mu \in F} F s(\mu) \Lambda^{n-d(\mu)}$. So if $\mu, \nu \in F$ with $s(\mu) = s(\nu)$ and $d(\mu\alpha) = n$, then $\mu\alpha, \nu\alpha \in G$. By applying (CK4) at $s(\mu)$ for each $\mu, \nu \in F$, we can write

$$a = \sum_{\mu,\nu \in G} b_{\mu,\nu} t_\mu t_\nu^* \quad \text{and} \quad a_0 = \sum_{\mu,\nu \in G, d(\mu)=d(\nu)} b_{\mu,\nu} t_\mu t_\nu^*,$$

where $b_{\mu,\nu} \neq 0$ implies $d(\mu) = n$ and $s(\mu) = s(\nu)$.

For each $v \in s(G)$, apply Lemma 6.2 with $l = \bigvee_{\nu \in G} d(\nu)$ to find $\lambda_v \in v\Lambda$ such that $d(\lambda_v) \geqslant l$ and

$$(\alpha\lambda_v)(0, l) \neq (\beta\lambda_v)(0, l) \quad \text{for distinct } \alpha, \beta \in Gv,$$

and let $Q_v := \sum_{\alpha \in Gv, d(\alpha)=n} t_{\alpha\lambda_v} t^*_{\alpha\lambda_v}$. Then (CK3) implies that the Q_v are mutually orthogonal projections. Hence

$$\Big\| \sum_{v \in s(G)} Q_v a Q_v \Big\| \leqslant \|a\|. \tag{6.2}$$

We show that

$$\sum_{v \in s(G)} Q_v a Q_v = \sum_{v \in s(G)} Q_v a_0 Q_v. \tag{6.3}$$

Fix $\mu, \nu \in G$ with $s(\mu) = s(\nu)$ and $d(\mu) = n$. A quick calculation using (CK4) gives

$$Q_v t_\mu t^*_\nu = \delta_{v, s(\mu)} t_{\mu\lambda_{s(\mu)}} t^*_{\nu\lambda_{s(\mu)}}. \tag{6.4}$$

Suppose $d(\nu) \neq n$ and fix $\alpha \in G \cap \Lambda^n s(\mu)$. Then $(\alpha\lambda_{s(\alpha)})(0, l) \neq (\nu\lambda_{s(\nu)})(0, l)$, and hence $t^*_{\nu\lambda_{s(\mu)}} t_{\alpha\lambda_{s(\alpha)}} = 0$. This and (6.4) give $Q_v t_\mu t^*_\nu Q_v = 0$ for all v, and (6.3) follows.

Finally, we show that

$$\Big\| \sum_{v \in s(G)} Q_v a_0 Q_v \Big\| = \|a_0\|. \tag{6.5}$$

Routine calculations using the Cuntz–Krieger relations and that the t_v are all non-zero show that $\{t_\mu t^*_\nu \colon \mu, \nu \in G \cap \Lambda^n, s(\mu) = s(\nu)\}$ is a family of non-zero matrix units spanning an isomorphic copy of $\bigoplus_{v \in s(G)} M_{Gv \cap \Lambda^n}(\mathbb{C})$, and that

$$\{t_{\mu\lambda_{s(\mu)}} t^*_{\nu\lambda_{s(\nu)}} \colon \mu, \nu \in G \cap \Lambda^n,\ s(\mu) = s(\nu)\}$$

is a family of non-zero matrix units for the same finite-dimensional C^*-algebra. Hence $t_\mu t^*_\nu \mapsto t_{\mu\lambda_{s(\mu)}} t^*_{\nu\lambda_{s(\nu)}}$ determines an isomorphism of finite-dimensional subalgebras of $C^*(\Lambda)$, so is isometric. Calculations like (6.4) show that

$$\sum_{v \in s(G)} Q_v t_\mu t^*_\nu Q_v = t_{\mu\lambda_{s(\mu)}} t^*_{\nu\lambda_{s(\mu)}}$$

whenever $\mu, \nu \in G \cap \Lambda^n$ with $s(\mu) = s(\nu)$, and (6.5) follows.

Combining (6.3), (6.2) and (6.5) proves the proposition. □

For the following proof, recall from the opening of [**14**, § 3] that there is a strongly continuous action $\gamma \colon \mathbb{T}^k \to \operatorname{Aut}(C^*(\Lambda))$ characterized by

$$\gamma_z(s_\lambda) = z^{d(\lambda)} s_\lambda = z_1^{d(\lambda)_1} z_2^{d(\lambda)_2} \cdots z_k^{d(\lambda)_k} s_\lambda$$

for all $\lambda \in \Lambda$. Averaging over this action yields a faithful conditional expectation $\Phi \colon C^*(\Lambda) \to \overline{\operatorname{span}}\{s_\mu s^*_\nu \colon d(\mu) = d(\nu)\}$ (see [**14**, Lemma 3.3]) such that $\Phi(s_\mu s^*_\nu) = \delta_{d(\mu), d(\nu)} s_\mu s^*_\nu$ for all $\mu, \nu \in \Lambda$.

Proof of Theorem 6.1. Follow the first paragraph of the proof of [**14**, Theorem 3.4] to see that π is injective on $C^*(\Lambda)^\gamma = \overline{\operatorname{span}}\{s_\mu s_\nu^*\colon d(\mu) = d(\nu)\}$.

Proposition 6.4 implies that the formula

$$\sum_{\mu,\nu\in F} a_{\mu,\nu} t_\mu t_\nu^* \mapsto \sum_{\mu,\nu\in F, d(\mu)=d(\nu)} a_{\mu,\nu} t_\mu t_\nu^*$$

is well defined on finite linear combinations (if two linear combinations are equal, Proposition 6.4 implies that the norm of the difference of their images is zero), and norm decreasing, and hence extends by continuity to a linear map $\Psi\colon \pi(C^*(\Lambda)) \to \overline{\operatorname{span}}\{t_\mu t_\nu^*\colon d(\mu) = d(\nu)\}$ such that $\Psi(t_\mu t_\nu^*) = \delta_{d(\mu),d(\nu)} t_\mu t_\nu^*$.

To complete the proof, we argue as in the last two lines of the proof of [**14**, Theorem 3.4]: let Φ be the faithful conditional expectation on $C^*(\Lambda)$ described above. By linearity and continuity, $\pi \circ \Phi = \Psi \circ \pi$. Suppose that $\pi(a) = 0$. Then $\Psi(\pi(a^*a)) = 0$ and hence $\pi(\Phi(a^*a)) = 0$. Since π is injective on $C^*(\Lambda)^\gamma$, it follows that $\Phi(a^*a) = 0$. Since Φ is a faithful expectation, we then have $a^*a = 0$ and hence $a = 0$. □

Let Λ be a row-finite graph without sources. As in [**15**], we say that Λ is *cofinal* if for every pair $v, w \in \Lambda^0$ there exists $n \in \mathbb{N}^k$ such that $v\Lambda s(\lambda) \neq \emptyset$ for all $\lambda \in w\Lambda^n$.

Remark 6.5. For row-finite graphs without sources, [**15**, Proposition A.2] implies that this notion of cofinality is equivalent to [**15**, Definition 3.3], and hence by [**15**, Theorem 5.1] to the usual one involving infinite paths.

Modulo the different formulation of cofinality, the following characterization of simplicity appeared in [**21**], and was generalized to locally convex k-graphs in [**22**] and finitely aligned k-graphs in [**15**, **24**].

Theorem 6.6. *Let Λ be a row-finite k-graph with no sources. Then $C^*(\Lambda)$ is simple if and only if Λ is both aperiodic and cofinal.*

In the proof we use the infinite-path representation. By [**14**, Proposition 2.3], for $x \in \Lambda^\infty$, $\lambda \in \Lambda r(x)$ and $n \in \mathbb{N}^k$, there are unique elements $\sigma^n(x)$ and λx of Λ^∞ such that

$$\sigma^n(x)(p, q) = x(n + p, n + q) \quad \text{and} \quad (\lambda x)(p, q) = (\lambda x(0, q))(p, q)$$

for $p \leqslant q \in \mathbb{N}^k$. Let $\{\xi_x\colon x \in \Lambda^\infty\}$ be the usual orthonormal basis for $\ell^2(\Lambda^\infty)$. Then for each $\lambda \in \Lambda$ there is a partial isometry S_λ on $\ell^2(\Lambda^\infty)$ such that $S_\lambda \xi_x = \xi_{\lambda x}$, and $\{S_\lambda\colon \lambda \in \Lambda\}$ is a Cuntz–Krieger Λ-family which gives a representation π_S of $C^*(\Lambda)$ on $\ell^2(\Lambda^\infty)$.

The following lemma is a special case of the implication (ii) $\Longrightarrow$ (i) in [**15**, Theorem 5.1], but the proof simplifies significantly in our setting.

Lemma 6.7. *Let Λ be a row-finite k-graph with no sources that is not cofinal. Then there exist a vertex $v \in \Lambda^0$ and an infinite path $x \in \Lambda^\infty$ such that $v\Lambda x(n) = \emptyset$ for all $n \in \mathbb{N}^k$.*

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Proof. Since Λ is not cofinal, there exist $v, w \in \Lambda^0$ such that, for each $n \in \mathbb{N}^k$, there exists $\lambda \in w\Lambda^n$ with $v\Lambda s(\lambda) = \emptyset$. Choose $n^{(i)} \to \infty$ in $\mathbb{N}^k$, and $\lambda_i \in w\Lambda^{n^{(i)}}$ such that $v\Lambda s(\lambda_i) = \emptyset$. Let $1_k = (1, \ldots, 1) \in \mathbb{N}^k$. Since Λ is row finite, there exists $\mu_1 \in v\Lambda^{1_k}$ such that $S_1 := \{j \in \mathbb{N} \colon \lambda_j(0, 1_k) = \mu_1\}$ is infinite. An induction argument now shows that there is a sequence $(\mu_i)_{i=1}^{\infty}$ in $v\Lambda$ such that, for every $i \geqslant 2$, we have $\mu_i \in \mu_{i-1}\Lambda^{1_k}$, and $S_i := \{j \in S_{i-1} \colon \lambda_j(0, i \cdot 1_k) = \mu_i\}$ is infinite. In particular, for any $i \in \mathbb{N}$ and $j \in S_i$ we have $v\Lambda s(\lambda_j) = \emptyset$, and hence $v\Lambda s(\mu_i) = \emptyset$. Since $d(\mu_i) \to (\infty, \ldots, \infty)$, [**14**, Remarks 2.2] imply that there is an infinite path x such that $x(0, d(\mu_i)) = \mu_i$ for all i. Now since $v\Lambda x(d(\mu_i)) = v\Lambda s(\mu_i) = \emptyset$ for all i, we have $v\Lambda x(n) = \emptyset$ for all n, so the infinite path x has the required properties. $\square$

Proof of Theorem 6.6. First suppose that Λ is aperiodic and cofinal, and let I be a non-zero ideal in $C^*(\Lambda)$. To see that $I = C^*(\Lambda)$, we fix $\mu \in \Lambda$ and aim to show that s_μ belongs to I. Since Λ is aperiodic, the Cuntz–Krieger Uniqueness Theorem (Theorem 6.1) implies that I contains a vertex projection s_v. Applying cofinality with this v and $w = s(\mu)$ gives $n \in \mathbb{N}^k$ such that for each $\lambda \in s(\mu)\Lambda^n$ there exists $\nu_\lambda \in v\Lambda s(\lambda)$. Then

$$s_\mu = \sum_{\lambda \in s(\mu)\Lambda^n} s_{\mu\lambda} s_{s(\lambda)} s_\lambda^* = \sum_{\lambda \in s(\mu)\Lambda^n} s_{\mu\lambda}(s_{\nu_\lambda}^* s_v s_{\nu_\lambda}) s_\lambda^* \in I,$$

as required.

For the other direction, we first suppose that Λ is not aperiodic, so that there exist $v \in \Lambda^0$ and distinct $m, n \in \mathbb{N}^k$ such that, for every $\lambda \in v\Lambda$ with $d(\lambda) \geqslant m \vee n$,

$$\lambda(m, m + d(\lambda) - (m \vee n)) = \lambda(n, n + d(\lambda) - (m \vee n)). \tag{6.6}$$

Then, for every $x \in v\Lambda^\infty$ and $l \in \mathbb{N}^k$, we can apply (6.6) to $\lambda = x(0, (m \vee n) + l)$ and deduce that $x(m, m + l) = x(n, n + l)$, whence $\sigma^m(x) = \sigma^n(x)$.

We now fix $\lambda \in v\Lambda^{m \vee n}$, let $\mu = \lambda(0, m)$ and $\nu = \lambda(0, n)$, and aim to prove that $a := s_\lambda s_\lambda^* - s_\mu s_\nu^* s_\lambda s_\lambda^*$ is non-zero and belongs to $\ker \pi_S$; since we know that $\ker \pi_S$ does not contain any vertex projections, this will prove that $\ker \pi_S$ is a non-trivial ideal. To see that $\pi_S(a) = 0$, fix $x \in \Lambda^\infty$ and compute

$$\pi_S(a)\xi_x = (S_\lambda S_\lambda^* - S_\mu S_\nu^* S_\lambda S_\lambda^*)\xi_x. \tag{6.7}$$

If $x(0, d(\lambda)) \neq \lambda$, then (6.7) vanishes. If $x(0, d(\lambda)) = \lambda$, then $x \in v\Lambda^\infty$, the argument in the previous paragraph gives $\sigma^m(x) = \sigma^n(x)$, and hence

$$\nu\sigma^n(x) = x = \mu\sigma^m(x) = \mu\sigma^n(x),$$

which implies $S_\mu S_\nu^* \xi_x = \xi_{\mu\sigma^n(x)} = \xi_x$ and $\pi_S(a)\xi_x = 0$. Thus $\pi_S(a) = 0$.

To see that $a \neq 0$, we choose $z \in \mathbb{T}^k$ such that $z^{m-n} = -1$. Then $\gamma_z(s_\mu s_\nu^* s_\lambda s_\lambda^*) = -s_\mu s_\nu^* s_\lambda s_\lambda^*$, and hence

$$\pi_S(a + \gamma_z(a)) = \pi_S(2 s_\lambda s_\lambda^*) = 2 S_\lambda S_\lambda^* \neq 0,$$

forcing $a \neq 0$. Thus $C^*(\Lambda)$ is not simple.

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Now suppose that Λ is not cofinal. By Lemma 6.7, there exist $v \in \Lambda^0$ and $x \in \Lambda^\infty$ such that $v\Lambda x(n) = \emptyset$ for all $n \in \mathbb{N}^k$. Let

$$[x]_\sigma := \{y \in \Lambda^\infty \colon \text{there exist } p, q \in \mathbb{N}^k \text{ such that } \sigma^p(x) = \sigma^q(y)\}.$$

We claim that

$$y \in [x]_\sigma \Longrightarrow r(y) \neq v. \tag{6.8}$$

To see this, fix $y \in [x]_\sigma$ and $p, q \in \mathbb{N}^k$ such that $\sigma^p(x) = \sigma^q(y)$. Then $x(p) = y(q)$ and hence $v\Lambda y(q) = \emptyset$ by choice of x. In particular, $y(0, q) \notin v\Lambda y(q)$, so $r(y) \neq v$, as claimed.

We now consider the subspace $\mathcal{H}_x := \overline{\operatorname{span}}\{\xi_y \colon y \in [x]_\sigma\}$ of $\ell^2(\Lambda^\infty)$. For $y \in [x]_\sigma$ and $s(\lambda) = r(y)$, we have $\lambda y \in [x]_\sigma$, and hence $\mathcal{H}_x$ is invariant for S_λ. On the other hand, $S_\lambda^* \xi_y$ vanishes unless $y(0, d(\lambda)) = \lambda$, and then $S_\lambda^* \xi_y = \xi_{\sigma^{d(\lambda)}(y)}$, which also belongs to $\mathcal{H}_x$. Thus $\mathcal{H}_x$ is reducing for π_S, and $\phi_x \colon a \mapsto \pi_S(a)|_{\mathcal{H}_x}$ is a homomorphism of $C^*(\Lambda)$ into $\mathcal{B}(\mathcal{H}_x)$. Since $\phi_x(s_{r(x)})\xi_x = \xi_x \neq 0$, $\ker \phi_x \neq C^*(\Lambda)$. Equation (6.8), on the other hand, implies that $\phi_x(s_v)\xi_y = 0$ for all $y \in [x]_\sigma$, so $s_v \in \ker \phi_x$. Thus $C^*(\Lambda)$ is not simple. $\square$

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