On the Φ-Invariant of Two Quadrics

By T. Scott

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1. *Introduction.*

The geometry associated with the five invariants of two quadrics is well known.[1] There appears to be an omission, however, in the treatment of the Φ-invariant. Salmon gives the vanishing of Φ merely as a *necessary* condition for the possibility of the construction of a tetrahedron self-conjugate for one of the quadrics and having its six edges tangential to the other.[2] Sommerville proves its *sufficiency*, and shows that this condition is poristic, giving rise to two systems of ∞^1 tetrahedra of the requisite type.[3] No investigation appears to have been made of the locus of the vertices of the ∞^1 tetrahedra of each system and the dual problem regarding the nature of the developable surface arising from their ∞^1 faces.

The purpose of this paper is to derive the equations of these constructs, showing the curve to be of order eight, and the developable to be of class eight. The treatment throughout is based on the Clebsch-Aronhold symbolism,[4] viz.,

$$
\begin{array}{llll}
 & f \equiv a_x^2, & f' \equiv b_x^2. & \\
 & \Sigma \equiv u_\alpha^2, & \Sigma' \equiv u_\beta^2, & (\alpha \equiv aa'a'',\ \beta \equiv bb'b''). \\
(1.1) & \Pi \equiv (Ap)^2, & \Pi' \equiv (Bp)^2, & (A \equiv aa',\quad B \equiv bb'). \\
 & k \equiv (A\beta x)^2, & k' \equiv (B\alpha x)^2, & [(A\beta x) \equiv \dot{a}_\beta \dot{a}'_x \quad \equiv \bar{a}_\beta \bar{a}'_x]. \\
 & \chi \equiv (Abu)^2, & \chi' \equiv (Bau)^2. &
\end{array}
$$

The five invariants are then represented as follows:—

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[1] (1) Salmon, *A Treatise on the Analytic Geometry of Three Dimensions* (revised by Rogers), Ch. IX.

(2) Sommerville, *Analytical Geometry of Three Dimensions*, Ch. XV.

(3) H. W. Turnbull, "Some Geometrical Interpretations of the Concomitants of Two Quadrics," *Proc. Cambridge Phil. Soc.*, XIX (1919), 196-206.

[2] (1) p. 204, § 201.

[3] (2) p. 312, § 15.22.

[4] (3) p. 198, §§ 4, 5.

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(1.2) $$\Delta \equiv a_\alpha^2, \ \Theta \equiv b_\alpha^2, \ \Phi \equiv (AB)^2, \ \Theta' \equiv a_\beta^2, \ \Delta' \equiv b_\beta^2.$$

In the following account it will be proved that, when Φ vanishes, the developable containing the faces of one system of the above-mentioned tetrahedra is formed by the ∞^1 planes common to the quadric and quartic envelopes

$$\chi = 0, \qquad \Delta\Sigma'^2 - 16\,\Theta'\Sigma\Sigma' + 12\,\Delta'\Sigma\chi' = 0;$$

while the locus of the ∞^1 vertices is the curve of intersection of the quadric and quartic surfaces

$$k = 0, \qquad \Delta f'^2 - 16\,\Theta ff' + 8fk' = 0.$$

The equations of the developable and curve associated with the second system of tetrahedra will also be given in similar form.

2. *The vanishing of Φ as a necessary condition.*

Let $PQRS$ be a tetrahedron self-congugate for f' and having its six edges touching f. If u represents the plane QRS then the conic envelope of lines in u belonging to Π is given by

$$(Auv)^2 = 0 \ (v \text{ current}).$$

The triangle QRS therefore circumscribes the conic $(Auv)^2$ and is self-conjugate for the section of f' by u. That is, $(Auv)^2$ is inpolar to f', and so u satisfies the equation

$$(Abu)^2 = 0.$$

Thus, the plane QRS touches the quadric envelope χ. Similarly, all four faces of the tetrahedron $PQRS$ touch χ, and since the tetrahedron circumscribes χ and is self-conjugate for f', χ is inpolar to f', and so $(Abb')^2 = 0$.

Therefore, the vanishing of $(AB)^2$ is a *necessary* condition for the possibility of the above construction.

For convenience such a construct will now be termed a Φ-*tetrahedron* of the two quadrics.

3. *Considerations of duality.*

It can easily be shown that the vertices P, Q, R and S of a Φ-tetrahedron lie on the quadric k, the vanishing of Φ being the condition for k and χ to be polar reciprocal quadrics with respect to f'. For the polar reciprocal of χ with respect to f' is given by

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$$b'_x\,(Abb')\;(Abb'')\;b''_x = 0,$$

i.e. $$\tfrac{1}{2}\dot{b}'_x\,(Abb')\;(A\dot{b}b'')\;b''_x = 0,$$

which by determinantal permutations gives

$$\tfrac{1}{2}\,(Abb')^2\,b''^2_x - \tfrac{1}{2}\dot{a}'_x\,(Abb')\,(\dot{a}b''bb')b''_x = 0,$$

reducing to $$\tfrac{1}{2}\,(AB)^2 b_x^2 - \tfrac{1}{6}\,(A\beta x)^2 = 0,$$

(3.1) $$3\Phi f' - k = 0,$$

and for Φ zero this gives $k = 0$.

Thus, each Φ-tetrahedron has its vertices on k, its faces touching χ, its edges belonging to Π, and is self-conjugate for Π'. The vanishing of Φ is obviously also a necessary condition for a Φ-tetrahedron to exist with the same relationships to k', χ', Π' and Π.

4. *A further property of k and χ.*

Before the sufficiency of the condition is considered, the following additional property associated with k and χ will be established as it is required at a later stage:

$$\begin{aligned}(A\beta.\;Bu)^2 &= [\dot{a}_\beta\,(\dot{a}'Bu)]^2\\ &= [\dot{b}_\beta\,(A\dot{b}'u)]^2 + 2\,(AB)\,(A\dot{b}'u)\,\dot{b}_\beta u_\beta, \qquad \text{since } (AB)^2 = 0\\ &= 2b_\beta^2\,(Ab'u)^2 - 2b_\beta\,(Ab'u)\,b'_\beta\,(Abu) + 4(AB)\,(Ab'u)\,b_\beta u_\beta\\ &= 2b_\beta^2\,(Ab'u)^2 - \tfrac{1}{2}b'^2_\beta\,(Abu)^2 - (Ab'u)^2 b_\beta^2\\ &= \tfrac{1}{2}\Delta'\chi.\end{aligned}$$

Hence, if the plane u belongs to χ (when Φ is zero), it also satisfies

(4.1) $$(A\beta.\;Bu)^2 = 0.$$

Therefore, by the Clebsch Transference Principle every plane belonging to χ cuts k in a conic which is outpolar to the conic section of f' by u.

5. *Investigation of the sufficiency of the condition* $\Phi = 0$.

In the following it will now be assumed that the quadrics f and f' are subject only to the condition that the invariant Φ vanishes. A plane u will cut f in the conic section given by

$$(Auv)^2 = 0.$$

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Let the plane u belong to χ. Then the polar reciprocal cone of $(Auv)^2$ with respect to f' is obtained from the equation

$$b_x\,(Abu)\,(Ab'u)\,b'_x = 0,$$

where $$b_x\,(Abu)\,(Ab'u)\,b'_x = \dot{b}_x\,(Abu)\,(A\dot{b}'u)\,b'_x, \quad \text{since } (Abu)^2 = 0$$
$$= -\,(uABx)\,(uAb)\,b'_x$$
$$= -\tfrac{1}{2}\,(uABx)^2. \qquad \text{(i)}$$

Let the point y be the pole of u with respect to f'. From (3.1) y lies on the quadric k, and from the foregoing construction is also the vertex of the polar reciprocal cone of the conic $(Auv)^2$ with respect to f'.

Thus, $$(uBAx)^2 = b_y\,(bBAx)\,(b'BAx)\,b'_y$$
$$= 2b_y\,(bBAx)\,(b'Ba)\,a'_x b'_y$$
$$= \tfrac{2}{3}\dot{b}_y\,(bBAx)\,(b'\dot{b}_1\dot{b}_2 a)\,a'_x b'_y\,, \text{ where } B = b_1 b_2$$
$$= \tfrac{2}{3}\dot{b}'_y\,(bBAx)\,(bB\dot{a})\,a'_x b'_y$$
$$= \tfrac{2}{3}\dot{b}'_y \bar{a}_\beta \bar{a}'_x \dot{a}_\beta a'_x b'_y$$
$$= \tfrac{2}{3}\,(A\beta x)\,a_\beta a'_x b'^2_y - \tfrac{2}{3}\,(A\beta x)\,a_y b'_\beta a'_x b'_y$$
$$= \tfrac{1}{3}\,(A\beta x)^2 b^2_y - \tfrac{1}{6} b'^2_\beta \dot{a}'_x a_y a'_x \dot{a}_y$$
$$= \tfrac{1}{3}\,(A\beta x)^2 b^2_y - \tfrac{1}{12} b^2_\beta\,(Ayx)^2. \qquad \text{(ii)}$$

Also $$(uBAx)\,(BA) = b_y\,(bBAx)\,(BA)$$
$$= -\,2b_y\,(abB)\,a'_x\,(aa'b_1b_2)$$
$$= -\,\tfrac{2}{3}\dot{b}_y\,(abB)\,a'_x\,(aa'\dot{b}_1\dot{b}_2)$$
$$= -\,\tfrac{2}{3}\dot{a}'_y a_\beta a'_x \dot{a}_\beta$$
$$= -\,\tfrac{1}{3}\,(A\beta y)\,(A\beta x). \qquad \text{(iii)}$$

By combining (i), (ii) and (iii) and using the identity

$$(uABx) \equiv -\,(uBAx) - (BA)\,u_x,$$

we have $$(uABx)^2 \equiv (uBAx)^2 + 2\,(uBAx)\,(BA)\,u_x, \qquad \text{since } \Phi = 0$$
$$\equiv \tfrac{1}{3}\,(A\beta x)^2 b^2_y - \tfrac{1}{12} b^2_\beta\,(Ayx)^2 - \tfrac{2}{3}\,(A\beta y)\,(A\beta x)\,b_y b_x.$$

Thus, the polar reciprocal cone of $(Auv)^2$ with respect to f', where u is a plane of χ and Φ vanishes, is given by the equation

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$$(5.1)\qquad 4b_y^2(A\beta x)^2 - b_\beta^2(Ayx)^2 - 8(A\beta y)(A\beta x)\,b_y b_x = 0 \quad (x \text{ current}).$$

To construct a Φ-tetrahedron let us take, as in §2, a triangle QRS self-conjugate for the section of f' by u and circumscribing $(Auv)^2$, where u is a plane touching χ, and $P(y)$ is the pole of u with respect to f'. In order to complete the construction it will be necessary to choose u and the triangle QRS so that PQ, PR and PS are generators common to the cones $(Ayx)^2$ and the polar reciprocal of $(Auv)^2$ with respect to f'. This condition is essential since PQ, PR and PS must be lines of Π and also the respective polar lines of RS, SQ and QR, which touch $(Auv)^2$.

From (5.1) it is seen that the points common to the quadric k, the polar reciprocal cone of $(Auv)^2$ with respect to f' and the plane u (viz. $b_y b_x = 0$) also lie on the cone $(Ayx)^2$. Thus, if a plane u of χ can be obtained containing a triangle QRS whose sides touch $(Auv)^2$ and whose vertices lie on the section of k by u, then since the triangle QRS is self-conjugate for f' and $P(y)$ is the pole of u with respect to f', the lines PQ, PR and PS will be generators of the polar reciprocal cone of $(Auv)^2$ with respect to f' and also generators of $(Ayx)^2$.

The problem of determining a Φ-tetrahedron is thus reduced from a problem in [3] to one in [2], viz. to obtain the planes u touching χ, which have a triangle QRS circumscribing the conic $(Auv)^2$, inscribed in the section of k by u and self-conjugate for the section of f' by u.

6. *Lemma concerning a configuration of three conics in a plane.*

To solve the problem outlined at the end of §5 the following three ternary forms and their duals will be considered:—

$$f_1 \equiv a_x^2,\ \Sigma_1 \equiv u_\alpha^2\ ;\ f_2 \equiv b_x^2,\ \Sigma_2 \equiv u_\beta^2\ ;\ f_3 \equiv c_x^2,\ \Sigma_3 \equiv u_\gamma^2.$$

It is to be noted that the symbols a, α; b, β; c, γ here used are in no wise related to the corresponding symbols used in connection with the quaternary forms.

The following identities connecting the non-symbolical and symbolical forms of the concomitants of two conics will be found necessary, and can easily be verified:

$$f \equiv \sum_{i,j=1}^{3} a_{ij}x_i x_j \equiv a_x^2,$$

$$\Sigma \equiv \sum_{i,j=1}^{3} A_{ij}u_i u_j \equiv \tfrac{1}{2}u_\alpha^2,$$

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(6.1)
$$\Delta \equiv \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \equiv \tfrac{1}{6} a_a^2 ,$$

$$\Theta \equiv \sum_{i,j=1}^{3} b_{ij} A_{ij} \equiv \tfrac{1}{2} b_a^2 .$$

Necessary and sufficient conditions for the existence of a triangle circumscribed about the conic f_1, inscribed in f_2 and self-conjugate for f_3 will now be determined.

Obviously, the following conditions will be necessary:—

(6.2) $\qquad b_\gamma^2 = 0$, since f_2 is outpolar to Σ_3,

(6.3) $\qquad c_a^2 = 0$, since Σ_1 is inpolar to f_3.

There is a third necessary condition, however, arising from the fact that a triangle is inscribed in f_2 and circumscribed about f_1. In non-symbolical form the necessary (and sufficient) condition for this last construction is $\Theta^2 = 4\Delta\Theta'$.[1]

From (6.1) the symbolical form of this third necessary condition is therefore

(6.4) $$b_a^2 \, b'^2_{a'} = \tfrac{4}{3} a_a^2 \, a'^2_\beta ,$$

which by the interchange of equivalent symbols reduces to the slightly modified form

(6.4(a)) $$b_a^2 \, b'^2_{a'} = (aa'\beta)^2 .$$

It can easily be shown that the condition (6.4(a)) also covers the case where Σ_1 degenerates to a point pair.

It will now be proved that conditions (6.2), (6.3) and (6.4) are also sufficient for the possibility of the above construction, and yield a unique triangle satisfying the conditions.

The configuration now to be considered consists of the conic envelope Σ_1, and conic loci f_2 and f_3 subject to conditions (6.2), (6.3) and (6.4(a)). Since the envelope Σ_1 is taken as fundamental, its point form becomes $S_1 \equiv (aa'x)^2$, where $(aa'x)^2 \equiv \tfrac{4}{3} a_a^2 \, f_1$, and $(aa'\beta)^2$ is the Θ'-invariant of S_1 and Σ_2. Also, since b_γ^2 is zero, the polar reciprocal of f_2 with respect to f_3 is their Φ-conic,[2] viz.

$$\Phi_{23} \equiv (bcu)^2 = 0.$$

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[1] (4) Sommerville, *Analytical Conics*, p. 279, § 23.

[2] (4) p. 286, Ex. 2., but the result can easily be proved symbolically.

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There are four tangents common to the conic envelopes Σ_1 and Φ_{23}, and these determine the range of conic envelopes

$$\Sigma_\lambda \equiv (bcu)^2 + \lambda u_a^2 = 0.$$

It can be shown by applying the Clebsch Transference Principle to the binary domain that the point equation of Σ_λ is

$$S_\lambda \equiv (bc.\, b'c'.\, x)^2 + 2\lambda\, (bc.\, a.\, x)^2 + \lambda^2\, (aa'x)^2 = 0.$$

The Θ'-invariant of S_λ and Σ_2 is therefore

$$(bc.\, b'c'.\, \beta)^2 + 2\lambda\, (bc.\, a.\, \beta)^2 + \lambda^2\, (aa'\beta)^2,$$

and so for a triangle to be inscribed in f_2 and circumscribed about Σ_λ the following necessary and sufficient condition of type (6.4(a)) will have to be proved:—

$$(6.5) \quad [(bcb')^2 + \lambda b_a^2]\, [(b''c'b''')^2 + \lambda b''^2_{a'}] = (bc.\, b'c'.\, \beta)^2 + 2\lambda\, (bc.\, a.\, \beta)^2 + \lambda^2\, (aa'\beta)^2,$$

where b, b''; c, c'; a, a' etc. are equivalent pairs of symbols.

The left-hand side of the above equation reduces to

$$(c_\beta^2 + \lambda b_a^2)\, (c'^2_{\beta'} + \lambda b'^2_{a'}) = c_\beta^2 c'^2_{\beta'} + 2\lambda b_a^2 c_\beta^2 + \lambda^2 b_a^2 b'^2_{a'},$$

while the right-hand side becomes

$$\begin{aligned}
&[(bb'c')\, c_\beta - (cb'c')\, b_\beta]^2 + 2\lambda\, [b_a c_\beta - c_a b_\beta]^2 + \tfrac{4}{3}\lambda^2 a_a^2 a'^2_\beta \\
&= c_\beta^2 c'^2_{\beta'} + b_\beta^2 b'^2_\gamma - 2c_\beta b_\beta\, (bb'c')\, (cb'c') + 2\lambda\, (b_a^2 c_\beta^2 + c_a^2 b_\beta^2 - 2b_a c_\beta b_\beta c_a) \\
&\qquad + \tfrac{4}{3}\lambda^2 a_a^2 a'^2_{\beta'} \\
&= c_\beta^2 c'^2_\beta + b_\beta^2 b'^2_\gamma - \tfrac{2}{3} b_\beta^2 b'^2_\gamma + 2\lambda\, (b_a^2 c_\beta^2 + c_a^2 b_\beta^2 - \tfrac{2}{3} b_\beta^2 c_a^2) + \tfrac{4}{3}\lambda^2 a_a^2 a'^2_\beta \\
&= c_\beta^2 c'^2_{\beta'} + 2\lambda b_a^2 c_\beta^2 + \tfrac{4}{3}\lambda^2 a_a^2 a'^2_\beta,
\end{aligned}$$

since relations (6.2) and (6.3) are satisfied by the three conics f_1, f_2 and f_3.

Also, from (6.4) $\qquad b_a^2 b'^2_{a'} = \tfrac{4}{3} a_a^2 a'^2_\beta$.

Thus, (6.5) is established, and so for all values of λ an infinity of triangles can be inscribed in f_2 and circumscribed about Σ_λ.

Now let the four tangents common to Σ_1 and Φ_{23} meet in the point pairs A, A'; B, B'; and C, C', which are three degenerate conic envelopes of the range Σ_λ. By application of (6.5) to the point pair A, A' a triangle can therefore be inscribed in f_2 and circumscribed about the degenerate conic envelope whose tangent lines pass through A, A'. Thus, either A or A' must lie on f_2, and similarly, B or B' and C or C' must also lie on f_2. As it is impossible for three collinear

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points to lie on a conic, there is no loss of generality in taking the points A, B and C as the three points on f_2. Therefore, the triangle ABC is inscribed in f_2 and circumscribed about Φ_{23} and Σ_1. Moreover, since Φ_{23} is the polar reciprocal of f_2 with respect to f_3 and f_2 is outpolar to f_3, this triangle is also self-conjugate for f_3.

The triangle ABC is therefore a unique triangle circumscribing Σ_1, inscribed in f_2 and self-conjugate for f_3, provided (6.2), (6.3) and (6.4) are satisfied.

7. *Application to the Φ-tetrahedron of two quadrics.*

In the case of the three-dimensional configuration it was shown in §5 that in order to construct a Φ-tetrahedron, when Φ vanishes, it is necessary to obtain a plane u belonging to χ, having a triangle circumscribing $(Auv)^2$ inscribed in the section of k by u, and self-conjugate for the section of f' by u.

From (4.1), provided u belongs to χ, then $(Auv)^2$ is inpolar to f', and the section of k by u is outpolar to the section of f' by u. Thus, if $(Auv)^2$ is taken to correspond to Σ_1, the section of k by u to correspond to f_2, and the section of f' by u to correspond to f_3, where Σ_1, f_2 and f_3 represent the conics dealt with in §6, then (6.2) and (6.3) are satisfied by the conic sections of f, k and f' by u. By the application of the Clebsch Transference Principle to relation (6.4) will be found the ∞^2 planes u, which are such that a triangle can be inscribed in the section of k by u, and circumscribed about the section of f by u. This condition is therefore

$$(7.1) \qquad [(A\beta.\ A'u)^2]\,[(A''\beta'.\ A'''u)^2] = \tfrac{4}{3}u_\alpha^2\,(\alpha.\ A\beta.\ A'\beta'.\ u)^2,$$

which represents a surface of class four. The ∞^1 planes common to this surface and the quadric envelope χ therefore produce conic sections which satisfy conditions (6.2), (6.3) and (6.4) simultaneously, and so constitute the developable surface of class eight whose planes are the faces of one system of Φ-tetrahedra, viz. that system whose edges touch f, and whose members are self-conjugate for f'. Dually, the ∞^1 vertices of this system will describe an octavic curve as locus.

Expressed in terms of the irreducible concomitants of two quadrics the developable is given by the equations

$$(7.2) \qquad \chi = 0 \text{ and } \Delta\Sigma'^2 - 16\Theta'\Sigma\Sigma' + 12\Delta'\Sigma\chi' = 0.$$

and the curve by the equations

$$(7.3) \qquad k = 0 \text{ and } \Delta f'^2 - 16\Theta f f' + 8fk' = 0.$$

The reduction of (7.1) and its dual to forms (7.2) and (7.3) is rather long and involved, and consequently the detailed proof is included as an Appendix.

Similarly, for the second system of Φ-tetrahedra whose edges touch f' and whose members are self-conjugate for f, the developable and curve are given respectively by the equations

$$(7.4) \qquad \chi' = 0 \quad \text{and} \quad \Delta'\Sigma^2 - 16\,\Theta\Sigma\Sigma' + 12\,\Delta\Sigma'\chi = 0,$$

$$(7.5) \qquad k' = 0 \quad \text{and} \quad \Delta' f^2 - 16\,\Theta' f f' + 8 f' k = 0.$$

8. *Conclusion.*

Certain aspects of the foregoing theory can be applied to the case where Φ is the $(AB)^2$ invariant of a quadratic complex $(Ap)^2$ and a quadric $(Bp)^2$ with further modifications where a Battaglini complex is involved.

In the case of the Φ-invariant of two general quadratic complexes, $(Ap)^2$ and $(Bp)^2$, an entirely different method of approach is necessary. These considerations are deferred meantime.

Appendix.

The following is the detailed working involved in deriving relations (7.2)—(7.5) from equation (7.1), viz.

$$(A\beta.\, A'u)^2\,(A''\beta'.\, A'''u)^2 = \tfrac{4}{3}u_\alpha^2\,(a.\, A\beta.\, A'\beta'.u)^2.$$

Firstly, $(A\beta.\, A'u)^2 = [\dot{a}_\beta\,(\dot{a}'A'u)]^2$

$$= 2a_\beta^2 u_\alpha^2 - 2a_\beta a'_\beta\,(aA'u)\,(a'A'u)$$

$$= 2a_\beta^2 u_\alpha^2 - \tfrac{2}{3}\dot{a}_\beta a'_\beta\,(aA'u)\,(a'\dot{a}_1\dot{a}_2u), \text{ where } A' \equiv a_1a_2,$$

$$= 2a_\beta^2 u_\alpha^2 - \tfrac{2}{3}a'^2_\beta u_\alpha^2 + \tfrac{2}{3}u_\beta a'_\beta u_\alpha a'_\alpha$$

$$= \tfrac{4}{3}a_\beta^2 u_\alpha^2 + \tfrac{1}{6}a_\alpha^2 u_\beta^2.$$

(A.1) Thus, $(A\beta.\, A'u)^2\,(A''\beta'.\, A'''u)^2 = \tfrac{16}{9}a_\beta^2 a'^2_{\beta'} u_\alpha^2 u_{\alpha'}^2 + \tfrac{1}{36}a_\alpha^2 a'^2_{\alpha'} u_\beta^2 u_{\beta'}^2 + \tfrac{4}{9}a_\alpha^2 a'^2_\beta u_{\alpha'}^2 u_{\beta'}^2.$

Secondly, $(a.\, A\beta.\, A'\beta'.\, u)^2 = [\dot{a}_{1\beta_1}\bar{a}_{2\beta_2}\,(\dot{a}_1{}'\bar{a}_2{}'au)]^2,$

where $A\beta \equiv \dot{a}_{1\beta_1}\dot{a}_1{}'$ and $A'\beta' \equiv \bar{a}_{2\beta_2}\bar{a}_2{}'.$

This reduces to

$$[a_{1\beta_1}a_{2\beta_2}(a'_1a'_2au) - a'_{1\beta_1}a_{2\beta_2}(a_1a'_2au) + a'_{1\beta_1}a'_{2\beta_2}(a_1a_2au) - a_{1\beta_1}a'_{2\beta_2}(a'_1a_2au)]^2$$

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$$= 4a_\beta^2 a'^2_{\beta'} u_\alpha^2 - 2a_{1\beta_1} a'_{1\beta_1} a^2_{2\beta_2} (a'_1 a'_2 au)(a_1 a'_2 au) + 2a_{1\beta_1} a'_{1\beta_1} a_{2\beta_2} a'_{2\beta_2} (a'_1 a'_2 au)(a_1 a_2 au)$$

$$- 2a^2_{1\beta_1} a_{2\beta_2} a'_{2\beta_2} (a'_1 a'_2 au)(a'_1 a_2 au) - 2a'^2_{1\beta_1} a_{2\beta_2} a'_{2\beta_2} (a_1 a'_2 au)(a_1 a_2 au)$$

$$+ 2a_{1\beta_1} a'_{1\beta_1} a_{2\beta_2} a'_{2\beta_2} (a_1 a'_2 au)(a'_1 a_2 au) - 2a_{1\beta_1} a'_{1\beta_1} a'^2_{2\beta_2} (a_1 a_2 au)(a'_1 a_2 au).$$

In the above expression, by interchanging equivalent symbols a_1 with a_2 and a'_1 with a'_2 in the fourth term; a'_1 with a_2 and a'_2 with a_1 in the fifth term; a_1 with a'_1 in the sixth term; and a_2 with a'_2 in the seventh term, it may be shown that

$$\text{(A.2)} \quad (\alpha . A\beta . A'\beta' . u)^2 = 4a_\beta^2 a'^2_{\beta'} u_\alpha^2 - 8a_{1\beta_1} a'_{1\beta_1} (a'_1 a'_2 au)(a_1 a'_2 au) a^2_{2\beta_2} + 4a_{1\beta_1} a'_{1\beta_1} a_{2\beta_2} a'_{2\beta_2} (a'_1 a'_2 au)(a_1 a_2 au)$$

$$= 4(X - 2Y + Z),$$

where $X = a_\beta^2 a'^2_{\beta'} u_\alpha^2$,

$$Y = a_{1\beta_1} a'_{1\beta_1} (a'_1 a'_2 au)(a_1 a'_2 au) a^2_{2\beta_2}$$

$$= \tfrac{1}{3} \dot{a}_{1\beta_1} a'_{1\beta} (a'_1 \dot{a}'_2 \dot{a} u)(a_1 a'_2 au) a^2_{2\beta_2}$$

$$= \tfrac{1}{3} a'^2_{1\beta_1} a^2_{2\beta_2} u_\alpha^2 - \tfrac{1}{3} u_{\beta_1} a'_{1\beta_1} a'_{1\alpha} u_\alpha a^2_{2\beta_2}$$

$$= \tfrac{1}{3} a_\beta^2 a'^2_{\beta'} u_\alpha^2 - \tfrac{1}{12} a_\alpha^2 a'^2_\beta u^2_{\beta'},$$

$$Z = a_{1\beta_1} a'_{1\beta_1} a_{2\beta_2} a'_{2\beta_2} (a'_1 a'_2 au)(a_1 a_2 au)$$

$$= \tfrac{1}{6} a_{1\beta_1} \dot{a}'_{1\beta_1} a_{2\beta_2} \dot{a}'_{2\beta_2} (a'_1 a'_2 au)(a_1 a_2 \dot{a} u)$$

$$= \tfrac{1}{6} [a_{1\beta_1} \dot{a}_{1\beta_1} a_{2\beta_2} \dot{a}_{2\beta_2} u_\alpha^2 - a_{1\beta_1} \dot{u}_{\beta_1} a_{2\beta_2} \dot{a}_{2\beta_2} u_\alpha a_{1\alpha} + a_{1\beta_1} \dot{u}_{\beta_1} a_{2\beta_2} \dot{a}_{1\beta_2} u_\alpha a_{2\alpha}]$$

$$= \tfrac{1}{6} [a^2_{1\beta_1} a^2_{2\beta_2} u_\alpha^2 - a_{1\beta_1} a_{2\beta_1} a_{2\beta_2} a_{1\beta_2} u_\alpha^2 - a_{1\beta_1} u_{\beta_1} a^2_{2\beta_2} u_\alpha a_{1\alpha}$$
$$+ a_{1\beta_1} a_{2\beta_1} a_{2\beta_2} u_{\beta_2} u_\alpha a_{1\alpha} + a_{1\beta_1} u_{\beta_1} a_{2\beta_2} a_{1\beta_2} u_\alpha a_{2\alpha} - a^2_{1\beta_1} a_{2\beta_2} u_{\beta_2} u_\alpha a_{2\alpha}].$$

The above six terms of Z are reduced as follows:—

1st term $= a_\beta^2 a'^2_{\beta'} u_\alpha^2$.

2nd term $= a_{1\beta_1} a_{2\beta_1} a_{2\beta_2} a_{1\beta_2} u_\alpha^2$

$$= a^2_{1\beta_1} a^2_{2\beta_2} u_\alpha^2 + a_{1\beta_1} \dot{a}_{2\beta_1} a_{2\beta_2} \dot{a}_{1\beta_2} u_\alpha^2$$

$$= a^2_{1\beta_1} a^2_{2\beta_2} u_\alpha^2 + (a_1 b_1 b_2 b_3)(a_2 a_1 \dot{b}_2 \dot{b}_3) a_{2\beta_2} \dot{b}_{1\beta_2} u_\alpha^2, \text{ where } \beta_1 \equiv b_1 b_2 b_3.$$

$$= a^2_{1\beta_1} a^2_{2\beta_2} u_\alpha^2 + \tfrac{3}{4} (a_1 \dot{b}_1 b_2 b_3)(a_2 a_1 b_2 b_3) a_{2\beta_2} \dot{b}_{1\beta_2} u_\alpha^2$$

$$= a^2_{1\beta_1} a^2_{2\beta_2} u_\alpha^2 - \tfrac{3}{4} (AB)^2 b^2_{1\beta_2} u_\alpha^2$$

$$= a_\beta^2 a'^2_{\beta'} u_\alpha^2, \text{ since } (AB)^2 = 0.$$

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3rd term $= a_{1\beta_1} u_{\beta_1} a^2{}_{2\beta_2} u_a a_{1a}$

$= \frac{1}{4} u^2_{\beta_1} a^2{}_{2\beta_2} a^2{}_{1a}$

$= \frac{1}{4} a^2_a a'^2_\beta u^2_{\beta'}.$

4th term $= a_{1\beta_1} a_{2\beta_1} a_{2\beta_2} u_{\beta_2} u_a a_{1a}$

$= \frac{1}{4} u_{\beta_1} a_{2\beta_1} a_{2\beta_2} u_{\beta_2} a^2{}_{1a}$

$= \frac{1}{4} u^2{}_{\beta_1} a^2{}_{2\beta_2} a^2{}_{1a} + \frac{1}{4} u_{\beta_1} \dot{a}_{2\beta_1} a_{2\beta_2} \dot{u}_{\beta_2} a^2{}_{1a}$

$= \frac{1}{4} u^2{}_{\beta_1} a^2{}_{2\beta_2} a^2{}_{1a} + \frac{1}{4} (ub_1 b_2 b_3)(a_2 u\dot{b}_2 \dot{b}_3) a_{2\beta_2} \dot{b}_{1\beta_2} a^2{}_{1a}$

$= \frac{1}{4} u^2{}_{\beta_1} a^2{}_{2\beta_2} a^2{}_{1a} + \frac{3}{4} (ub_1 b_2 b_3)(a_2 u b_2 b_3) a_{2\beta_2} b_{1\beta_2} a^2{}_{1a}$

$= \frac{1}{4} u^2{}_{\beta_1} a^2{}_{2\beta_2} a^2{}_{1a} + \frac{3}{16} (ua_2 b_2 b_3)(a_2 u b_2 b_3)\, b^2{}_{1\beta_2} a^2{}_{1a}$

$= \frac{1}{4} a^2_a a'^2_\beta u^2_{\beta'} - \frac{3}{16} (Ba'u)^2 b^2_\beta a^2_a .$

5th term $= a_{1\beta_1} u_{\beta_1} a_{2\beta_2} a_{1\beta_2} u_a a_{2a}$

$= \frac{1}{4} a_{1\beta_1} u_{\beta_1} u_{\beta_2} a_{1\beta_2} a^2{}_{2a}$

$= \frac{1}{4} a^2_a a'^2_\beta u^2_{\beta'} - \frac{3}{16} (Ba'u)^2 b^2_\beta a^2_a$ (as for the 4th term).

6th term $= a^2{}_{1\beta_1} a_{2\beta_2} u_{\beta_2} u_a a_{2a}$

$= \frac{1}{4} a_{1\beta_1} a^2{}_{2a} u^2{}_{\beta_2}$

$= \frac{1}{4} a^2_a a'^2_\beta u^2_{\beta'}.$

By combining the above six terms, we have $Z = -\frac{1}{16} a^2_a b^2_\beta (Ba'u)^2$.

Thus, from (A.2)

$$\tfrac{4}{3} u^2_a (a \,.\, A\beta \,.\, A'\beta' \,.\, u)^2 = \tfrac{16}{3} u^2_a (X - 2Y + Z)$$

$$= \tfrac{16}{3} u^2_a \,[a^2_\beta a'^2_{\beta'} u^2_{a'} - 2\,(\tfrac{1}{3} a^2_\beta a'^2_{\beta'} u^2_{a'} - \tfrac{1}{12} a^2_{a'} a'^2_\beta u^2_{\beta'}) - \tfrac{1}{16} a^2_{a'} b^2_\beta (Ba'u)^2]$$

(A.3) $$= \tfrac{16}{9} a^2_\beta a'^2_{\beta'} u^2_a u^2_{a'} + \tfrac{8}{9} a^2_a a'^2_\beta u^2_{a'} u^2_{\beta'} - \tfrac{1}{3} a^2_a b^2_\beta u^2_{a'} (Ba'u)^2.$$

From (A.1) and (A.3) condition (7.1) therefore reduces to

$$\tfrac{16}{9} a^2_\beta a'^2_{\beta'} u^2_a u^2_{a'} + \tfrac{1}{36} a^2_a a'^2_{a'} u^2_\beta u^2_{\beta'} + \tfrac{4}{9} a^2_a a'^2_\beta u^2_{a'} u^2_\beta = \tfrac{16}{9} a^2_\beta a'^2_{\beta'} u^2_a u^2_{a'} + \tfrac{8}{9} a^2_a a'^2_\beta u^2_{a'} u^2_{\beta'} - \tfrac{1}{3} a^2_a b^2_\beta u^2_{a'} (Ba'u)^2,$$

i e. $$a^2_a a'^2_{a'} u^2_\beta u^2_{\beta'} - 16 a^2_a a'^2_\beta u^2_{a'} u^2_{\beta'} + 12 a^2_a b^2_\beta u^2_{a'} (Ba'u)^2 = 0,$$

which gives $$\Delta^2 \Sigma'^2 - 16 \Delta \Theta' \Sigma \Sigma' + 12 \Delta \Delta' \Sigma \chi' = 0.$$

The equations (7.2) determining the developable surface previously considered are therefore

$$\chi = 0 \qquad \text{and} \qquad \Delta \Sigma'^2 - 16 \Theta' \Sigma \Sigma' + 12 \Delta' \Sigma \chi' = 0.$$

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To obtain the equations of the curve locus of the vertices of the ∞^1 Φ-tetrahedra it is merely necessary to reciprocate this developable with respect to f', since the vertices are the poles of the faces with respect to f'.

The equation of the polar reciprocal of the quartic envelope specified in (7.2) is thus given by substituting $b_x b$ for u in the latter equation.

The polar reciprocal of Σ is given by the following expression equated to zero:

$$\begin{aligned} b_x b_a b'_a b'_x &= b_a^2 b'^2_x + \dot{b}_x b_a \dot{b}'_a b'_x \\ &= b_a^2 b'^2_x - \tfrac{1}{2}(Bax)^2 \\ &= \Theta f' - \tfrac{1}{2}k'. \end{aligned}$$

For Σ', it is
$$\begin{aligned} b_x b_\beta b'_\beta b'_x &= \tfrac{1}{4} b_\beta^2 b'^2_x \\ &= \tfrac{1}{4}\Delta' f', \end{aligned}$$

and for χ',
$$\begin{aligned} b_x(Bab)(Bab')\,b'_x &= \tfrac{1}{3}\dot{b}_x(Bab)(\dot{B}ab')\,b'_x \\ &= \tfrac{1}{3}a_\beta^2 b'^2_x - \tfrac{1}{3}a_\beta b'_\beta a_x b'_x \\ &= \tfrac{1}{3}a_\beta^2 b'^2_x - \tfrac{1}{12}b'^2_\beta a_x^2 \\ &= \tfrac{1}{3}\Theta' f' - \tfrac{1}{12}\Delta' f. \end{aligned}$$

By substitution of these results in the second equation of (7.2) the quartic surface of the polar reciprocal with respect to f' is found to be
$\tfrac{1}{16}\Delta^2\Delta'^2 f'^2 - 4\Delta\Delta'\Theta' f'(\Theta f' - \tfrac{1}{2}k') + 12\Delta\Delta'(\Theta f' - \tfrac{1}{2}k')(\tfrac{1}{3}\Theta' f' - \tfrac{1}{12}\Delta' f) = 0$,
which reduces to $\Delta f'^2 - 16\Theta f f' + 8fk' = 0$, and so the equations determining the locus of vertices are, as in (7.3),

$$k = 0 \text{ and } \Delta f'^2 - 16\Theta f f' + 8fk' = 0.$$

For the second system of Φ-tetrahedra, (7.4) and (7.5) are true in similar manner.

HERIOT-WATT COLLEGE,
EDINBURGH.

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