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THE IDEAL STRUCTURE OF SEMIGROUPS OF TRANSFORMATIONS WITH RESTRICTED RANGE

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Abstract

Let Y be a fixed nonempty subset of a set X and let $T(X, Y)$ denote the semigroup of all total transformations from X into Y. In 1975, Symons described the automorphisms of $T(X, Y)$. Three decades later, Nenthein, Youngkhong and Kemprasit determined its regular elements, and more recently Sanwong, Singha and Sullivan characterized all maximal and minimal congruences on $T(X, Y)$. In 2008, Sanwong and Sommanee determined the largest regular subsemigroup of $T(X, Y)$ when $|Y| \neq 1$ and $Y \neq X$; and using this, they described the Green's relations on $T(X, Y)$. Here, we use their work to describe the ideal structure of $T(X, Y)$. We also correct the proof of the corresponding result for a linear analogue of $T(X, Y)$.

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1. Introduction

Let X be a nonempty set and let $T(X)$ denote the semigroup (under composition) of all total transformations of X. For each α in $T(X)$, we let $X\alpha = \operatorname{ran}\alpha$ denote the *range* of α and we define the *rank* of α to be $r(\alpha) = |\operatorname{ran}\alpha|$. If $\emptyset \neq Y \subseteq X$, we write

$$T(X, Y) = \{\alpha \in T(X) : X\alpha \subseteq Y\}.$$

Clearly $T(X, Y)$ is a subsemigroup of $T(X)$, and if $Y = X$ then $T(X, Y) = T(X)$. Also, if $|Y| = 1$ then $T(X, Y)$ contains exactly one element: the constant map with range Y. Hence, throughout the following, we assume that Y is a proper subset of X with at least two elements.

In [9], Symons described all the automorphisms of $T(X, Y)$. Several years later, its regular elements were characterized in [4]. Also, in [6], the authors determined the largest regular subsemigroup of $T(X, Y)$ and, using this, they described Green's relations on $T(X, Y)$. More recently, in [5], Sanwong *et al.* characterized all maximal and minimal congruences on $T(X, Y)$.

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In [8] Sullivan described Green's relations and ideals in a linear analogue of $T(X, Y)$. Namely, if W is a nonzero proper subspace of a vector space V, we let $T(V, W)$ denote the semigroup (under composition) of all linear $\alpha : V \to V$ such that $V\alpha \subseteq W$. That is, we use the 'V' and 'W' in $T(V, W)$ to denote the fact that we are considering *linear* transformations. By [8, Corollary 12], $T(V, W)$ is rarely isomorphic to the semigroup $T(U)$ of all linear transformations of an arbitrary vector space U. In addition, whereas $T(V, W)$ always contains a zero element (namely, the map $V \to \{0\}$), the same is not true for $T(X, Y)$ if $|Y| \geq 2$. Hence, these two semigroups are not isomorphic and so they are worthy of study in their own right.

In Section 4, using the work in [6], we describe the ideal structure of $T(X, Y)$ and, as a consequence, we prove that this semigroup is almost never isomorphic to $T(Z)$ for any set Z. Also, in Section 5, we show how certain algebraic semigroups can be 'anti-embedded' in some $T(X, Y)$. However, before we present these nonlinear results, we correct the proof of [8, Theorem 11] which describes all of the ideals of $T(V, W)$: the argument we give for this in Section 3 then suggests how to derive the corresponding result for $T(X, Y)$.

In effect, this paper completes a project in which Green's relations and ideals are determined for semigroups which appear to be related but are almost never isomorphic or anti-isomorphic: namely, the semigroup $T(X, Y)$ and its linear analogue $T(V, W)$, as well as the semigroups

$$K(V, W) = \{\alpha \in T(V) : W \subseteq \ker \alpha\},$$
$$E(X, \sigma) = \{\alpha \in T(X) : \sigma \subseteq \pi_\alpha\},$$

where σ is a fixed equivalence on X and $\pi_\alpha = \alpha \circ \alpha^{-1}$ (see [3, 7]).

2. Green's relations on $T(X, Y)$

Throughout this paper, we write id_A for the identity transformation on a set A and we let A_b denote the constant mapping with domain A and range $\{b\}$. We also write $A \dot{\cup} B$ for the *disjoint union* of sets A and B. In addition, we adopt the convention introduced by Clifford and Preston in [1, Vol. 2, p. 241]: that is, if $\alpha \in T(X)$ then we write

$$\alpha = \begin{pmatrix} A_i \\ x_i \end{pmatrix}$$

and take as understood that the subscript i belongs to some (unmentioned) index set I, that the abbreviation $\{x_i\}$ denotes $\{x_i : i \in I\}$, and that $\operatorname{ran} \alpha = \{x_i\}$ and $x_i\alpha^{-1} = A_i$.

Green's relations on $T(X)$ are well known: if $\alpha, \beta \in T(X)$, then $\alpha\mathcal{L}\beta$ if and only if $\operatorname{ran} \alpha = \operatorname{ran} \beta$; $\alpha\mathcal{R}\beta$ if and only if $\pi_\alpha = \pi_\beta$; $\alpha\mathcal{D}\beta$ if and only if $r(\alpha) = r(\beta)$; and $\mathcal{J} = \mathcal{D}$ (see [1, Vol. 1, Lemmas 2.5, 2.6 and 2.8 and Theorem 2.9]). In [6, Theorem 2.4], the authors determined the largest regular subsemigroup of $T(X, Y)$ when $X \neq Y$ and $|Y| \neq 1$: the set F given by

$$F = \{\alpha \in T(X, Y) : X\alpha \subseteq Y\alpha\},$$

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which is needed to describe Green's relations on $T(X, Y)$. This was done by Sanwong and Sommanee in [6, Theorems 3.2, 3.3, 3.7 and 3.9], and we quote their results for convenience.

LEMMA 1. *Let $\gamma \in F$ and $\beta \in T(X, Y)$. Then $\beta = \lambda\gamma$ for some $\lambda \in T(X, Y)$ if and only if* $\operatorname{ran}\beta \subseteq \operatorname{ran}\gamma$. *Consequently, if $\alpha, \beta \in T(X, Y)$, then $\alpha\mathcal{L}\beta$ in $T(X, Y)$ if and only if $\alpha = \beta$ or* ($\operatorname{ran}\alpha = \operatorname{ran}\beta$ *and $\alpha, \beta \in F$).*

LEMMA 2. *If $\alpha, \beta \in T(X, Y)$, then $\beta = \alpha\mu$ for some $\mu \in T(X, Y)$ if and only if $\pi_\alpha \subseteq \pi_\beta$. Consequently, $\alpha\mathcal{R}\beta$ in $T(X, Y)$ if and only if $\pi_\alpha = \pi_\beta$.*

LEMMA 3. *If $\alpha, \beta \in T(X, Y)$, then $\alpha\mathcal{D}\beta$ in $T(X, Y)$ if and only if $\pi_\alpha = \pi_\beta$ or $(r(\alpha) = r(\beta)$ and $\alpha, \beta \in F)$.*

LEMMA 4. *If $\alpha, \beta \in T(X, Y)$, then $\beta = \lambda\alpha\mu$ for some $\lambda, \mu \in T(X, Y)$ if and only if $r(\beta) \leq |Y\alpha|$. Consequently, $\alpha\mathcal{J}\beta$ in $T(X, Y)$ if and only if $\pi_\alpha = \pi_\beta$ or $r(\alpha) = |Y\alpha| = |Y\beta| = r(\beta)$.*

By Hall's theorem [2, Proposition II.4.5], any regular subsemigroup of $T(X)$ inherits characterizations of its relations $\mathcal{L}$ and $\mathcal{R}$ from those on $T(X)$. Thus, by Lemmas 1 and 2, if $\alpha, \beta \in F$, then $\alpha\mathcal{L}\beta$ in F if and only if $\operatorname{ran}\alpha = \operatorname{ran}\beta$, and $\alpha\mathcal{R}\beta$ in F if and only if $\pi_\alpha = \pi_\beta$.

As observed in [6, Corollary 3.11], $\mathcal{J} = \mathcal{D}$ on F. In fact, the next result shows that if $\alpha, \beta \in F$, then $\alpha\mathcal{J}\beta$ in F if and only if $r(\alpha) = r(\beta)$: this is comparable with the $\mathcal{J}$-relation on $T(X)$.

LEMMA 5. *If $\alpha, \beta \in F$, then $\beta = \lambda\alpha\mu$ for some $\lambda, \mu \in F$ if and only if $r(\beta) \leq r(\alpha)$. Consequently, $\alpha\mathcal{J}\beta$ in F if and only if $r(\alpha) = r(\beta)$.*

PROOF. Suppose that $\beta = \lambda\alpha\mu$ for some $\lambda, \mu \in F$. By Lemma 4, $r(\beta) \leq |Y\alpha|$. Since $\alpha \in F$, then $X\alpha \subseteq Y\alpha \subseteq X\alpha$, and so $|Y\alpha| = |X\alpha| = r(\alpha)$. Thus, $r(\beta) \leq r(\alpha)$. Conversely, suppose that the latter holds and let $\operatorname{ran}\beta = \{b_i\}$ and $\operatorname{ran}\alpha = \{a_i\} \,\dot\cup\, \{a_j\}$, where $\{b_i\} = Y\beta = X\beta \subseteq Y$ and $\{a_i\} \,\dot\cup\, \{a_j\} = Y\alpha = X\alpha \subseteq Y$. For each i, let $b_i\beta^{-1} = B_i$ and $a_i\alpha^{-1} = A_i$, and choose $y_i \in A_i \cap Y$ (possible since $a_i \in Y\alpha$). Define $\lambda \in T(X)$ by

$$\lambda = \begin{pmatrix} B_i \\ y_i \end{pmatrix}.$$

Clearly, $X\lambda = \{y_i\} \subseteq Y$. Since $\{b_i\} = Y\beta$, it follows that $B_i \cap Y \neq \emptyset$ for every i. Therefore, $Y\lambda = \{y_i\} = X\lambda$, and hence $\lambda \in F$. Now fix $i_0 \in I$ and let $Y \setminus X\alpha = \{a_k\}$ (note that this set may be empty). Write $\{a_j\} \,\dot\cup\, \{a_k\} \,\dot\cup\, (X \setminus Y) = C$ and define $\mu \in T(X)$ by

$$\mu = \begin{pmatrix} a_i & C \\ b_i & b_{i_0} \end{pmatrix}.$$

Then $X\mu = Y\mu = \{b_i\} \subseteq Y$, and so $\mu \in F$. Also $\beta = \lambda\alpha\mu$.

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Next we show that if $\alpha\mathcal{J}\beta$ in F then $r(\alpha) = r(\beta)$ (the converse follows from the first part of this lemma). Suppose that $\beta = \lambda\alpha\mu$ and $\alpha = \lambda'\beta\mu'$ for some $\lambda, \lambda', \mu, \mu' \in F^1$. Then

$$|X\beta| = |(X\lambda)\alpha\mu| \leq |(X\alpha)\mu| \leq |X\alpha|,$$

even if $\lambda = 1$ or $\mu = 1$. Similarly, $|X\alpha| \leq |(X\lambda')\beta\mu'| \leq |X\beta|$, and hence $r(\alpha) = r(\beta)$. □

Although the $\mathcal{R}$-relation on $T(X, Y)$ can be described just like the corresponding one on $T(X)$, the other Green's relations differ substantially from the corresponding ones on $T(X)$. In particular, from Lemma 4, we conclude that $\alpha\mathcal{J}\beta$ in $T(X, Y)$ implies that $r(\alpha) = r(\beta)$, but the converse does not hold when $X \neq Y$ and $|Y| \neq 1$. To see this, choose two distinct elements y_1, y_2 in Y and write $Y = A \dot{\cup} B$, with $y_1 \in A$ and $y_2 \in B$. Also, let $X \setminus Y = C$. Now define $\alpha, \beta \in T(X)$ by

$$\alpha = \begin{pmatrix} A \dot{\cup} B & C \\ y_1 & y_2 \end{pmatrix}, \quad \beta = \begin{pmatrix} A \dot{\cup} C & B \\ y_2 & y_1 \end{pmatrix}.$$

Clearly, $\alpha, \beta \in T(X, Y)$ and $r(\alpha) = r(\beta)$, since $\operatorname{ran}\alpha = \operatorname{ran}\beta = \{y_1, y_2\} \subseteq Y$. On the other hand, $|Y\alpha| \neq |Y\beta|$ and $\pi_\alpha \neq \pi_\beta$, and this implies that α and β are not $\mathcal{J}$-related in $T(X, Y)$.

In passing, we observe that in [6, Theorem 3.12], the authors proved that if Y is finite, then $\mathcal{D} = \mathcal{J}$ on $T(X, Y)$, but the same does not hold in general (see [6, Example 3.10]).

3. Ideals in $T(V, W)$

Before determining all of the ideals in $T(X, Y)$, we correct the proof of the corresponding result for $T(V, W)$ in [8, Theorem 11]. The argument for that result appeals to [8, Lemma 10] where, using the notation of its proof, $\{w_m\} \dot{\cup} \{w_n\}$ is a linearly independent subset of W and $u \in V \setminus W$, so $\{w_m\} \dot{\cup} \{u + w_n\}$ is linearly independent in V and each $u + w_n \notin W$. However, it is asserted that $\dim(W\gamma) < \dim(V\gamma)$ for some $\gamma \in T(V, W)$, which may be false. For example, $(u + w_1) - (u + w_2) \in W$ if $1, 2 \in N$ (see [8, p. 450]), and this may change the relative dimensions of $W\gamma$ and $V\gamma$. The result in [8, Theorem 11] is correct, but it requires a different lemma (recall that, as assumed in [8, p. 442], to avoid trivialities, W is a nonzero proper subspace of V). In what follows, we use the notation of [8], but change it slightly to avoid any confusion with our notation in Section 4.

As in [8, p. 442], we let $Q = \{\alpha \in T(V, W) : V\alpha \subseteq W\alpha\}$. By [8, Lemma 1], Q is the largest regular subsemigroup of $T(V, W)$.

Lemma 6. *If $\beta \in Q$ and $r < \dim(W\beta) = s$, then there exists $\lambda \in T(V, W)$ such that $\lambda\beta \notin Q$ and $\dim(W\lambda\beta) = r$.*

Proof. If $\beta \in Q$ and $\dim(W\beta) = s \geq r'$, we can write

$$\beta = \begin{pmatrix} u_p & w_j \\ 0 & w_j' \end{pmatrix},$$

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where $|J| = s$. Choose $K \,\dot\cup\, \{1\} \subseteq J$ with $|K| = r$, let $u \in V \setminus W$, write $V = \langle v_\ell \rangle \oplus \langle u \rangle \oplus \langle w_k \rangle$ where $W \subseteq \langle v_\ell \rangle \oplus \langle w_k \rangle$, and define $\lambda \in T(V, W)$ by

$$\lambda = \begin{pmatrix} v_\ell & u & w_k \\ 0 & w_1 & w_k \end{pmatrix}.$$

Then $W\lambda\beta = \langle w_k' \rangle \neq \langle w_1' \rangle \oplus \langle w_k' \rangle = V\lambda\beta$, so $\lambda\beta \notin Q$ and $\dim(W\lambda\beta) = r$. □

We now prove [8, Theorem 11]: in essence, the only difference between what follows and the argument for [8, Theorem 11] lies in the choice of the subset Σ of the ideal $\mathbb{I}$ in $T(V, W)$. For convenience, we recall some notation in [8, p. 448]: namely, for each $1 \leq r \leq \dim W$, T_r denotes the set $\{\alpha \in T(V, W) : r(\alpha) < r\}$, and if Σ is a nonempty subset of $T(V, W)$, then

$$r(\Sigma) = \min\{r : r > \dim(W\alpha) \text{ for all } \alpha \in \Sigma\},$$
$$K(\Sigma) = \{\beta \in T(V, W) : \ker \beta \supseteq \ker \alpha \text{ for some } \alpha \in \Sigma\}.$$

THEOREM 7. *The ideals of $T(V, W)$ are precisely the sets $T_r \cup K(\Sigma)$ and $T_{r'} \cup K(\Sigma)$, where $r = r(\Sigma)$ and Σ is a nonempty subset of $T(V, W)$.*

PROOF. Let $\mathbb{I}$ be an ideal of $T(V, W)$. If $\mathbb{I} = \{0\}$, we let $\Sigma = \mathbb{I}$, so $r(\Sigma) = 1$, $T_1 = \{0\}$; and, if $\beta \in K(\{0\})$ then $\ker \beta = V$, so $\beta = 0$ and thus $K(\{0\}) = \{0\}$. That is, $\{0\} = T_1 \cup K(\{0\})$.

Suppose $\alpha \in \mathbb{I}$ is nonzero and write

$$\alpha = \begin{pmatrix} u_p & w_j & v_k \\ 0 & w_j' & w_k \end{pmatrix}$$

where $W \subseteq \langle u_p \rangle \oplus \langle w_j \rangle$ and $W \cap \langle v_k \rangle = \{0\}$. If $J = \emptyset$, then $K \neq \emptyset$ and $W\alpha = \{0\} \neq \langle w_k \rangle = V\alpha$, so $\alpha \in \mathbb{I} \setminus Q$. On the other hand, if $J \neq \emptyset$, choose $1 \in J$ and $u \in V \setminus W$, write $V = \langle u \rangle \oplus \langle v_m \rangle$ where $W \subseteq \langle v_m \rangle$, and let

$$\lambda = \begin{pmatrix} v_m & u \\ 0 & w_1 \end{pmatrix}.$$

Then $W\lambda\alpha = \{0\} \neq \langle w_1' \rangle = V\lambda\alpha$, so $\lambda\alpha \in \mathbb{I}$ and $\lambda\alpha \notin Q$. That is, in each case, if $\Sigma = \mathbb{I} \setminus Q$ then $\Sigma \neq \emptyset$ and we assert that $\mathbb{I}$ equals $T_r \cup K(\Sigma)$ or $T_{r'} \cup K(\Sigma)$, where $r = r(\Sigma)$.

First suppose that $\dim(W\beta) < r$ for all $\beta \in \mathbb{I}$. In this case, suppose that $\beta \in \mathbb{I}$. Now, if $r(\beta) < r$, then $\beta \in T_r$ and, if $\dim(W\beta) < r \leq r(\beta)$, then $W\beta \neq V\beta$, so $\beta \in \Sigma$ and hence $\beta \in K(\Sigma)$. Thus, in this case, $\mathbb{I} \subseteq T_r \cup K(\Sigma)$. Conversely, suppose that $\beta \in T_r$. If $\dim(W\alpha) < r(\beta) < r$ for all $\alpha \in \Sigma$, we contradict the choice of $r = r(\Sigma)$. Therefore, $r(\beta) \leq \dim(W\alpha)$ for some $\alpha \in \Sigma \subseteq \mathbb{I}$, and hence $\beta \in \mathbb{I}$ by [8, Lemma 4]. Clearly, $K(\Sigma) \subseteq \mathbb{I}$ by [8, Lemma 3], so we conclude that $\mathbb{I} = T_r \cup K(\Sigma)$.

Next suppose that $r \leq \dim(W\pi)$ for some $\pi \in \mathbb{I}$. In this case, if $W\pi \neq V\pi$, then $\pi \in \Sigma$ and we contradict the choice of r. Hence $W\pi = V\pi$ and thus $\pi \in Q$, where $r(\pi) = s \geq r$. Now, if $s \geq r'$, then Lemma 6 says that there exists $\lambda \in T(V, W)$ such that $\lambda\pi \in \mathbb{I} \setminus Q = \Sigma$ and $\dim(W\lambda\pi) = r$, which contradicts the choice of r.

Hence, in this case, $r = s$ and thus $\pi \in T_{r'}$. Clearly this conclusion holds for any $\beta \in \mathbb{I}$ such that $r \leq \dim(W\beta)$. On the other hand, if $\beta \in \mathbb{I}$ and $\dim(W\beta) < r$, then we have already seen that $\beta \in T_r \cup K(\Sigma)$. So, in this case, $\mathbb{I} \subseteq T_{r'} \cup K(\Sigma)$. Conversely, if $\beta \in T_{r'}$ then $r(\beta) \leq r = \dim(W\pi)$ for the same π as before, so $\beta \in \mathbb{I}$ by [8, Lemma 4]. Like before, $K(\Sigma) \subseteq \mathbb{I}$, and we now conclude that $\mathbb{I} = T_{r'} \cup K(\Sigma)$. □

4. Ideals in $T(X, Y)$

As in Section 3, for each cardinal r, we let r' denote the successor of r. It is well known that the ideals of $T(X)$ are precisely the sets $\{\alpha \in T(X) : r(\alpha) < r\}$, where $1 < r \leq |X|'$, and hence they form a chain under containment. The same is true for the ideals in F, as we now show.

THEOREM 8. *The ideals in F are exactly the sets*

$$F_r = \{\alpha \in F : r(\alpha) < r\},$$

where $1 < r \leq |Y|'$. Moreover, F_r is a principal ideal of F if and only if r is a successor cardinal.

PROOF. It is easy to see that F_r is nonempty. For, given $y \in Y$, $r(X_y) = 1 < r$ and so $X_y \in F_r$. Now let $\alpha \in F_r$ and $\beta \in F$. Then $\alpha\beta$, $\beta\alpha \in F$ and

$$r(\alpha\beta) = |X\alpha\beta| \leq |X\alpha| = r(\alpha) < r.$$

Also $X\beta\alpha \subseteq X\alpha$, and so $r(\beta\alpha) \leq r(\alpha) < r$. Therefore $\alpha\beta$, $\beta\alpha \in F_r$, and hence F_r is an ideal of F. Conversely, let $\mathbb{I}$ be an ideal of F and let r be the least cardinal greater than $r(\alpha)$ for every $\alpha \in \mathbb{I}$ (this is possible since the cardinals are well ordered). Then $\mathbb{I} \subseteq F_r$. To see that $F_r \subseteq \mathbb{I}$, let $\beta \in F_r$. Then there exists $\alpha \in \mathbb{I}$ such that $r(\beta) \leq r(\alpha)$; otherwise, $r(\alpha) < r(\beta) < r$ for every $\alpha \in \mathbb{I}$, and this contradicts our choice of r. By Lemma 5, $r(\beta) \leq r(\alpha)$ implies that $\beta = \lambda\alpha\mu$ for some $\lambda, \mu \in F$. Since $\mathbb{I}$ is an ideal of F, $\beta \in \mathbb{I}$, and so $F_r = \mathbb{I}$.

Next we determine all the principal ideals of F. To do this, let r be a successor cardinal, say $r = s'$, and choose $\alpha \in F_r$ with $r(\alpha) = s$. If $r(\beta) > s$ for some $\beta \in F_r$, then $r(\beta) \geq s' = r$, a contradiction. Thus, for every $\beta \in F_r$, $r(\beta) \leq s = r(\alpha)$ and, by Lemma 5, $\beta \in J(\alpha)$, the principal ideal of F generated by α. Hence, $F_r \subseteq J(\alpha)$. Since the reverse inclusion also holds, F_r is principal. Conversely, suppose that $F_r = J(\alpha)$ for some $\alpha \in F_r$. Let $r(\alpha) = s$ and assume that $s < t < r$ for some cardinal t. Clearly, $t = r(\gamma)$ for some $\gamma \in F$ (since $t < r \leq |Y|'$). By Lemma 5, $J(\alpha) \subseteq J(\gamma) \subseteq F_r$, contradicting our supposition. In other words, r is the least cardinal greater than s, and so $r = s'$. □

We proceed to describe the ideals of $T(X, Y)$. To do this, let $1 < r \leq |Y|'$ and write

$$T_r = \{\alpha \in T(X, Y) : r(\alpha) < r\}.$$

Let $\alpha \in T_r$ and $\beta \in T(X, Y)$. Then $X\beta\alpha \subseteq X\alpha$, and so $r(\beta\alpha) \leq r(\alpha) < r$. Also $r(\alpha\beta) = |X\alpha\beta| \leq |X\alpha| = r(\alpha) < r$. Therefore, T_r is an ideal of $T(X, Y)$.

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Now let $\mathfrak{S}$ be a nonempty subset of $T(X, Y)$ and let

$$r(\mathfrak{S}) = \min\{r : |Y\alpha| < r \text{ for every } \alpha \in \mathfrak{S}\},$$
$$\Pi(\mathfrak{S}) = \{\beta \in T(X, Y) : \pi_\alpha \subseteq \pi_\beta \text{ for some } \alpha \in \mathfrak{S}\}.$$

LEMMA 9. *For each nonempty subset $\mathfrak{S}$ of $T(X, Y)$, $T_{r(\mathfrak{S})} \cup \Pi(\mathfrak{S})$ and $T_{r(\mathfrak{S})'} \cup \Pi(\mathfrak{S})$ are ideals of $T(X, Y)$.*

PROOF. Given $\beta, \mu \in T(X, Y)$, $\pi_\beta \subseteq \pi_{\beta\mu}$. Thus, $\Pi(\mathfrak{S})$ is a right ideal of $T(X, Y)$. Now, let $\lambda \in T(X, Y)$ and $\beta \in \Pi(\mathfrak{S})$. Then $\pi_\alpha \subseteq \pi_\beta$ for some $\alpha \in \mathfrak{S}$ and, by Lemma 2, $\beta = \alpha\mu$ for some $\mu \in T(X, Y)$. Therefore, since $X\lambda \subseteq Y$,

$$r(\lambda\beta) = |X\lambda\beta| \leq |Y\beta| = |Y\alpha\mu| \leq |Y\alpha| < r(\mathfrak{S}).$$

Hence, $\lambda\beta \in T_{r(\mathfrak{S})}$. By the remark above, $T_{r(\mathfrak{S})}$ is an ideal of $T(X, Y)$. Thus, given $\beta \in T_{r(\mathfrak{S})} \cup \Pi(\mathfrak{S})$ and $\lambda, \mu \in T(X, Y)^1$, we have $\lambda\beta\mu \in T_{r(\mathfrak{S})} \cup \Pi(\mathfrak{S})$, and so $T_{r(\mathfrak{S})} \cup \Pi(\mathfrak{S})$ is an ideal of $T(X, Y)$. Since $T_{r(\mathfrak{S})'}$ is an ideal of $T(X, Y)$ and $T_{r(\mathfrak{S})} \subseteq T_{r(\mathfrak{S})'}$, it follows that $T_{r(\mathfrak{S})'} \cup \Pi(\mathfrak{S})$ is also an ideal of $T(X, Y)$. □

Next we show that the above ideals are the only ones in $T(X, Y)$. Although the following argument is similar to the one given for $T(V, W)$ in Section 3, we provide most of the details in this nonlinear context. As before, we start with a technical result.

LEMMA 10. *If $\beta \in F$ and $r < |Y\beta| = s$, then there exists $\lambda \in T(X, Y)$ such that $\lambda\beta \notin F$ and $|Y\lambda\beta| = r$.*

PROOF. If $\beta \in F$ and $|Y\beta| = s \geq r'$, we can write

$$\beta = \begin{pmatrix} A_j \\ y_j' \end{pmatrix}$$

where $|J| = s$ and $Y \cap A_j \neq \emptyset$ for each j. Choose $K \,\dot{\cup}\, \{1\} \subseteq J$ with $|K| = r$, and let $y_i \in Y \cap A_i$ for each $i \in K \cup \{1\}$. Also, choose $2 \in K$ and write $L = K \setminus \{2\}$ (which may be empty). Finally, choose $u \in X \setminus Y$, let $B = X \setminus [\{u\} \cup \{y_\ell\}]$ and define $\lambda \in T(X, Y)$ by

$$\lambda = \begin{pmatrix} B & u & y_\ell \\ y_2 & y_1 & y_\ell \end{pmatrix}.$$

Then $Y\lambda\beta = \{y_2'\} \,\dot{\cup}\, \{y_\ell'\} \neq X\lambda\beta$, so $\lambda\beta \notin F$ and $|Y\lambda\beta| = r$. □

Recall that, as stated in Section 1, Y is a proper subset of X with at least two elements. We let $C(Y)$ denote the set of all constants in $T(X, Y)$ and observe that this is the smallest ideal of $T(X, Y)$.

THEOREM 11. *The ideals of $T(X, Y)$ are precisely the sets $T_r \cup \Pi(\mathfrak{S})$ and $T_{r'} \cup \Pi(\mathfrak{S})$, where $r = r(\mathfrak{S})$ and $\mathfrak{S}$ is a nonempty subset of $T(X, Y)$.*

PROOF. Let $\mathbb{I}$ be an ideal of $T(X, Y)$. If $\mathbb{I} = C(Y)$, we let $\mathfrak{S} = \mathbb{I}$, so $r(\mathfrak{S}) = 2$ and $T_2 = C(Y)$; and, if $\beta \in \Pi(\mathfrak{S})$, then β is constant and thus $\Pi(\mathfrak{S}) = \mathfrak{S}$. That is, $C(Y) = T_2 \cup \Pi(\mathfrak{S})$, where $\mathfrak{S} = C(Y)$.

Suppose that $\alpha \in \mathbb{I}$ is nonconstant and write

$$\alpha = \begin{pmatrix} A_j & A_k \\ y'_j & y'_k \end{pmatrix}$$

where $Y \cap A_j \neq \emptyset$ for each j and $Y \cap \bigcup A_k = \emptyset$. If $K \neq \emptyset$ then $Y\alpha = \{y'_j\} \neq X\alpha$, so $\alpha \notin F$. On the other hand, if $K = \emptyset$ then $|J| \geq 2$. Now choose $1, 2 \in J$ and $y_i \in A_i \cap Y$ for $i = 1, 2$, let $u \in X \setminus Y$ and define $\lambda \in T(X, Y)$ by

$$\lambda = \begin{pmatrix} u & X \setminus \{u\} \\ y_1 & y_2 \end{pmatrix}.$$

Then $Y\lambda\alpha = \{y'_2\} \neq \{y'_1, y'_2\} = X\lambda\alpha$, so $\lambda\alpha \in \mathbb{I}$ and $\lambda\alpha \notin F$. That is, in each case, if $\mathfrak{S} = \mathbb{I} \setminus F$ then $\mathfrak{S} \neq \emptyset$ and we assert that $\mathbb{I}$ equals $T_r \cup \Pi(\mathfrak{S})$ or $T_{r'} \cup \Pi(\mathfrak{S})$, where $r = r(\mathfrak{S})$.

First suppose that $|Y\beta| < r$ for all $\beta \in \mathbb{I}$. In this case, suppose that $\beta \in \mathbb{I}$. Now, if $r(\beta) < r$, then $\beta \in T_r$ and, if $|Y\beta| < r \leq r(\beta)$, then $Y\beta \neq X\beta$, so $\beta \in \mathfrak{S}$ and hence $\beta \in \Pi(\mathfrak{S})$. Thus, in this case, $\mathbb{I} \subseteq T_r \cup \Pi(\mathfrak{S})$. Conversely, suppose that $\beta \in T_r$. Then, as in the linear case, $r(\beta) \leq |Y\alpha|$ for some $\alpha \in \mathfrak{S} \subseteq \mathbb{I}$, and hence $\beta \in \mathbb{I}$ by Lemma 4. Clearly, $\Pi(\mathfrak{S}) \subseteq \mathbb{I}$ by Lemma 2, so we conclude that $\mathbb{I} = T_r \cup \Pi(\mathfrak{S})$.

Next suppose that $r \leq |Y\gamma|$ for some $\gamma \in \mathbb{I}$. In this case, if $Y\gamma \neq X\gamma$, then $\gamma \in \mathfrak{S}$ and we contradict the choice of r. Hence $Y\gamma = X\gamma$ and thus $\gamma \in F$, where $r(\gamma) = s \geq r$. Now, if $s \geq r'$, then Lemma 10 says that there exists $\lambda \in T(X, Y)$ such that $\lambda\gamma \in \mathbb{I} \setminus F = \mathfrak{S}$ and $|Y\lambda\gamma| = r$, which contradicts the choice of r. Hence, in this case, $r = s$ and $\gamma \in T_{r'}$. The rest of the proof proceeds in the same way as for Theorem 7, so we omit the details. □

COROLLARY 12. *If $|Y| \geq 3$, then $T(X, Y)$ is not isomorphic to $T(Z)$ for any set Z.*

PROOF. Suppose that $|Y| \geq 3$, write Y as a disjoint union of three sets, say $A \dot\cup B \dot\cup C$, and let $y_1, y_2, y_3 \in Y$ be distinct. By our assumption, $X \setminus Y \neq \emptyset$. Define $\alpha_1, \alpha_2 \in T(X, Y)$ by

$$\alpha_1 = \begin{pmatrix} A \dot\cup B & C & X \setminus Y \\ y_1 & y_2 & y_3 \end{pmatrix}, \quad \alpha_2 = \begin{pmatrix} A & B \dot\cup C & X \setminus Y \\ y_1 & y_2 & y_3 \end{pmatrix}.$$

Clearly, $|Y\alpha_1| = 2 < 3 = |X\alpha_1|$ and so, if $\mathfrak{S}_1 = \{\alpha_1\}$, then $r(\mathfrak{S}_1) = 3$ and $\alpha_1 \in T_3 \cup \Pi(\mathfrak{S}_1)$ and this is an ideal of $T(X, Y)$ by Lemma 9. Likewise, if $\mathfrak{S}_2 = \{\alpha_2\}$ then $T_3 \cup \Pi(\mathfrak{S}_2)$ is an ideal of $T(X, Y)$ and $\alpha_2 \in T_3 \cup \Pi(\mathfrak{S}_2)$. Now, $\alpha_1 \notin T_3 \cup \Pi(\mathfrak{S}_2)$ since $r(\alpha_1) = 3$ and $\pi_{\alpha_2} \not\subseteq \pi_{\alpha_1}$, so $T_3 \cup \Pi(\mathfrak{S}_1) \not\subseteq T_3 \cup \Pi(\mathfrak{S}_2)$. Similarly, $r(\alpha_2) = 3$ and $\pi_{\alpha_1} \not\subseteq \pi_{\alpha_2}$ imply $\alpha_2 \notin T_3 \cup \Pi(\mathfrak{S}_1)$, and hence $T_3 \cup \Pi(\mathfrak{S}_2) \not\subseteq T_3 \cup \Pi(\mathfrak{S}_1)$. In other words, we have shown that, if $|Y| \geq 3$, then $T(X, Y)$ contains two ideals which are not comparable under containment, and so it cannot be isomorphic to $T(Z)$ for any set Z. □

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It is obvious that, if $|X| \geq 2$, then the largest proper ideal of $T(X)$ is $\{\alpha \in T(X) : r(\alpha) < |X|\}$. However, to determine the maximal ideals in $T(X, Y)$, we need a technical lemma, which we motivate by observing that, for each $\alpha \in T(X, Y)$, $|Y\alpha| \leq |X\alpha| \leq |Y|$.

Lemma 13. *No proper ideal of $T(X, Y)$ contains any element γ with $|Y\gamma| = |X\gamma| = |Y|$.*

Proof. Let $\mathbb{J}$ be an ideal of $T(X, Y)$ and suppose that there exists $\gamma \in \mathbb{J}$ such that $|Y\gamma| = |X\gamma| = |Y|$. Given $\beta \in T(X, Y)$, we have ran $\beta \subseteq Y$, and so $r(\beta) \leq |Y| = |Y\gamma|$. By Lemma 4, $\beta = \lambda\gamma\mu$ for some $\lambda, \mu \in T(X, Y)$, and so $\beta \in \mathbb{J}$. Therefore, $\mathbb{J} = T(X, Y)$. □

Theorem 14. *If $|Y| = p \geq 2$, then the largest proper ideal of $T(X, Y)$ is the set $T_p \cup \mathfrak{S}$, where $\mathfrak{S} = \{\alpha \in T(X, Y) : |Y\alpha| < |X\alpha| = p\}$ (which may be empty).*

Proof. First suppose that $\mathfrak{S} = \emptyset$. By the remark before Lemma 9, T_p is an ideal of $T(X, Y)$. Clearly, it is a proper ideal and, by Lemma 13, every proper ideal of $T(X, Y)$ is contained in T_p. Hence, in this case, T_p is the largest proper ideal of $T(X, Y)$.

If $\mathfrak{S} \neq \emptyset$, then let $\alpha \in \mathfrak{S}$ and write $Y\alpha = \{a_j\}$. Since $|Y\alpha| < p = |X\alpha|$, we can write $X\alpha = \{a_j\} \dot\cup \{a_i\}$ for some subset $\{a_i\}$ of Y, where $|J| + |I| = p$. Clearly, $\{a_i\} = X\alpha \setminus Y\alpha \subseteq (X \setminus Y)\alpha$, and so $|X \setminus Y| \geq |I|$.

If p is infinite, then $|X \setminus Y| \geq |I| = p = |Y|$ and so, for every cardinal q such that $q < p$, we can write $Y = \{y_m\} \dot\cup \{y_n\}$ and $X \setminus Y = \{x_n\} \dot\cup \{x_\ell\}$, where $|M| = q$, $|N| = p$ and $|L| = |X \setminus Y|$. Choose $1 \in M$ and define $\beta \in T(X, Y)$ by

$$\beta = \begin{pmatrix} y_m & \{y_n\} & x_n & \{x_\ell\} \\ y_m & y_1 & y_n & y_1 \end{pmatrix}.$$

Since $Y\beta = \{y_m\}$ and $X\beta = \{y_m\} \dot\cup \{y_n\} = Y$, it follows that $|Y\beta| = q$ and $\beta \in \mathfrak{S}$. That is, for each cardinal $q < p$, there exists $\beta \in \mathfrak{S}$ with $|Y\beta| = q$ and so $r(\mathfrak{S}) = p$.

Now suppose that $p \geq 2$ is finite and write $Y = \{y_1, \ldots, y_{p-1}, y_p\}$. Let $X \setminus Y = \{x_k\}$ (nonempty since we assume $Y \subsetneq X$) and define $\beta \in T(X, Y)$ by

$$\beta = \begin{pmatrix} y_1 & \cdots & y_{p-1} & y_p & \{x_k\} \\ y_1 & \cdots & y_{p-1} & y_1 & y_p \end{pmatrix}.$$

Clearly, $p - 1 = |Y\alpha| < |X\alpha| = p$, and so $r(\mathfrak{S}) = p$.

By Lemma 9, $T_p \cup \Pi(\mathfrak{S})$ is an ideal of $T(X, Y)$. It is not difficult to see that $T_p \cup \Pi(\mathfrak{S}) = T_p \cup \mathfrak{S}$. For example, clearly, $T_p \cup \mathfrak{S} \subseteq T_p \cup \Pi(\mathfrak{S})$. Given $\beta \in \Pi(\mathfrak{S})$, then $\pi_\alpha \subseteq \pi_\beta$ for some $\alpha \in \mathfrak{S}$. But this implies that $p > |Y\alpha| \geq |Y\beta|$. If $r(\beta) < p$, then $\beta \in T_p$. If not, then $\beta \in \mathfrak{S}$, and the equality follows. Also, if $\mathbb{J}$ is a proper ideal of $T(X, Y)$ then, by Lemma 13, $\mathbb{J} \subseteq T(X, Y) \setminus \{\alpha \in T(X, Y) : |X\alpha| = |Y\alpha| = p\}$: that is, $\mathbb{J} \subseteq T_p \cup \mathfrak{S}$ and this is the largest proper ideal of $T(X, Y)$. □

EXAMPLE 15. As in the proof of Theorem 14, it is easy to see that if Y is finite, then $\mathfrak{S}$ is nonempty. Now suppose $|Y| = p \geq \aleph_0$ and $|X \setminus Y| < p$. Then $|X| = p$. Clearly, there exists $\alpha \in T(X, Y)$ such that $|X\alpha| = p$. For example, write $Y = \{y_j\}$ and $X = \{x_j\}$ with $|J| = p$, and define $\alpha \in T(X, Y)$ by

$$\alpha = \begin{pmatrix} x_j \\ y_j \end{pmatrix}.$$

But, given $\beta \in T(X, Y)$ with $|X\beta| = p$, we know that $|Y\beta| = p$ (since $|(X \setminus Y)\beta| \leq |X \setminus Y| < p$), and so $\mathfrak{S} = \emptyset$ in this case.

5. An embedding problem

It is well known that any semigroup S can be embedded in $T(S^1)$, where S^1 equals S with an identity adjoined. This is achieved via the mapping $\rho : S \to T(S^1)$, $a \to \rho_a$, where $\rho_a : S^1 \to S^1$, $x \to xa$, for each $a \in S$. However, if we want ρ to embed some S into $T(S^1, Y)$ for some proper subset Y of S^1, then we must have $Sa \cup \{a\} = \operatorname{ran} \rho_a \subseteq Y$ for all $a \in S$, and hence $Y = S$. On the other hand, if we do not add an identity to S, then we need S to be 'cancellative' in some way: compare the embedding of a right cancellative semigroup S into the semigroup of all injective transformations of S in [1, Vol. 1, Lemma 1.0].

If $|Y| \geq 3$, then $T = T(X, Y)$ is *right reductive* (see [1, Vol. 1, p. 9]). In fact, it is $\mathfrak{S}$-*right-reductive* for some nonempty subset $\mathfrak{S}$ of T: that is, if $\alpha\gamma = \beta\gamma$ for all $\gamma \in \mathfrak{S}$, then $\alpha = \beta$. For example, let $\mathfrak{S}_3$ denote the set of all $\gamma \in T$ with the form

$$\gamma = \begin{pmatrix} A & B & C \\ y_1 & y_2 & y_3 \end{pmatrix}$$

where precisely one of A, B and C contains no element of Y. Suppose that $\alpha, \beta \in T$ and $\alpha\gamma = \beta\gamma$ for all $\gamma \in \mathfrak{S}_3$, and assume that $x\alpha = y_1 \neq y_2 = x\beta$ for some $x \in X$. Now, since $|Y| \geq 3$ and there exists $u \in X \setminus Y$, we can write $X = A \,\dot{\cup}\, \{y_2\} \,\dot{\cup}\, \{u\}$ and let

$$\gamma = \begin{pmatrix} A & y_2 & u \\ y_1 & y_2 & y_3 \end{pmatrix} \in \mathfrak{S}_3.$$

Then $x\alpha\gamma = y_1$ and $x\beta\gamma = y_2$, contradicting the supposition. That is, $x\alpha = x\beta$ for all $x \in X$, and thus $\alpha = \beta$.

Next recall that $T_3 = \{\alpha \in T : r(\alpha) < 3\}$ is an ideal of T, and observe that $\mathfrak{S}_3^2 \subseteq T_3$. In fact, if we write an arbitrary $\alpha \in T$ as

$$\alpha = \begin{pmatrix} A_j & A_k \\ y_j & y_k \end{pmatrix}$$

where $Y \cap A_j \neq \emptyset$ for each j and $Y \cap \bigcup A_k = \emptyset$, then it can be seen that $r(\alpha\gamma) \leq 2$ for each $\gamma \in \mathfrak{S}_3$. That is, for each $\alpha \in T$, $\alpha\mathfrak{S}_3 \subseteq T_3$. Consequently, if $L = \mathfrak{S}_3 \cup T_3$, then L is a left ideal of $T(X, Y)$ and $\alpha L \subseteq T_3 \subsetneq L$ for all $\alpha \in T$.

With the above in mind, we say that, if M, N are semigroups, then $\theta : M \to N$ is an *anti-embedding* if θ is injective and $(xy)\theta = (y\theta)(x\theta)$ for all $x, y \in M$. We now modify the *regular anti-representation* of a semigroup (see [1, Vol. 1, p. 9]) to anti-embed certain semigroups into $T(X, Y)$ for some sets X and Y.

THEOREM 16. *Suppose $K \subseteq L$ are left ideals of a semigroup S such that $aL \subseteq K$ for all $a \in S$. If S is L-right-reductive, then S can be anti-embedded into $T(L, K)$.*

PROOF. Let $\lambda : S \to T(L)$, $a \to \lambda_a$, where $\lambda_a : L \to L$, $x \to ax$, for each $a \in S$. Clearly, λ is well defined (since $aL \subseteq L$ for each $a \in S$) and $(ab)\lambda = (b\lambda)(a\lambda)$ for all $a, b \in S$. Also, if $\lambda_a = \lambda_b$, then $ax = bx$ for all $x \in L$ and so $a = b$ by supposition. In addition, ran $\lambda_a = aL \subseteq K$, so each $\lambda_a \in T(L, K)$. □

The dual of the above result embeds certain semigroups into $T(X, Y)$ for some sets X and Y and, for interest, we now state it explicitly. However, we note that if $1 < |Y|$ and $Y \subsetneq X$, then $T(X, Y)$ is not $\mathfrak{S}$-*left-reductive* for any nonempty subset $\mathfrak{S}$ of T; that is, there exist distinct $\alpha, \beta \in T(X, Y)$ such that $\gamma\alpha = \gamma\beta$ for every $\gamma \in \mathfrak{S}$. To see this, choose $x_1 \in X \setminus Y$ and distinct $y_1, y_2 \in Y$, and let $\alpha, \beta \in T(X)$ be such that $x_1\alpha = y_1$, $x_1\beta = y_2$, and $x\alpha = y_1 = x\beta$ for every $x \in X \setminus \{x_1\}$. Clearly, α, β are distinct elements of $T(X, Y)$ and, since $\alpha|Y = \beta|Y$, we have $\gamma\alpha = \gamma\beta$ for every $\gamma \in \mathfrak{S}$.

THEOREM 17. *Suppose that $K \subseteq R$ are right ideals of a semigroup S such that $Ra \subseteq K$ for all $a \in S$. If S is R-left-reductive, then S can be embedded into $T(R, K)$.*

EXAMPLE 18. We give one example of a semigroup which satisfies the algebraic conditions of Theorem 16 but differs from every $T(X, Y)$ with $|Y| \geq 2$. Suppose that $X = \{a, b, c, d\}$, and let a_b denote the partial transformation with domain $\{a\}$ and range $\{b\}$. Also let $I_2 = \{\alpha \in I(X) : r(\alpha) < 2\}$: that is, the smallest nonzero ideal of $I(X)$, the symmetric inverse semigroup on X [1, Vol. 1, p. 29]. Now write

$$K = I_2, \quad L = K \cup \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right\}, \quad S = L \cup \{\mathrm{id}_{\{c,d\}}\}.$$

Clearly, S is a semigroup with $\emptyset$ as a zero element, and $S^2 \neq \{\emptyset\}$ (that is, the operation on S is nontrivial). Also $K \subsetneq L$, and K, L are left ideals of S such that $\alpha L \subseteq K$ for all $\alpha \in S$ (moreover, $\alpha L \neq \{\emptyset\}$ for some $\alpha \in S$).

To show that S is L-right-reductive, suppose that $a_b\gamma = \beta\gamma$ for all $\gamma \in L$. In particular, if $\gamma = b_a$ then $a_b \cdot b_a \neq \emptyset$ implies that $\beta \cdot b_a \neq \emptyset$, so $b \in \operatorname{ran} \beta$ and such $\beta \in S$ cannot have rank two; hence, by comparing domains, we see that $\beta = a_b$, as required. Also, if $a_c\gamma = \beta\gamma$ for all $\gamma \in L$, then $c \in \operatorname{ran} \beta$ and $a \in \operatorname{dom} \beta$; and, if $r(\beta) = 2$ then $\beta d_d \neq \emptyset$ for $d_d \in L$, whereas $a_c \cdot d_d = \emptyset$. Thus $\beta = a_c$, as required. Likewise, if $b_b\gamma = \beta\gamma$ for all $\gamma \in L$, then $b_b \cdot b_a \neq \emptyset$, so $b \in \operatorname{ran} \beta$ and we deduce that $\beta = b_b$. Similarly, if $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\gamma = \beta\gamma$ for all $\gamma \in L$, then $c, d \in \operatorname{ran} \beta$ and $a, b \in \operatorname{dom} \beta$, and thus β must equal $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$. Similarly, we can show that if $\alpha, \beta \neq \emptyset$ in S and $\alpha\gamma = \beta\gamma$ for all $\gamma \in L$, then $\alpha = \beta$. In addition, it is obvious that $\emptyset\gamma = \beta\gamma$ for all $\gamma \in L$ precisely when $\beta = \emptyset$. Finally, recall that $T(X, Y)$ does not contain a zero if $|Y| \geq 2$.

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