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ON A CONJECTURE REGARDING THE MOUSE ORDER FOR WEASELS

JAN KRUSCHEWSKI AND FARMER SCHLUTZENBERG

Abstract. We investigate Steel's conjecture in 'The Core Model Iterability Problem' [10], that if $\mathcal{W}$ and $\mathcal{R}$ are $\Omega+1$-iterable, 1-small weasels, then $\mathcal{W} \leq^* \mathcal{R}$ iff there is a club $C \subset \Omega$ such that for all $\alpha \in C$, if α is regular, then $\alpha^{+\mathcal{W}} \leq \alpha^{+\mathcal{R}}$. We will show that the conjecture fails, assuming that there is an iterable premouse M which models KP and which has a Σ_1-Woodin cardinal. On the other hand, we show that assuming there is no transitive model of KP with a Woodin cardinal the conjecture holds. In the course of this we will also show that if M is a premouse which models KP with a largest, regular, uncountable cardinal δ, and $\mathbb{P} \in M$ is a forcing poset such that $M \models$ "$\mathbb{P}$ has the δ-c.c.", and $g \subset \mathbb{P}$ is M-generic, then $M[g] \models$ KP. Additionally, we study the preservation of admissibility under iteration maps. At last, we will prove a fact about the closure of the set of ordinals at which a weasel has the S-hull property. This answers another question implicit in remarks in [10].

§1. Introduction. In the book 'The Core Model Iterability Problem' [10, p. 28], John Steel conjectured the following:

Conjecture 1. Let $\mathcal{W}$ and $\mathcal{R}$ be 1-small weasels which are $\Omega+1$-iterable. Then the following are equivalent:

1. $\mathcal{W} \leq^* \mathcal{R}$, and
2. there is a club $C \subset \Omega$ such that $(\alpha^+)^{\mathcal{W}} \leq (\alpha^+)^{\mathcal{R}}$ for all regular cardinals $\alpha \in C$.

In the terminology of the book a weasel is premouse of ordinal height Ω, where Ω is a fixed measurable cardinal. The relation $\leq^*$ is the mouse order, i.e., if $\mathcal{W}$ and $\mathcal{R}$ are weasels which are sufficiently iterable to successfully coiterate (by results of the book $\Omega+1$-iterability suffices), then $\mathcal{W} \leq^* \mathcal{R}$ if and only if $\mathcal{R}$ wins the coiteration, i.e. if $(\mathcal{T},\mathcal{U})$ is the successful coiteration of $(\mathcal{W},\mathcal{R})$ and $\mathrm{lh}(\mathcal{T}) = \theta+1$ and $\mathrm{lh}(\mathcal{U}) = \gamma+1$, then $\mathcal{M}^{\mathcal{T}}_\theta \trianglelefteq \mathcal{M}^{\mathcal{U}}_\gamma$.

Steel showed in [10] that Conjecture 1 holds for weasels small enough that linear iterations suffice for comparison. His proof is based on universal linear iterations.

In the following we will prove Conjecture 1 under the assumption that neither $\mathcal{W}$ nor $\mathcal{R}$ have an initial segment which models the theory KP + "there is a Woodin cardinal", see Theorem 26. In particular, Conjecture 1 holds if there is no transitive model of KP + "there is a Woodin cardinal".

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On the other hand, assuming the existence of an iterable admissible passive premouse M with a $\underset{\sim}{\Sigma}_1$-Woodin cardinal,[1] we will show that there is an inner model that thinks there is a counterexample to Conjecture 1, see Section 4.2.

For the construction of the counterexample from the assumption just described we will need to investigate the extender algebra over admissible mice. In the course of this, we will show that if M is an iterable premouse modelling KP with a largest, regular, uncountable cardinal δ, $\mathbb{P} \in M$ is a forcing poset such that $M \models$ "$\mathbb{P}$ has the δ-c.c.", and $g \subseteq \mathbb{P}$ is M-generic, $M[g] \models$ KP, see Theorem 10.

In Section 5 we will answer another question implicit in remarks from [10, p. 32] about the S-hull property in 1-small weasels. We will show that for any $\Omega + 1$-iterable weasel $\mathcal{W}$ such that Ω is S-thick in $\mathcal{W}$ the set of points that have the S-hull property is almost closed, see Theorem 37 and Definition 36.

We will use the notation from [9]. A k-maximal iteration tree is as in Definition 3.4 of [11].

§2. Admissible premice.

DEFINITION 2. A passive[2] premouse $M = (\lfloor M \rfloor, \in, \mathbb{E}^M, \emptyset)$ is called *admissible* if $M \models$ KP.[3]

REMARK. Note that an active premouse M, i.e. so that $F^M \neq \emptyset$, cannot model KP. We leave this as an exercise.

For $n \leq \omega$, we say an n-sound premouse M is n-countably iterable if every countable elementary substructure of M is $(n, \omega_1, \omega_1 + 1)^*$-iterable. If M is ω-countably iterable we also say that M is countably iterable.

The following is a nice criterion for the admissibility of a passive premouse, whose proof we leave to the reader.

LEMMA 3. *Let M be a passive premouse. Then $M \models KP$ if and only if for all $f \in \Sigma_1^M(M)$ such that f is a function with* $\mathrm{dom}(f) \in M$, $f \in M$.

Moreover, if M has a largest cardinal δ, then $M \models KP$ if and only if for all $f \in \Sigma_1^M(M)$ such that f is a function with $\mathrm{dom}(f) = \delta$, $f \in M$.

REMARK. If M is a 0-countably iterable premouse without a largest cardinal, then M is admissible. The proof of this uses the Condensation Lemma.

LEMMA 4. *Suppose M is an admissible passive premouse. If $\rho_1^M < \mathrm{OR}^M$, then ρ_1^M is the largest cardinal of M.*

PROOF. Let $\rho := \rho_1^M$. Assume for the sake of contradiction that ρ^{+M} exists, i.e., $\rho^{+M} < \mathrm{OR}^M$. Let

$$H' := \mathrm{Hull}_1^M(\rho \cup \{p_1^M, \rho\}).$$

[1] See Definition 28 for the definition of a $\underset{\sim}{\Sigma}_1$-Woodin cardinal.

[2] Note that an active premouse cannot model KP with the active extender as a predicate and without the active extender as a predicate it models ZF$^-$, so trivially KP.

[3] For a definition of the theory KP see Definition 2.3 in [1].

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Let $\xi := \sup(H' \cap \rho^{+M})$. Suppose first that $\xi = \rho^{+M}$. In this case by the upwards absoluteness of Σ_1 formulas

$$M \models \forall \beta < \xi \exists \alpha \exists \zeta (\zeta \in \mathrm{Hull}_1^{M|\alpha}(\rho \cup \{p_1^M, \rho\}) \wedge \beta < \zeta < \xi).$$

Note that the part of the formula in parentheses is Σ_1. Thus, by Δ_0-collection, there is some $\gamma < \mathrm{OR}^M$ such that

$$M \models \forall \beta < \xi \exists \zeta (\zeta \in \mathrm{Hull}_1^{M|\gamma}(\rho \cup \{p_1^M, \rho\}) \wedge \beta < \zeta < \xi).$$

Thus, there is $f \in J(M|\gamma) \trianglelefteq M$ such that $f : \rho \to \rho^{+M}$ is cofinal. But then there is also $f' : \rho \to \rho^{+M}$ onto and $f' \in J(M|\gamma)$. Contradiction!

Let us suppose now that $\xi < \rho^{+M}$. Note that since $\mathrm{Hull}_1^M(\rho \cup \{p_1^M\}) \subset H'$ and $\mathrm{Hull}_1^M(\rho \cup \{p_1^M\})$ is unbounded in OR^M, H' is unbounded in OR^M. Again we have that

$$M \models \forall \beta < \xi \exists \alpha \exists \zeta (\zeta \in \mathrm{Hull}_1^{M|\alpha}(\rho \cup \{p_1^M, \rho\}) \wedge \beta < \zeta < \xi).$$

Thus, by Δ_0-collection there is some $\gamma < \mathrm{OR}^M$ such that

$$M \models \forall \beta < \xi \exists \zeta (\zeta \in \mathrm{Hull}_1^{M|\gamma}(\rho \cup \{p_1^M, \rho\}) \wedge \beta < \zeta < \xi).$$

Since H' is unbounded in OR^M, there is some such $\gamma \in H'$. But then since $\rho \in H'$ there is $f \in H'$ such that $f : \rho \to \xi$ is cofinal and thus also some $f' : \rho \to \xi$ which is surjective and $f' \in H'$. But this means that $\xi \in H'$. Contradiction! ⊣

2.1. Forcing over admissible premice. In this subsection, we will prove that if M is an admissible passive premouse with a largest cardinal δ which is Woodin in M, $\mathbb{B}$ is the extender algebra as defined inside M, and $g \subset \mathbb{B}$ is M-generic, then $M[g] \models \mathrm{KP}$. This will be Corollary 11 which is an instance of the more general Theorem 10.

Note that if M is an admissible passive premouse with a largest, regular, and uncountable cardinal δ and $\mathbb{P} \in M$ is a forcing poset, then we may assume without loss of generality that $\mathbb{P} \subset \delta$, as there is a surjection $f : \delta \to \mathbb{P}$ in M. We will do so throughout without further mention.

Lemma 5. *Let M be an admissible passive premouse with a largest, regular, and uncountable cardinal δ and let $\mathbb{P} \in M$ be a forcing poset such that $M \models$ "$\mathbb{P}$ has the δ-c.c." Then there is no $A \in \Sigma_1^M(M)$ such that $A \subset \mathbb{P}$ is an antichain in $\mathbb{P}$ which is unbounded in δ.*

Proof. Suppose there is $A \in \Sigma_1^M(M)$ is such that $A \subset \mathbb{P}$ is an antichain in $\mathbb{P}$ which is unbounded in δ. Since $\mathbb{P}$ has the δ-c.c. in M, and δ is regular in M, this means that $A \notin M$. Let φ be a Σ_1-formula such that for some $p \in M$,

$$x \in A \iff M \models \varphi(x, p).$$

Define for $\alpha < \mathrm{OR}^M \setminus \delta$ such that $p \in M|\alpha$,

$$A_\alpha := \{\xi \in \delta : M|\alpha \models \varphi(\xi, p)\}.$$

Note that $A = \bigcup_{\alpha < \mathrm{OR}^M} A_\alpha$ and $A_\alpha \in M$ for all α such that $\alpha \in \mathrm{OR}^M \setminus \delta$ and $p \in M|\alpha$. In particular, since $A_\alpha \subset A$ is an antichain, $\mathbb{P}$ has the δ-c.c. in M, and δ is regular in M, A_α is bounded in δ for all such α.

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Note that since $A = \bigcup_{\alpha<\mathrm{OR}^M} A_\alpha$ is by assumption unbounded in δ,

$$M \models \forall\beta < \delta\exists\alpha(\exists\gamma(\gamma > \beta \wedge \gamma \in A_\alpha)).$$

Since the last part of this formula is Σ_1, we have by Δ_0-collection that there is some $\alpha' \in M$ such that α' works for all $\beta < \delta$ uniformly, i.e.

$$M \models \exists\alpha'\forall\beta < \delta(\exists\gamma(\gamma > \beta \wedge \gamma \in A_{\alpha'})).$$

This contradicts the fact that every A_α is bounded in δ! ⊣

LEMMA 6. *Suppose that M is an admissible passive premouse with a largest, regular, and uncountable cardinal δ. Then δ is a $\Sigma_1^M(M)$-regular cardinal, i.e. for all $\eta < \delta$ and $f \in \Sigma_1^M(M)$ such that $f : \eta \to \delta$, $\mathrm{ran}(f)$ is bounded in δ, i.e., $\sup(\mathrm{ran}(f)) < \delta$.*

The proof of the Lemma is very similar to the proof of Lemma 5. Thus, we leave it as an exercise for the reader.

DEFINITION 7. Let M be an admissible passive premouse, $\mathbb{P} \in M$ be a forcing poset, and $g \subset \mathbb{P}$ be M-generic. Let $\xi < \mathrm{OR}^M$ such that $\mathbb{P} \in M|\xi + 1$. For $\beta \geq \omega \cdot \xi$ it is a standard fact that $\Vdash^{\mathbb{P}}_{M||\beta} \in M$, where $\Vdash^{\mathbb{P}}_{M||\beta}$ is the syntactical forcing relation with $M||\beta$ as a ground model using $\mathbb{P}$-names in $M||\beta$. Moreover, the function F is $\Delta_1^M(\{\mathbb{P}\})$, where $F(\beta) = \Vdash^{\mathbb{P}}_{M||\beta}$ for $\beta \geq \omega \cdot \xi$. We let $M[g]||\beta = (L_\beta[\mathbb{E}^M, g], \in, \mathbb{E}^M \restriction \beta, g)$ and have that $\lfloor M[g]||\beta \rfloor$ is the collection of the evaluations of the $\mathbb{P}$-names in $M||\beta$. We let $M||\beta[g] = M[g]||\beta$.[4]

Suppose that $g \subset \mathbb{P}$ is M-generic and that for a Σ_0-formula ψ, $M[g] \models \exists x\psi(x, z_1, ..., z_n)$, where $\{z_1, ..., z_n\} \subset M[g]$. Let $\varphi \equiv \exists x\psi$ be the corresponding Σ_1 formula and let $\dot{z}_1, ..., \dot{z}_n$ be $\mathbb{P}$-names for $z_1, ..., z_n$. We let

$$W^M_{\varphi,\dot{z}_1,...,\dot{z}_n} := \{p \in \mathbb{P} : \exists\beta < \mathrm{OR}^M\ \exists\dot{x} \in (M|\beta)^{\mathbb{P}}(p \Vdash^{\mathbb{P}}_{M|\beta} \psi(\dot{x}, \dot{z}_1, ..., \dot{z}_n))\}.$$

We call the set $W^M_{\varphi,\dot{z}_1,...,\dot{z}_n}$ the *set of conditions strongly forcing* φ (with parameters $\dot{z}_1, ..., \dot{z}_n$).[5]

REMARK. Since $M \models$ "Pairing" it follows easily (using $\mathbb{P}$-names) that $M[g] \models$ "Pairing". Therefore, we may arrange that *n* has any specific value which we want it to have. In particular, 1.

We will write $W_{\varphi,\dot{z}_1,...,\dot{z}_n}$ for $W^M_{\varphi,\dot{z}_1,...,\dot{z}_n}$ if it is clear from the context what M is. Since $F \in \Delta_1^M(\{\mathbb{P}\})$, it follows easily that $W_{\varphi,\dot{z}} \in \Sigma_1^M(\{\dot{z}, \varphi, \mathbb{P}\})$. Moreover, since $\mathbb{P} \in M$, there is $\beta < \mathrm{OR}^M$ such that $\mathbb{P} \in M|\beta$ so that $W_{\varphi,\dot{z}}$ is not trivially empty. However, in general we cannot assume that $W_{\varphi,\dot{z}} \in M$.

We also write $p \Vdash^{w.\mathbb{P}}_M \varphi(\dot{z})$ for $p \in W_{\varphi,\dot{z}}$.

For $\beta < \mathrm{OR}^M$ we might interpret the Σ_1 formula defining $W_{\varphi,\dot{z}}$ over the structure $M|\beta$. We will denote the set defined over $M|\beta$ in this way by $W^{M|\beta}_{\varphi,\dot{z}}$.

We have the following forcing theorem for Σ_1 statements for admissible premice.

[4]One could define this also for $M|\beta$ that is including the active extender. However, this kind of generality is not necessary for our context.

[5]The W stands for witness.

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THEOREM 8. *Suppose that M is an admissible passive premouse. Let $\varphi(y)$ be the formula $\exists x\psi(x,y)$, where ψ is a Σ_0 formula. Let $\mathbb{P}\in M$ be a forcing poset, $\dot{z}\in M^{\mathbb{P}}$, and suppose that $g\subset\mathbb{P}$ is M-generic. Then the following are equivalent*:

- $M[g]\models\varphi(\dot{z}^g)$,
- *there is* $p\in W_{\varphi,\dot{z}}\cap g$, *and*
- *there is* $p\in g$ *such that* $p\Vdash^{\mathbb{P}}_M\varphi(\dot{z})$.

REMARK. Here $p\Vdash^{\mathbb{P}}_M\varphi(\dot{z})$ refers to the classical notion of forcing an existential statement, i.e., for every $q\leq p$ there is some $r\leq q$ such that there is $\dot{x}\in M^{\mathbb{P}}$ such that $r\Vdash^{\mathbb{P}}_M\psi(\dot{x},\dot{z})$. In order to show that this is equivalent to the existence of some $p\in W_{\varphi,\dot{z}}\cap g$, we must use the admissibility of M. The rest of the proof is standard.

It is not in general the case that if N is admissible and $g\subset\mathbb{P}$ is N-generic for some forcing poset $\mathbb{P}\in N$, then $N[g]$ is admissible. See Proposition 13 and the remark preceding it for examples of admissible structures where the generic extension fails to be admissible. For $N[g]$ to be admissible, it is sufficient that g meets all dense open subsets of $\mathbb{P}$ that are unions of a $\Sigma_1^N(N)$ and a $\Pi_1^N(N)$ class over N. See [5] for more on this. With Theorem 10 we will give another criterion for ensuring that the forcing extension of an admissible premouse is a model of KP.

LEMMA 9. *Let M be an admissible passive premouse with a largest, regular, and uncountable cardinal δ. Let $\mathbb{P}\in M$ be a forcing such that $M\models$ "$\mathbb{P}$ has the δ-c.c.". Let $g\subset\mathbb{P}$ be M-generic. Let $\lambda<\delta$ and suppose that $M[g]\models\forall\alpha<\lambda\exists x\psi(x,\alpha,z)$, where ψ is a Σ_0-formula and $z\in M[g]$. Let $\varphi\equiv\exists x\psi$ and $\dot{z}\in M^{\mathbb{P}}$ be a name for z.*

Then there is $A\in M$ such that $A\subset\lambda\times\delta$ and if for $\alpha<\lambda$, $A_\alpha:=\{\beta:(\alpha,\beta)\in A\}$, then $A_\alpha\subset W_{\varphi,\check{\alpha},\dot{z}}$ is a maximal antichain in $W_{\varphi,\check{\alpha},\dot{z}}$, in particular, for every $p\in W_{\varphi,\check{\alpha},\dot{z}}$ there is some $q\in A_\alpha$ such that $q\,||\,p$.

PROOF. We construct the set A recursively along the ordinals of M. More specifically, we will define via a Σ_1-recursion a sequence $\langle(A^i,\beta_i):i<\mathrm{OR}^M\rangle$ with each $A^i\in M$ and $\beta_i\in\mathrm{OR}^M$, and $\langle(A^i,\beta_i):i<\lambda\rangle\in M$ for each $\lambda<\mathrm{OR}^M$. We will then set $A=\bigcup_{i<\mathrm{OR}^M}A^i$. Let β_0 be the least β such that

$$M|\beta\models\exists\alpha<\lambda\exists p\in\mathbb{P}(\exists\dot{x}\in M^{\mathbb{P}}(p\Vdash^{\mathbb{P}}_{M|\beta}\psi(\dot{x},\check{\alpha},\dot{z}))),$$

i.e., β is such that there is some $\alpha<\lambda$ such that $W^{M|\beta}_{\varphi,\check{\alpha},\dot{z}}\neq\emptyset$. Let A^0 be the set of all $(\alpha,p)\in A^0$ such that $\alpha<\lambda$, $W^{M|\beta_0}_{\varphi,\check{\alpha},\dot{z}}\neq\emptyset$, and p is the $<_M$-least element in $W^{M|\beta_0}_{\varphi,\check{\alpha},\dot{z}}$. Here, $<_M$ denotes the canonical Σ_1-definable well-order of M. Note that $A^0\in M$, since it is definable over $M|\beta_0$. Moreover, A^0 is a bounded subset of δ. (Actually, of δ^2, but via coding we may assume that $A^0\subset\delta$). Before we continue the construction let us introduce the following notation: For $\alpha<\lambda$ and $i<\mathrm{OR}^M$ let $A^i_\alpha:=\{q\in\mathbb{P}:(\alpha,q)\in A^i\}$, where A^i is to be defined.

Let $\gamma+1<\mathrm{OR}^M$ and suppose that $\langle(A^i,\beta_i):i\leq\gamma\rangle$ is defined such that $\langle(A^i,\beta_i):i\leq\gamma\rangle\in\Sigma_1^M(\{\lambda,\psi,\dot{z},\mathbb{P}\})$. We define $\beta_{\gamma+1}$ as the least β such that

$$M|\beta\models\exists\alpha<\lambda\exists p\in\mathbb{P}((\forall q\in A^\gamma_\alpha(p\perp q))\wedge p\in W_{\varphi,\check{\alpha},\dot{z}}),$$

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if there is such β. In other words β is such that for some α there is an element $p \in W^{M|\beta}_{\varphi,\check{\alpha},\dot{z}}$ incompatible with the elements of A^γ_α. In case $\beta_{\gamma+1}$ is undefined we stop the recursion. Note that in this case $A^\gamma \in M$.

In case $\beta_{\gamma+1}$ is defined, we set $A^{\gamma+1}$ to be the union of A^γ with the set of all (α, p) such that p is the $<_M$-least q such that

$$M|\beta_{\gamma+1} \models q \in W_{\varphi,\check{\alpha},\dot{z}} \wedge \forall r \in A^\gamma_\alpha (r \perp q),$$

if there is such q. Since $A^{\gamma+1}$ is definable over $M|\beta_{\gamma+1}$, $A^{\gamma+1} \in M$.

Now suppose that $\gamma < \mathrm{OR}^M$ is a limit ordinal and $\langle A^i : i < \gamma\rangle$ is defined. We set $\beta_\gamma = 0$ and $A^\gamma = \bigcup_{i<\gamma} A^i$. Note that since $\langle A_i : i < \gamma\rangle \in \Sigma^M_1(M)$ by KP, $A^\gamma \in M$.

We aim to see that the recursive definition of $\langle (A^i, \beta_i) : i < \mathrm{OR}^M\rangle$ stops before the ordinal height of OR^M, i.e. there is some $\gamma < \mathrm{OR}^M$ such that β_γ is not defined. Suppose that this is not the case. We claim that $\bar{A} := \bigcup_{i<\mathrm{OR}^M} A^i \in \Sigma^M_1(M)$ is an unbounded subset of δ. For suppose not. Then, since $\rho^M_1 = \delta$, $\bar{A} \in M$. But the definition of $\bar{A}$ gives a cofinal and total $f : \bar{A} \to \mathrm{OR}^M$ such that $f \in \Sigma^M_1(M)$. Contradiction!

In particular, the following holds in M,

$$\forall \beta < \delta \exists (\alpha, \gamma)(\alpha < \lambda \wedge \gamma \in (\beta, \delta) \wedge \exists \beta_i (\text{“}\gamma \text{ is added to } A_\alpha \text{ at stage } \beta_i\text{”})).$$

But by Δ_0-collection, there is some $\beta_i < \mathrm{OR}^M$ such that for all $\beta < \delta$, there is some $\gamma \in (\beta, \delta)$ added to A^i_α for some $\alpha < \lambda$. But this is inside $M|\beta_i + \omega$. Thus, as δ is a regular cardinal of M, there is some α to which unboundedly many $\gamma < \delta$ are added. Contradiction, since $\mathbb{P}$ has the δ-c.c. in M.

Let $\gamma < \mathrm{OR}^M$ be least such that β_γ is undefined. Note that γ is by definition a successor ordinal, i.e. $\gamma = \eta + 1$ for some $\eta < \mathrm{OR}^M$. Set $A := A^\eta$. By construction $A \in M$. For $\alpha < \lambda$, set $A_\alpha := \{p \in \mathbb{P} : (\alpha, p) \in A\}$. Since $\beta_{\eta+1}$ is undefined, we have that for all $\alpha < \lambda$, $A_\alpha \subset W_{\varphi,\check{\alpha},\dot{z}}$ is a maximal antichain in $W_{\varphi,\check{\alpha},\dot{z}}$. ⊣

THEOREM 10. *Let M be an admissible passive premouse with a largest, regular, and uncountable cardinal δ and $\mathbb{P} \in M$ be such that $M \models$ “$\mathbb{P}$ has the δ-c.c.” Then $M[g] \models KP$ for any M-generic $g \subset \mathbb{P}$.*

PROOF. We will verify that $M[g]$ satisfies Δ_0-collection and leave the remaining axioms of KP as an exercise.

Let $g \subset \mathbb{P}$ be M-generic and suppose for the sake of contradiction that $M[g] \not\models$ “Δ_0-collection”. Let ψ be a Σ_0-formula and $y \in M[g]$ such that they constitute a counterexample to Δ_0-collection, i.e.

$$M[g] \models \forall \alpha < \delta \exists x (\psi(\alpha, x, y)),$$

but there exists no $z \in M[g]$ such that

$$M[g] \models \forall \alpha < \delta \exists x \in z (\psi(\alpha, x, y)).$$

We distinguish two cases. First, suppose that there is some $\lambda < \delta$ such that there exists no $z \in M[g]$ such that

$$M[g] \models \forall \alpha < \lambda \exists x \in z (\psi(\alpha, x, y)).$$

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Then, by the previous Lemma 9 there is $A \in M$ such that for $\alpha < \lambda$, $A_\alpha := \{p \in \mathbb{P} : (\alpha, p) \in A\}$ is a maximal antichain in $W_{\varphi,\check{\alpha},\dot{y}}$, where $\varphi \equiv \exists x\psi$ and $\dot{y} \in M^{\mathbb{P}}$ is a $\mathbb{P}$-name for y.

Claim 1. *For $\alpha < \lambda$, $A_\alpha \cap g \neq \emptyset$.*

Proof. Fix some $\alpha < \lambda$. Note that since $M[g] \models \exists x\psi(\alpha, x, y)$, there is some $b \in M[g]$ such that $M[g] \models \psi(\alpha, b, y)$. By Theorem 8 we can fix some $p \in g \cap W_{\varphi,\check{\alpha},\dot{y}}$. We claim that this implies that $A_\alpha \cap g \neq \emptyset$. To this end let $D_\alpha := \{r \in \mathbb{P} : \forall q \in A_\alpha(r \perp q)\}$. Note that since $A_\alpha \in M$, $D_\alpha \in M$. Moreover, $A_\alpha \cup D_\alpha \in M$ is a pre-dense subset of $\mathbb{P}$ in M and thus, $g \cap (A_\alpha \cup D_\alpha) \neq \emptyset$ by the M-genericity of g. Suppose for the sake of contradiction that $g \cap D_\alpha \neq \emptyset$ and let $q \in g \cap D_\alpha$. Since $q, p \in g$, there is $r \in g$ such that $r \leq q, p$. However, this means that $r \in W_{\varphi,\check{\alpha},\dot{y}}$, since $r \leq p$. But r is incompatible with every element of A_α, so A_α is not a maximal antichain in $W_{\varphi,\check{\alpha},\dot{y}}$. Contradiction! Thus, $A_\alpha \cap g \neq \emptyset$. ⊣

We have established that for all $\alpha < \lambda$, $A_\alpha \cap g \neq \emptyset$. Moreover, by Theorem 8, we have that

$$M \models \forall a \in A \exists \dot{x} (\exists p \in \mathbb{P} \exists \alpha < \lambda (a = (\alpha, p) \wedge p \Vdash^{\mathbb{P}}_M \psi(\check{\alpha}, \dot{x}, \dot{y}))).$$

This is the antecedence of an instance of the Δ_0-collection scheme since the part in parentheses is Σ_1 (note that we are using here the fact that $\Vdash^{\mathbb{P}}_M$ for Σ_1 statements is Σ_1 definable over M which is true by Theorem 8). Thus, there is $z \in M$ such that

$$M \models \forall a \in A \exists \dot{x} \in z (\exists p \in \mathbb{P} \exists \alpha < \lambda (a = (\alpha, p) \wedge p \Vdash^{\mathbb{P}}_M \psi(\check{\alpha}, \dot{x}, \dot{y}))).$$

Let $z^g := \{\dot{x}^g : x \in z \cap M^{\mathbb{P}}\}$. Note that $z^g \in M[g]$, as $z \cap M^{\mathbb{P}} \in M \cap M^{\mathbb{P}}$. It follows that

$$M[g] \models \forall \alpha < \lambda \exists x \in z^g (\psi(\alpha, x, y)),$$

a contradiction!

Let us now turn towards the second case, i.e., we assume that for all $\lambda < \delta$ there is $z \in M[g]$ such that

$$M[g] \models \forall \alpha < \lambda \exists x \in z (\psi(\alpha, x, y)).$$

Let us associate to ψ a function $f \in \Sigma_1^{M[g]}(M[g])$ such that $f : \delta \to \mathrm{OR}^{M[g]}$ and for $\alpha < \delta$, $f(\alpha)$ is the least β such that there is $x \in M||\beta[g]$ such that $M[g] \models \psi(\alpha, x, y)$. By our assumption $f \upharpoonright \alpha \in M[g]$ for every $\alpha < \delta$, but $f \notin M[g]$. Let us define an auxiliary function F with domain δ such that $F(\alpha) = f \upharpoonright \alpha$. Note that $F \in \Sigma_1^{M[g]}(M[g])$. Let φ'_F be a Σ_1 formula and $z \in M[g]$ such that

$$(c, d) \in F \iff M[g] \models \varphi'_F(c, d, z),$$

and let $\varphi_F(a, b, z)$ be the formula

$$\exists c \exists d (\varphi'_F(c, d, z) \wedge a \leq c \wedge b = d \upharpoonright a). \tag{2.1}$$

Let

$$X := \{p \in \mathbb{P} : \exists \beta < \mathrm{OR}^M (p \Vdash^{\mathbb{P}}_{M|\beta} \forall \alpha < \check{\delta} \exists y \varphi_F(\alpha, y, \dot{z}))\}.$$

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Note that $X \in \Sigma_1^M(M)$ and let φ_X be the defining formula and $a \in M$ the corresponding parameter.

We now aim to construct in a similar way as in the proof of Lemma 9 a maximal antichain A in X. If $X \neq \emptyset$, let $A_0 := \{p\}$ for some $p \in X$. Otherwise, set $A_0 = \emptyset$. Suppose that $\langle A_i : i < \gamma \rangle$ is defined via a Σ_1-recursion for some $\gamma < \mathrm{OR}^M$. If γ is a limit ordinal, let $A_\gamma = \bigcup_{i<\gamma} A_i$. Then, $A_\gamma \in M$ and $\langle A_i : i \leq \gamma \rangle \in \Sigma_1^M(M)$. If γ is not a limit, let $\beta_\gamma < \mathrm{OR}^M$ be the least β such that

$$M|\beta \models \exists p \in \mathbb{P}(\varphi_X(p, a) \wedge \forall q \in A_{\gamma-1}(q \perp p)),$$

if there exists such β. In the case that there is no such β, stop the recursion. In case β_γ is defined, let p_γ be the $<_M$-least p such that

$$M|\beta_\gamma \models \varphi_X(p, a) \wedge \forall q \in A_{\gamma-1}(q \perp p).$$

Let $A_\gamma := A_{\gamma-1} \cup \{p_\gamma\}$. It is not hard to verify that $\langle A_i : i \leq \gamma \rangle \in \Sigma_1^M(M)$ and $A_\gamma \in M$.

Similar to the proof of the Lemma 9 we aim to see that there is a least $\gamma < \mathrm{OR}^M$ such that β_γ is undefined, which will show that $A := A_{\gamma-1}$ is a maximal antichain in X and $A \in M$.

Suppose for the sake of contradiction that for all $\gamma < \mathrm{OR}^M$, β_γ is defined. Let $\bar{A} := \bigcup_{\gamma < \mathrm{OR}^M} A_\gamma$. Clearly, $\bar{A} \in \Sigma_1^M(M)$. Since $\bar{A}$ is by construction an antichain, by Lemma 5 $\bar{A}$ is bounded in δ. In particular, $\bar{A} \in M$. However, as in the previous proof, the definition of $\bar{A}$ give rise to a function $f \in \Sigma_1^M(M)$ such that $\mathrm{dom}(f) = \bar{A}$ and f is cofinal in OR^M. But this is a contradiction!

Let $D := \{p \in \mathbb{P} : \forall q \in A(q \perp p)\}$. As $A \in M$, $D \in M$ and thus, $A \cup D \in M$. $A \cup M$ is pre-dense and therefore, $g \cap (A \cup D) \neq \emptyset$. Note that $g \cap A = \emptyset$, so $g \cap D \neq \emptyset$.

Let $\tilde{p} \in D \cap g$. Note that this means that there is no extension p of $\tilde{p}$ such that

$$p \Vdash_M^{w.\mathbb{P}} \exists \beta((M|\beta)[\dot{g}] \models \forall \alpha < \check{\delta} \exists y \varphi_F(\alpha, y, \dot{z})). \tag{2.2}$$

We claim that there is $\vec{p} = \langle p_i : i < \delta \rangle \in M$ such that for all $i < \delta$, $p_i \leq \tilde{p}$ and $M \models \psi(i, \dot{z}, p_i)$, where $\psi(i, \dot{z}, p)$ is the statement

$$p \Vdash_M^{w.\mathbb{P}} \exists y \varphi_F(i, y, \dot{z}). \tag{2.3}$$

We may find such $\langle p_i : i < \delta \rangle$ in M, as Δ_0-collection holds in M and

$$M \models \forall i < \delta \exists p((p \Vdash_M^{w.\mathbb{P}} \exists y \varphi_F(i, y, \dot{z})) \wedge p \leq \tilde{p}).$$

For every $i < \delta$ there exists such $p \in \mathbb{P}$, since $\tilde{p} \in g$ and $M[g] \models \forall i < \delta \exists y \varphi_F(i, y, z)$ and so by Theorem 8 the existence of p follows.

Claim 2. *There is $i_0 < \delta$ such that for all $i \in [i_0, \delta)$ and all $q \in \mathbb{P}$ with $q \leq p_i$ and for all $j \in (i, \delta)$, there is $k \in [j, \delta)$ such that $q \mid\mid p_k$.*

Proof. This follows routinely from the δ-c.c. of $\mathbb{P}$ in M (otherwise, working in M, construct by recursion on β an antichain $\langle q_i \rangle_{i<\delta}$). ⊣

Now since $p_{i_0} \leq \tilde{p}$, the following claim gives a contradiction, completing the proof:

CLAIM 3. *Replacing p with p_{i_0}, line (2.2) holds.*

PROOF. Let h be $(M, \mathbb{P})$-generic with $p_{i_0} \in h$. By Claim 2, for every $\beta \in (i_0, \delta)$, $h \cap \{p_j \mid j \in (i, \delta)\} \neq \emptyset$. So there are cofinally many $i < \delta$ such that $p_i \in h$. By KP in M, we can fix $\xi < \mathrm{OR}^M$ such that $M|\xi$ satisfies $\psi(i, \dot{z}, p_i)$ for all $i < \delta$. Therefore, $M||\xi[h]$ satisfies $\exists y \varphi_F(i, y, z)$ for cofinally many $i < \delta$. Note that for all $j_0 < j_1 < \delta$ and all $p \in \mathbb{P}$, if $M|\xi$ satisfies $\psi(j_1, \dot{z}, p)$ then $M|\xi$ satisfies $\psi(j_0, \dot{z}, p)$.[6] Thus, $M||\xi[h]$ satisfies $\exists y \varphi_F(i, y, z)$ for all $i < \delta$. ⊣

As stated prior the claim, this completes the proof. ⊣

From Theorem 10 we immediately get the following corollary.

COROLLARY 11. *Suppose that M is an admissible passive premouse with a largest, regular, and uncountable cardinal δ which is Woodin in M and let $\mathbb{B} \in M$ be the extender algebra with δ-many generators as defined inside M. Let $g \subset \mathbb{B}$ be M-generic. Then $M[g] \models KP$.*

If we replaced $M \models$ "$\mathbb{P}$ has the δ-c.c." in the statement of Theorem 10 with $M \models$ "$\mathbb{P}$ is $< \delta$-closed," Theorem 10 is false as Proposition 13 shows.

LEMMA 12. *Suppose that M, M_1, and M_2 are sound premice such that OR^M is a regular, uncountable cardinal (in V), $\rho_\omega^{M_1} = \rho_\omega^{M_2} = \mathrm{OR}^M$, $M \trianglelefteq M_1$, $M \trianglelefteq M_2$, and Condensation holds of M_1 and M_2. Then, either $M_1 \trianglelefteq M_2$ or $M_2 \trianglelefteq M_1$.*

The Lemma follows from the proof of Lemma 3.1 in [4].

Let $M = L_{\omega_1^{\mathrm{CK}}}$ and $\mathbb{P}$ be Cohen forcing. By [2], there are $(M, \mathbb{P})$-generics g such that $M[g] \not\models \mathrm{KP}$.[7] The following proposition establishes a variant of this fact.

PROPOSITION 13. *Let $M = (\lfloor M \rfloor, \in, \mathbb{E}^M, \emptyset)$ be a 1-sound admissible passive premouse with a largest, regular cardinal δ such that $\rho_1^M < \mathrm{OR}^M$ and suppose that Condensation holds in M. Let $\mathbb{C}_\delta := (\delta^{<\delta})^M$. Then there is an M-generic $g \subset \mathbb{C}_\delta$ such that $(\lfloor M[g] \rfloor, \in) \not\models KP$.*

PROOF. Note that $\mathbb{C}_\delta \in M$ is such that

$$M \models \text{“}\mathbb{C}_\delta \text{ is a } < \delta\text{-closed, separable, and atom-less forcing”}.$$

By Lemma 4 and 1-soundness, $\delta = \rho_1^M$ and $M = H_1^M(\delta \cup \{p_1^M\})$. Thus, there is a partial surjective function $h : \delta \to M$ such that $h \in \Sigma_1^M(\{p_1^M\})$. Let

$$\tilde{D} := \{\xi < \delta : M \models \text{“}h(\xi) \subset \mathbb{C}_\delta \text{ is dense in } \mathbb{C}_\delta\text{”}\}.$$

Note that $\tilde{D} \notin M$, as otherwise M could construct an M-generic. Let

$$D := \langle \xi_i : i < \delta \rangle,$$

be the monotone enumeration of $\tilde{D}$. Note that $\tilde{D}$ has ordertype δ. Since $h \in \Sigma_1^M(M)$, $\tilde{D} \in \Sigma_1^M(M)$. However, $D \notin \Sigma_1^M(M)$, as otherwise by Δ_0-collection, $D \in M$.

[6] This is why we formulated φ_F as in line (2.1).

[7] The authors thank Philipp Schlicht for pointing out this result to them.

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The idea is now to construct an M-generic g which codes in a Σ_1-fashion the set D so that $D \in M[g]$ if $(\lfloor M[g] \rfloor, \in) \models \mathrm{KP}$.

Let us define $\tilde{g} = \langle p_i : i < \delta \rangle$ via a recursion on i such that for all $\alpha < \delta$, $\langle p_i : i < \alpha \rangle$ is uniformly Σ_1 over M in the parameters $\{D \restriction \alpha, \mathbb{C}_\delta\}$. We will have that $p_i \leq p_j$ for $j < i$. Set $p_0 := \langle \xi_0 \rangle$ and p_1 to be the $<_M$-least $p \in h(\xi_0)$ such that $p \leq p_0$. Note that there is such p, since $h(\xi_0)$ is dense in $\mathbb{C}_\delta$.

Suppose that $\langle p_i : i < \lambda \rangle$ is defined, where $\lambda < \delta$ is a limit ordinal, such that $p_i \leq p_j$ for $j \leq i < \lambda$. Note that $D \restriction \lambda \in M$, since $\tilde{D} \in \Sigma_1^M(M)$ and δ is $\Sigma_1^M(M)$-regular by Lemma 6. By our induction hypothesis the construction so far is uniformly Σ_1 in the parameters $D \restriction \lambda$ and $\mathbb{C}_\delta$. This implies that $\langle p_i : i < \lambda \rangle \in M$. Thus, we can set $p_\lambda = (\bigcup_{i<\lambda} p_i)^\frown \langle \xi_\lambda \rangle$. Let $p_{\lambda+1}$ be the $<_M$-least $p \in h(\xi_\lambda)$ such that $p \leq p_\lambda$.

We now turn towards the successor case. Suppose we have defined $\langle p_\beta \rangle_{\beta \leq \gamma}$ where γ is an odd successor (that is, $\gamma = \lambda + 2n + 1$ for some $n < \omega$ and λ is a limit less than γ or equal to 0). Set $p_{\gamma+1} := p_\gamma^\frown \langle \xi_{\lambda+n} \rangle$ and let $p_{\gamma+2}$ be the $<_M$-least $p \in h(\xi_{\lambda+n})$ such that $p \leq p_{\gamma+1}$.

Let g be the upwards-closure of $\tilde{g}$ in $\mathbb{C}_\delta$. By definition of $\tilde{g}$, g is M-generic. Note that since $\mathbb{E}^M \restriction \delta \in M[g]$ by Lemma 12 we can define $\mathbb{E}^M$ over $M[g]$ in a Σ_1 fashion as the extender sequence of the stack of sound premice extending $M|\delta$ which project to δ and for which condensation holds. Thus, it follows that $(M, \in, \mathbb{E}^M) \in \Sigma_1^{(\lfloor M[g] \rfloor, \in)}(\{M|\delta\})$. Note that therefore $\tilde{g}$ is definable over $(M[g], \in)$ from $\{\mathbb{C}_\delta, \delta, g, M|\delta\}$ via a Σ_1 recursion of length δ. Thus, if $(\lfloor M[g] \rfloor, \in) \models \mathrm{KP}$, then $\tilde{g} \in M[g]$ and so $D \in M[g]$.

Let $h' : \delta \to \mathrm{OR}^M$ be the partial function such that $h'(\alpha)$ is the least $\beta < \mathrm{OR}^M$ such that $h(\alpha) \in M|\beta$, if $h(\alpha)$ is defined. Clearly, $h' \in \Sigma_1^M(\{p_1^M\})$. We have that $h' \in \Sigma_1^{(\lfloor M[g] \rfloor, \in)}(\lfloor M[g] \rfloor)$. Moreover, as $\mathbb{C}_\delta$ is $< \delta$-closed and atom-less, $\mathrm{ran}(h' \restriction D)$ is cofinal in M. Since $\mathrm{OR}^M = \mathrm{OR}^{M[g]}$, $h' \restriction D$ is cofinal in $M[g]$. But now if $(\lfloor M[g] \rfloor, \in) \models \mathrm{KP}$, since $D \in M[g]$, $\mathrm{OR}^{M[g]} \in M[g]$. Contradiction! Thus, $(\lfloor M[g] \rfloor, \in) \not\models \mathrm{KP}$. ⊣

2.2. Preservation of admissibility between iterates of premice. Next we deal with the preservation of admissibility between iterates of premice. Parts of this will be used in the proof of Lemma 25.

LEMMA 14. *Suppose that M and N are premice and let $i : M \to N$ be $r\Sigma_3$-elementary. Then $M \models KP$ iff $N \models KP$.*

The lemma follows directly from the fact that the part of KP excluding the induction axioms has an $r\Pi_3$ axiomatization and is therefore preserved under $r\Sigma_3$-elementary embeddings between premice.

Thus, by Lemma 14, if $\mathrm{Ult}_n(M, E)$ is wellfounded, $\mathrm{Ult}_n(M, E) \models \mathrm{KP}$ if $n \geq 2$ and M is an admissible passive premouse.

LEMMA 15. *Let $k \leq \omega$ and let N be a k-sound premouse. Let $\mathcal{T}$ be a k-maximal iteration tree on N such that $\mathrm{lh}(\mathcal{T}) = \theta + 1$. Let $b := [0, \theta]^\mathcal{T}$ be the main branch of $\mathcal{T}$ and α be least such that $\alpha + 1 \in b$ and $(\alpha + 1, \theta]^\mathcal{T}$ does not drop in model. Let $\eta = \alpha + 1$. Then $\mathcal{M}_\eta^{*\mathcal{T}}$ is an admissible passive premouse with a largest, regular, and uncountable cardinal if and only if $\mathcal{M}_\theta^\mathcal{T}$ is an admissible passive premouse with a largest, regular, and uncountable cardinal.*

Proof. We prove the two directions separately.

Claim 1. *Suppose $\mathcal{M}_\eta^{*\mathcal{T}}$ is an admissible passive premouse with a largest, regular, and uncountable cardinal. Then so is $\mathcal{M}_\theta^{\mathcal{T}}$.*

Proof. Note that in the case that $\deg^{\mathcal{T}}(\theta) \geq 2$, the lemma holds by Lemma 14. Thus, we may assume that $\deg^{\mathcal{T}}(\theta) \leq 1$.

By standard facts $\mathcal{M}_\theta^{\mathcal{T}}$ is a passive premouse with a largest, regular, and uncountable cardinal. We have to verify that $\mathcal{M}_\theta^{\mathcal{T}}$ is admissible. By Lemma 3, it suffices to show that Δ_0-collection holds in $\mathcal{M}_\theta^{\mathcal{T}}$.

Case 1. $\theta = \zeta + 1$ *is a successor ordinal.*

We inductively assume that $\mathcal{M}_\theta^{*\mathcal{T}}$ is an admissible passive premouse with a largest, regular, and uncountable cardinal.

Let $n := \deg^{\mathcal{T}}(\theta)$, $M := \mathcal{M}_\theta^{*\mathcal{T}}$, $E := E_\zeta^{\mathcal{T}}$, and $M' := \mathrm{Ult}_n(\mathcal{M}_\theta^{*\mathcal{T}}, E)$. By our initial remarks $n \leq 1$. By standard arguments $\kappa := \mathrm{crit}(E)$ is strictly less than the largest cardinal of M. In particular, by Lemma 4, $\kappa < \rho_1^M$.

Since M is admissible we have by Δ_0-collection that for any $m < \omega$, $f \in \Sigma_1^M(M) \cap {}^{[\kappa]^m}M$ implies that $f \in M$. This means that the canonical factor map $\pi\colon \mathrm{Ult}_0(M, E) \to \mathrm{Ult}_1(M, E)$ is the identity. Thus, $\mathrm{Ult}_0(M, E) = \mathrm{Ult}_1(M, E)$ and the associated ultrapower maps are the same. In particular, $i_\theta^{*\mathcal{T}}\colon M \to M'$ is a 1-embedding and $i_\theta^{*\mathcal{T}}$ is cofinal in $\mathrm{OR}^{M'}$.

Subclaim 1. *For $a \in [\mathrm{lh}(E)]^{<\omega}$, $\mathrm{Ult}_0(M, E_a) \models KP$.*

Proof. Let $a \in [\mathrm{lh}(E)]^{<\omega}$ and suppose for contradiction that $\mathrm{Ult}_0(M, E_a) \not\models \mathrm{KP}$. By our initial remarks this means that Δ_0-collection fails in $\mathrm{Ult}_0(M, E_a)$, i.e., there is a Σ_0-formula φ, $\xi < \mathrm{OR}^{\mathrm{Ult}_0(M,E_a)}$, and $p \in \mathrm{Ult}_0(M, E_a)$ such that

$$\mathrm{Ult}_0(M, E_a) \models \forall\alpha < \xi \exists x \varphi(\alpha, x, p), \tag{2.4}$$

but there is no $z \in \mathrm{Ult}_0(M, E_a)$ such that

$$\mathrm{Ult}_0(M, E_a) \models \forall\alpha < \xi \exists x \in z \varphi(\alpha, x, p). \tag{2.5}$$

Let $i\colon M \to \mathrm{Ult}_0(M, E_a)$ be the ultrapower embedding. By our previous remarks we have that i is a 1-embedding and cofinal in $\mathrm{OR}^{\mathrm{Ult}_0(M,E_a)}$.

Let δ be the largest cardinal of M and $\delta' = i(\delta)$ be the largest cardinal of $\mathrm{Ult}_0(M, E_a)$. By Lemma 3 we may assume that ξ in (2.4) is equal to δ'. Let $j = |a|$. Let $g'\colon \delta' \to \mathrm{Ult}_0(M, E_a)$ be the canonical function derived from φ by taking the $<_{\mathrm{Ult}_0(M,E_a)}$-least witness. Note that $g' \in \Sigma_1^{\mathrm{Ult}_0(M,E_a)}(\{p\})$ as φ is Σ_0. We will now show that g' is bounded in $\mathrm{Ult}_0(M, E_a)$. This will contradict the failure of Δ_0-collection, completing the proof.

By the definition of $\mathrm{Ult}_0(M, E_a)$ we have for $\alpha < \delta'$ some $f_\alpha \in {}^{[\kappa]^j}\delta \cap M$ such that $\alpha = [a, f_\alpha]^M_{E_a}$. Also, there is $f_p \in ({}^{[\kappa]^j}M) \cap M$ such that $p = [a, f_p]^M_E$. Since we assumed that δ is regular in M and $\kappa < \delta$ we may assume that for $\alpha < \delta'$, $f_\alpha \in {}^{[\kappa]^j}\delta \cap (M|\delta)$. Moreover, since Łoś's Theorem holds for Σ_1-formulae, we have by (2.4) for all $\alpha < \delta'$,

$$A_\alpha := \{b \in [\kappa]^j : M \models \exists x \varphi(f_\alpha(b), x, f_p(b))\} \in E_a.$$

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Note that for any $h \in {}^{[\kappa]^j}\delta \cap (M|\delta)$ there is some $\alpha < \delta'$ such that $\alpha = [a, h]^M_E$. Thus,

$$M \models \forall h \in {}^{[\kappa]^j}\delta \cap (M|\delta) \exists A (A \in E_a \wedge \forall b \in A \exists x \varphi(h(b), x, f_p(b))).$$

Note that this does make sense since $E_a \in M$: By standard facts about k-maximal iteration trees E is close to M and therefore in particular, for every $b \in [\text{lh}(E)]^{<\omega}$, $E_b \in \Sigma_1^M(M)$. But since M is admissible and $\kappa < \delta$, this implies that $E_a \in M$.

Using Δ_0-collection, we have for every $h \in {}^{[\kappa]^j}M \cap (M|\delta)$ and A such that

$$M \models A \in E_a \wedge \forall b \in A \exists x \varphi(h(b), x, f_p(b)),$$

there is some $Y \in M$ such that

$$M \models A \in E_a \wedge \forall b \in A \exists x \in Y \varphi(h(b), x, f_p(b)).$$

Thus,

$$M \models \forall h \in {}^{[\kappa]^j}\delta \cap (M|\delta) \exists A \exists Y (A \in E_a \wedge \forall b \in A \exists x \in Y \varphi(h(b), x, f_p(b))).$$

By another application of Δ_0-collection this gives us $\beta < \text{OR}^M$ such that

$$M \models \forall h \in {}^{[\kappa]^j}\delta \cap (M|\delta) \exists A \in E_a (\forall b \in A \exists x \in (M|\beta) \varphi(h(b), x, f_p(b))).$$

It now easily follows that g' is bounded by $M'|i(\beta)$. $\dashv$

In the case that E is finitely generated this argument shows that $M' \models \text{KP}$. Thus, we may assume that E is not finitely generated. In this case, M' is the direct limit of

$$(\text{Ult}_0(M, E_a), \pi_{ab} : a, b \in [\nu(E)]^{<\omega} \wedge a \subset b),$$

where π_{ab} is the canonical factor embedding. For $a \in [\nu(E)]^{<\omega}$ let $X_a := \pi_{a\infty}[\text{Ult}_0(M, E_a)]$. By Łoś's Theorem it follows that $\pi_{a\infty} : \text{Ult}_0(M, E_a) \to M'$ is Σ_1-elementary.

Suppose now that $f' \in \Sigma_1^{M'}(M')$. There is some $a \in [\nu(E)]^{<\omega}$ such that $f' \in \Sigma_1^{M'}(\{p'\})$ for some $p' \in X_a$. Let $p = \pi_{a\infty}^{-1}(p')$ and let f be the function, which is defined over $\text{Ult}_0(M, E_a)$ via the parameter p, as f' is defined over M' via the parameter p'. By the claim we have that $f \in \text{Ult}_0(M, E_a)$. But this means that $f' \in M'$. Thus, by Lemma 3, M' is admissible.

Case 2. $\text{lh}(\mathcal{T}) = \theta$ *is a limit ordinal.*

This is proven much as in the case that $\theta = \zeta + 1$ is a successor ordinal and E_ζ is not finitely generated.

This finishes the proof of Claim 1. $\dashv$

Claim 2. *Suppose $\mathcal{M}^{\mathcal{T}}_\theta$ is an admissible passive premouse with a largest, regular, and uncountable cardinal. Then so is $\mathcal{M}^{*\mathcal{T}}_\eta$.*

Proof. Again we argue by induction on θ. Suppose first that $\theta = \zeta + 1$ and let $\mathcal{M}^{\mathcal{T}}_\theta = M'$ and $M = \mathcal{M}^{*\mathcal{T}}_\theta$. Again, by standard arguments we may assume that M is a premouse with a largest, regular, and uncountable cardinal δ. It suffices to see that M is admissible for which in turn it suffices to see that Δ_0-collection holds in M.

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Suppose for the sake of contradiction that this is not true and let $\eta < \mathrm{OR}^M$ witness the least failure of Δ_0-collection, i.e.,

1. there is a Σ_1-formula φ and $p \in M$ such that $M \models \forall\alpha < \eta\exists x\varphi(x, \alpha, p)$, but there is no $z \in M$ such that $M \models \forall\alpha < \eta\exists x \in z\varphi(x, \alpha, p)$, and
2. for all Σ_1-formulas ψ and $q \in M$ we have that if $\gamma < \eta$ and $M \models \forall\alpha < \gamma\exists x\psi(x, q)$, then there is $z \in M$ such that $M \models \forall\alpha < \gamma\exists x \in z\psi(x, q)$.

Let $n = \deg^{\mathcal{T}}_\theta$ so that $M' = \mathrm{Ult}_n(M, E)$, where $E = E^{\mathcal{T}}_\zeta$. As before we may assume that $n \leq 1$. Let $i := i^{*\mathcal{T}}_\theta : M \to M'$ be the ultrapower map and let $\delta' = i(\delta)$ be the largest cardinal of M'. Note that we can no longer assume that i is both a 0- and a 1-embedding, unlike in Claim 1.

CASE 1. $n = 1$.

Note that it suffices to see that $M' \models \forall\alpha < i(\eta)\exists x\varphi(x, \alpha, i(p))$ because then by the admissibility of M', $M' \models \exists z'\forall\alpha < i(\eta)\exists x \in z'\varphi(x, \alpha, i(p))$ and so by the elementarity of i there is a bound in M. Note that i is a 1-embedding and therefore $r\Sigma_2$-elementary. But this means that $M' \models \forall\alpha < i(\eta)\exists x\varphi(x, \alpha, i(p))$.

CASE 2. $n = 0$.

Let $\eta' := \sup(i[\eta])$ and note that η' is a limit ordinal. We claim that for every $\alpha < \eta'$, $M' \models \exists x\varphi(x, \alpha, i(p))$. Suppose otherwise and let $\alpha' < \eta'$ be a counterexample, i.e.

$$M' \models \forall x\neg\varphi(x, \alpha', i(p)).$$

Let $\alpha < \eta$ be such that $i(\alpha) > \alpha'$. Note that since η is the minimal failure of Δ_0-collection in M,

$$M \models \exists z\forall\beta < \alpha\exists x \in z\varphi(x, \beta, p).$$

But this is a Σ_1-statement, so that

$$M' \models \exists z\forall\beta < i(\alpha)\exists x \in z\varphi(x, \beta, i(p)).$$

But then in particular,

$$M' \models \exists x\varphi(x, \alpha', i(p)).$$

Contradiction! Therefore, by Δ_0-collection in M', there is $\beta' < \mathrm{OR}^{M'}$ such that $M' \models \forall\alpha < \eta'\exists x \in (M'|\beta')\varphi(x, \alpha, i(p))$. Since i is cofinal, we may assume without loss of generality that $\beta' \in \mathrm{ran}(i)$. Let $\beta < \mathrm{OR}^M$ be such that $i(\beta) = \beta'$. We claim that

$$M \models \forall\alpha < \eta\exists x \in (M|\beta)\varphi(x, \alpha, p),$$

which would be a contradiction! So suppose that there is $\alpha < \eta$ such that

$$M \models \forall x \in (M|\beta)\neg\varphi(x, \alpha, p).$$

This is a Δ_0 statement, so that

$$M' \models \forall x \in (M'|\beta')\neg\varphi(x, \alpha, i(p)).$$

Contradiction!

Now let us suppose that θ is a limit ordinal. In the case that there is some $\gamma \in b \cap \theta$ such that for all $\xi \in (\gamma, \theta) \cap b$, $\deg_\xi^{\mathcal{T}} \geq 1$, we can argue as in Case 1 of the successor case. If otherwise we can use the argument from Case 2 of the successor case, since $i_{\gamma\theta}^{\mathcal{T}}$ will be cofinal. ⊣

This completes the proof of the lemma. ⊣

If we would not require δ to be regular in the statement of Lemma 15, the lemma would be provably false as the following example shows.

Example 16. Let M be a 1-sound premouse such that $M \models \mathrm{KP}$. Suppose that M has a largest cardinal $\delta > \omega$ such that for some $\kappa < \delta$, $\mathrm{cof}^M(\delta) = \kappa$ and there is an M-total $E \in \mathbb{E}^M$ such that $\mathrm{crit}(E) = \kappa$. Let $\pi : M \to \mathrm{Ult}_1(M, E)$ be the ultrapower embedding.

Then π is discontinuous at δ so that $\pi(\delta) > \sup(\pi[\delta])$. However, since π is a 1-embedding, $\rho_1^M = \sup(\pi[\delta])$. But $\pi(\delta)$ is the largest cardinal of $\mathrm{Ult}_1(M, E)$. So by Lemma 4, $\mathrm{Ult}_1(M, E) \not\models \mathrm{KP}$.

In this subsection we have established the following (cf. [11, Chapter 7.2]).

Theorem 17. *Let $k \leq \omega$ and suppose that M is a k-sound, $(k, |M|^+ + 1)$-iterable admissible passive premouse with a largest, regular, and uncountable cardinal δ. Suppose that δ is Woodin in M and let $X \subset |M|$. Then there is a k-maximal iteration tree $\mathcal{T}$ on M with last model $\mathcal{M}_\infty^{\mathcal{T}}$, which does not drop in model anywhere, such that X is generic over $\mathcal{M}_\infty^{\mathcal{T}}$ for the extender algebra of $\mathcal{M}_\infty^{\mathcal{T}}$ and $\mathcal{M}_\infty^{\mathcal{T}}[X] \models$ KP.*

2.3. A version of the truncation Lemma. For the proof of Lemma 24, we need a version of the Truncation Lemma for premice. Recall the following coarse definition. If M is a possibly ill-founded structure in some signature extending $\mathcal{L}_{\dot\in}$, we call

$$\mathrm{wfp}(M) := \{x \in \lfloor M \rfloor \mid \in^M \restriction (\mathrm{trc}_{\in^M}(\{x\}))^2 \text{ is wellfounded}\}$$

the wellfounded part of M. By [1] and Problem 5.27 of [8], if $M \models \mathrm{KP}$, then $\mathrm{wfp}(M) \models \mathrm{KP}$. We aim to show something similar in the case that M is an ill-founded premouse.

Definition 18. Let $M = (\lfloor M \rfloor, \in^M, \mathbb{E}^M)$ be an $\mathcal{L}_{\dot\in,\dot E}$-structure. We say that $M \models$ "$V = L[E]$", if M models the Axiom of Extensionality, the Axiom of Foundation, and

$$\forall x \exists \alpha (x \in S_\alpha[\dot E]) \wedge \forall \alpha \exists x (x \notin S_\alpha[\dot E]), \tag{2.6}$$

where $S_\alpha[\dot E]$ is the refinement of the $J[\dot E]$-hierarchy as described in Chapter 5 of [8].

Note that the second conjunct of 2.6 makes sure that the model is an actual instance of its internal J-hierarchy.

Definition 19. Let $M = (\lfloor M \rfloor, \in^M, \mathbb{E}^M)$ be an $\mathcal{L}_{\in,\dot E}$-structure such that $M \models$ "$V = L[E]$" and $\mathrm{wfp}(M)$ is transitive. Letting $\mathrm{wfo}(M) = \mathrm{OR} \cap \mathrm{wfp}(M)$, we call

$$\mathrm{wfc}(M) := (S^{\mathbb{E}^M}_{\mathrm{wfo}(M)}, \in, \mathbb{E}^M \restriction \mathrm{wfo}(M))$$

the *wellfounded cut of M*.

The next lemma is the version of the Truncation Lemma we need for the proof of Lemma 24. The proof is essentially identical to the proof of Proposition 2.4 in [3].

LEMMA 20. *Let $M = (\lfloor M \rfloor, \in^M, \mathbb{E}^M)$ be an ill-founded $\mathcal{L}_{\in,\dot{E}}$-structure such that $M \models$ "$V = L[E]$" and wfp(M) is transitive. Then wfc$(M) \models KP$.*

PROOF. We may assume that $\omega \in \mathrm{wfc}(M)$, as otherwise $\mathrm{wfc}(M) = L_\omega$ which is clearly admissible. It suffices to see that $\mathrm{wfc}(M) \models \Delta_0$-Collection. By induction, it easily follows that for $\alpha < \mathrm{wfo}(M)$, $S_\alpha^{\mathbb{E}^M} = (S_\alpha)^M \in \mathrm{wfp}(M)$, so that $\mathrm{wfc}(M) \subseteq \mathrm{wfp}(M)$. Let φ be a Σ_0 formula and $a, p \in \mathrm{wfc}(M)$ such that

$$\mathrm{wfc}(M) \models \forall x \in a \exists y \varphi(x, y, p).$$

Note that by Σ_1 upwards absoluteness this holds in M. Let γ be a non-standard ordinal of M. In S_γ^M, we may define a function F with $\mathrm{dom}(F) = a$ such that for $x \in a$,

$$F(x) = \eta \iff S_\gamma^M \models x \in a \wedge \exists y \in S_{\eta+1}\varphi(x, y, p) \wedge \forall y \in S_\eta \neg\varphi(x, y, p).$$

Since $\mathrm{wfc}(M) \subseteq S_\gamma^M$, it follows by Σ_1 upwards absoluteness that $F(x) < \mathrm{wfo}(M)$. However, this means that $\eta := \bigcup_{x \in a} F(x) \subset \mathrm{wfo}(M)$. Since F is definable over M, we must have that $\eta < \mathrm{wfo}(M)$. This means that

$$\mathrm{wfc}(M) \models \forall x \in a \exists y \in S_\eta \varphi(x, y, p).$$

Thus, $\mathrm{wfc}(M) \models \mathrm{KP}$. ⊣

§3. Where the conjecture holds.

In this section, we will show under the assumption that there is no transitive model of KP with a Woodin cardinal that Conjecture 1 holds. This will be a consequence of Theorem 26. Lemma 24 is the key insight for proving Theorem 26. First, let us recall some well-known basic properties of weasels and their coiterations.

The following theorem from [10] guarantees that the coiteration of two $\Omega + 1$-iterable weasels of height Ω is successful.

THEOREM 21. *Let κ be an inaccessible cardinal. Let $\mathcal{M}$ and $\mathcal{N}$ be premice such that $\mathrm{OR}^{\mathcal{M}} = \mathrm{OR}^{\mathcal{N}} = \kappa$. Let $(\mathcal{T}, \mathcal{U})$ be a successful coiteration of $(\mathcal{M}, \mathcal{N})$. Then $\max\{\mathrm{lh}(\mathcal{T}), \mathrm{lh}(\mathcal{U})\} \leq \kappa + 1$. Moreover, setting $\mathrm{lh}(\mathcal{T}) = \theta + 1$ and $\mathrm{lh}(\mathcal{U}) = \gamma + 1$, either*

1. *$D^{\mathcal{T}} \cap [0, \theta]^{\mathcal{T}} = \emptyset$, $i_{0,\theta}^{\mathcal{T}}[\kappa] \subset \kappa$, $\mathcal{M}_\theta^{\mathcal{T}} \trianglelefteq \mathcal{M}_\gamma^{\mathcal{U}}$, and $\mathrm{OR}^{\mathcal{M}_\theta^{\mathcal{T}}} = \kappa$, or*
2. *$D^{\mathcal{U}} \cap [0, \gamma]^{\mathcal{U}} = \emptyset$, $i_{0,\gamma}^{\mathcal{U}}[\kappa] \subset \kappa$, $\mathcal{M}_\gamma^{\mathcal{U}} \trianglelefteq \mathcal{M}_\theta^{\mathcal{T}}$, and $\mathrm{OR}^{\mathcal{M}_\gamma^{\mathcal{U}}} = \kappa$.*

The next lemma collects basic facts about iteration trees whose proof is well-known.

LEMMA 22. *Let κ be a regular and uncountable cardinal. Let $\mathcal{M}$ be a premouse such that $\mathrm{OR}^{\mathcal{M}} = \kappa$ and let $\mathcal{T}$ be a 0-maximal iteration tree on $\mathcal{M}$ such that $\mathrm{lh}(\mathcal{T}) = \kappa + 1$. Let $b := [0, \kappa]^{\mathcal{T}}$. Then the following hold:*

1. *if $b \cap D^{\mathcal{T}} = \emptyset$ and $i_{0\kappa}^{\mathcal{T}}[\kappa] \subset \kappa$, then there is a club $C_1 \subset b$ such that for all $\alpha \in C_1 \cap b$, $i_{0\alpha}^{\mathcal{T}}[\alpha] \subset \alpha$,*

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2. *if* $\mathrm{OR}^{\mathcal{M}_\alpha^{\mathcal{T}}} \leq \kappa$ *for all* $\alpha < \kappa$, *then there is a club* $C_2 \subset b$ *such that for all* $\alpha \in C_2$, $\alpha = \sup\{\mathrm{lh}(E_\beta^{\mathcal{T}}) : \beta < \alpha\}$, *and*
3. *if* $\mathrm{OR}^{\mathcal{M}_\alpha^{\mathcal{T}}} \leq \kappa$ *for all* $\alpha < \kappa$, *and* $\mathrm{OR}^{\mathcal{M}_\kappa^{\mathcal{T}}} > \kappa$, *then there is a club* $C_3 \subset b$ *such that for all* $\alpha \in C_3$, $\alpha > \sup(D^{\mathcal{U}} \cap b)$, $i_{\alpha\kappa}^{\mathcal{T}}(\alpha) = \kappa$ *and* $\mathrm{crit}(i_{\alpha\kappa}^{\mathcal{T}}) = \alpha$.

We need the following version of the Σ_1^1-Bounding Theorem in order to prove Lemma 24.

THEOREM 23. *Let* $x \in {}^\omega\omega$ *and suppose that* $A \subset WO$ *is* $\Sigma_1^1(x)$*-definable, where* $WO \subset {}^\omega\omega$ *is the set of reals coding well orderings of* ω. *Then*

$$\sup\{\mathrm{otp}(R_y) : y \in A\} < \omega_1^{\mathrm{CK}}(x),$$

where $R_y \subset \omega \times \omega$ *is the relation coded by* y *and* $\omega_1^{\mathrm{CK}}(x)$ *is the least ordinal* α *such that* $L_\alpha[x] \models KP$.[8]

The proof of Theorem 23 is basically a refinement of the Kunen–Martin Theorem, which may be found in [6, p. 75].

LEMMA 24. *Let* M *be a premouse such that* δ *is a regular, uncountable cardinal of* M *and* $M \models$ "δ^+ *exists*". *Suppose that* M *is* $(0, \delta+1)$*-iterable, and there is no initial segment of* M *which models* $KP +$ *"there is a Woodin cardinal". Let* $\mathcal{U}$ *be a* 0*-maximal iteration tree on* M *such that*

- $\mathrm{lh}(\mathcal{U}) = \delta + 1$,
- $\sup\{\mathrm{lh}(E_\alpha^{\mathcal{U}}) : \alpha < \delta\} = \delta$,
- $b \cap D^U = \emptyset$, *where* $b := [0, \delta]^{\mathcal{U}}$, *i.e.,* $\mathcal{U}$ *is non-dropping on its main branch, and*
- $i_{0\delta}^{\mathcal{U}}(\delta) = \delta$.

Then $i_{0\delta}^{\mathcal{U}}(\delta^{+M}) = \delta^{+M}$.

PROOF. Suppose otherwise, and let $\mathcal{U}$ be a counterexample. Let $\xi < \delta^{+M}$ be such that $i_{0\delta}^{\mathcal{U}}(\xi) > \delta^{+M}$ and $\rho_\omega^{M|\xi} = \delta$. Note that such ξ exists since $i_{0\delta}^{\mathcal{U}}$ is continuous at δ^{+M}, as δ^{+M} and its images are not of measurable cofinality in any $\mathcal{M}_\alpha^{\mathcal{U}}$ for $\alpha \in [0, \delta)^{\mathcal{U}}$ and $\mathcal{U}$ is 0-maximal. Moreover, it is a standard fact that the ordinals η such that $\rho_\omega^{M|\eta} = \delta$ are cofinal in δ^{+M}. We also assume that $\xi > \mathrm{OR}^Q$, where $Q \lhd M$ is least such that $J(Q) \models$ "δ is not Woodin". Note that Q exists and $\mathrm{OR}^Q < \delta^{+M}$, since there is no initial segment of M which models KP and has δ as a Woodin.

Let $g \subset \mathrm{Col}(\omega, \delta)$ be V-generic. In particular, g is also M-generic. Let $x \in M[g] \cap {}^\omega\omega$ code $M|\xi + \omega$.

Let us define in $V[g]$ the set $\bar{A}$ such that $\mathcal{T} \in \bar{A}$ if and only if

1. $\mathcal{T}$ is a 0-maximal putative iteration tree on $M|\xi + \omega$ such that $\mathrm{lh}(\mathcal{T}) = \delta + 1$
2. $\mathcal{T}$ is non-dropping on its main branch,
3. $\sup\{\mathrm{lh}(E_\alpha^{\mathcal{T}}) : \alpha < \delta\} = \delta$, and
4. $i_{0\delta}^{\mathcal{T}}(\delta) = \delta$, thus in particular, $\mathcal{M}_\delta^{\mathcal{T}}|\delta \in \mathrm{wfp}(\mathcal{M}_\delta^{\mathcal{T}})$.

Note that if $\mathcal{T} \in \bar{A}$, then $\mathcal{T} \upharpoonright \delta$ is guided by Q-structures.

CLAIM 1. *If* $\mathcal{T} \in \bar{A}$, *then there is a Q-structure for* $\mathcal{T} \upharpoonright \delta$ *in* $V[g]$, *i.e., there is* $\alpha < \mathrm{OR}$ *such that* $J_\alpha(\mathcal{M}(\mathcal{T} \upharpoonright \delta)) \models$ "$\delta = \delta(\mathcal{T} \upharpoonright \delta)$ *is not Woodin*".

[8] Here $\mathrm{otp}(R_y)$ is the transitive collapse of the wellorder R_y.

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PROOF. Suppose that there is no Q-structure in $V[g]$. This means that for all $\alpha < \mathrm{OR}$, $J_\alpha(\mathcal{M}(\mathcal{T}\restriction\delta)) \models$ "δ is Woodin". Let α be least such that $J_\alpha(\mathcal{M}(\mathcal{T}\restriction\delta)) \models \mathrm{KP}$. Since δ is regular in $L(\mathcal{M}(\mathcal{T}\restriction\delta))$, there is $X \prec J_\alpha(\mathcal{M}(\mathcal{T}\restriction\delta))$ such that $X \cap \delta \in \delta$. Let N be the transitive collapse of X; then by condensation, $N \lhd \mathcal{M}(\mathcal{T}\restriction\delta)$. Note that since $\mathcal{M}(\mathcal{T}\restriction\delta) = \bigcup_{\alpha\in[0,\delta)^\mathcal{T}} \mathcal{M}_\alpha^\mathcal{T}|\mathrm{lh}(E_\alpha^\mathcal{T})$ and by elementarity there is no initial segment of $\mathcal{M}(\mathcal{T}\restriction\delta)$ which models KP and has a Woodin. However, N models KP and has a Woodin. Contradiction! ⊣

Since for $\mathcal{T} \in \bar{A}$ there is a Q-structure for $\mathcal{T}\restriction\delta$ and $\sup\{\mathrm{lh}(E_\alpha^\mathcal{T}) : \alpha < \delta\} = \delta$ (and thus Q-structures for $\mathcal{T}\restriction\alpha$ with $\alpha < \delta$ are of ordinal height less than δ) it follows by the usual absoluteness arguments that there is a unique cofinal well-founded branch b for $\mathcal{T}\restriction\delta$ in $V[g]$.

However, at this point we do not know whether for $\mathcal{T} \in \bar{A}$, the unique cofinal wellfounded branch of $\mathcal{T}\restriction\delta$ was actually chosen, i.e. whether $\mathcal{T}$ is an iteration tree. This is verified in the next claim.

CLAIM 2. *If $\mathcal{T} \in \bar{A}$, then $\mathcal{M}_\delta^\mathcal{T}$ is wellfounded.*

PROOF. Suppose otherwise. Let $\mathcal{T} \in \bar{A}$ be such that $\mathcal{M}_\delta^\mathcal{T}$ is ill-founded. By 4 we have that $\mathcal{M}_\delta^\mathcal{T}|\delta \in \mathrm{wfp}(\mathcal{M}_\delta^\mathcal{T})$. As we are assuming that $\mathcal{M}_\delta^\mathcal{T}$ is ill-founded, we may apply Lemma 20 and get that $\mathrm{wfc}(\mathcal{M}_\delta^\mathcal{T}) \models \mathrm{KP}$. To arrive at a contradiction, we distinguish two cases.

The first case is that $Q(c, \mathcal{T}\restriction\delta) \lhd \mathrm{wfc}(\mathcal{M}_\delta^\mathcal{T})$, where $c = [0,\delta]^\mathcal{T}$. Let b be the unique, cofinal, and wellfounded branch b for $\mathcal{T}\restriction\delta$ in $V[g]$. By 1-smallness, we have that $Q(c, \mathcal{T}\restriction\delta) = Q(b, \mathcal{T}\restriction\delta)$. Note that $Q(b, \mathcal{T}\restriction\delta) \neq \mathcal{M}_b^\mathcal{T}$, since otherwise there is a drop on b and thus $Q(c, \mathcal{T}\restriction\delta)$ is not sound, but $Q(c, \mathcal{T}\restriction\delta) \lhd \mathrm{wfc}(\mathcal{M}_\delta^\mathcal{T})$, which would be a contradiction! But then by the Zipper Lemma, we have that δ is Woodin in $J(Q(b, \mathcal{T}\restriction\delta))$. Contradiction!

The second case is that the Q-structure is not an element in $\mathrm{wfc}(\mathcal{M}_\delta^\mathcal{T})$. In this case we have that for every proper initial segment of $\mathrm{wfc}(\mathcal{M}_\delta^\mathcal{T})$, $\mathcal{M}_\delta^\mathcal{T}$ thinks that δ is Woodin in that segment. Moreover, by 1-smallness we have that $\mathrm{wfc}(\mathcal{M}_\delta^\mathcal{T}) = J_\beta(\mathcal{M}(\mathcal{T}\restriction\delta))$ for some β. But this means that δ is Woodin in $\mathrm{wfc}(\mathcal{M}_\delta^\mathcal{T})$.

Let b be unique cofinal and wellfounded branch b for $\mathcal{T}\restriction\delta$ in $V[g]$. By 1-smallness, $Q(b, \mathcal{T}\restriction\delta) = J_\alpha(\mathcal{M}(\mathcal{T}\restriction\delta))$ for some $\alpha \in \mathrm{OR}$. Since $J_\beta(\mathcal{M}(\mathcal{T}\restriction\delta)) \models$ "δ is Woodin", we must have that $\beta \leq \alpha$ and $J_\alpha(\mathcal{M}(\mathcal{T}\restriction\delta)) \trianglelefteq \mathcal{M}_b^\mathcal{T}$. But since $J_\beta(\mathcal{M}(\mathcal{T}\restriction\delta))$ satisfies KP + "$\delta(\mathcal{T})$ is Woodin" and M has no segment modelling this theory, we must have $\beta = \alpha = \mathrm{OR}(\mathcal{M}_b^\mathcal{T})$; that is, $J_\beta(\mathcal{M}(\mathcal{T}\restriction\delta)) = \mathcal{Q}(\mathcal{T}\restriction\delta, b) = \mathcal{M}_b^\mathcal{T}$. Since $M|(\xi+\omega)$ satisfies "there is no Woodin cardinal $\leq \delta$", b must drop. But since $\mathcal{M}_b^\mathcal{T} \models$ KP + "there is a Woodin", by Lemma 15, the last drop in model along b is to a segment modelling this same theory, contradicting the fact that M has no segment modelling this theory. ⊣

The set $\bar{A}$ gives rise to the set $A = \{\mathrm{OR}^{\mathcal{M}_\infty^\mathcal{T}} : \mathcal{T} \in \bar{A}\}$. Note that A is a set of countable ordinals in $V[g]$, since the trees are countable in $V[g]$ and also $M|\xi+\omega$ is countable in $V[g]$. Thus, we may code A as a set of reals A^* by setting $A^* := \{y \in \mathrm{WO} : \mathrm{otp}(R_y) \in A\}$.

CLAIM 3. *A^* is $\Sigma_1^1(x)$.*

Proof. The main difficulty is in expressing in a $\Sigma^1_1(x)$-fashion that for $\alpha < \delta$, $\mathcal{M}^{\mathcal{T}}_\alpha$ is wellfounded. Fix some $\alpha < \delta$. Note that if α is a successor, then $\mathcal{M}^{\mathcal{T}}_\alpha$ is always wellfounded, since M is $(0, \delta + 1)$-iterable in $V[g]$ as can be seen by standard arguments, and we may assume inductively that $\mathcal{U} \upharpoonright \alpha$ is guided by Q-structures. Suppose that $\alpha < \delta$ is a limit. Since $\sup\{\mathrm{lh}(E^{\mathcal{T}}_\alpha) : \alpha < \delta\} = \delta$, we have that $\delta(\mathcal{T} \upharpoonright \alpha) < \delta$. Note that $\mathrm{lh}(E^{\mathcal{T}}_\alpha) > \delta(\mathcal{T} \upharpoonright \alpha)$ and $\mathcal{M}^{\mathcal{T}}_\alpha$ is 1-small below δ. Thus, $Q(b, \mathcal{T} \upharpoonright \alpha)$, where $b = [0, \alpha]^{\mathcal{T}}$, must exist and $Q(b, \mathcal{T} \upharpoonright \alpha) \lhd \mathcal{M}^{\mathcal{T}}_\alpha | \mathrm{lh}(E^{\mathcal{T}}_\alpha)$. Thus, the wellfoundedness of $Q(b, \mathcal{T} \upharpoonright \alpha)$ can be expressed via the existence of an isomorphism between the ordinals of $Q(b, \mathcal{T} \upharpoonright \alpha)$ and an initial segment of the ordinals of the model coded by x. This is $\Sigma^1_1(x)$. ⊣

Claim 4. $\sup(A) < \omega_1^{\mathrm{CK}}(x)$.

Proof. This follows immediately from Theorem 23 and the previous claim. ⊣

Let $\mathcal{U}^*$ be the 0-maximal iteration tree on $M|(\xi + \omega)$ which is otherwise equivalent to $\mathcal{U}$, i.e. it has the same length and tree order, and uses the same extenders as $\mathcal{U}$. Since $\sup\{\mathrm{lh}(E^{\mathcal{U}}_\alpha) : \alpha < \delta\} = \delta$ and δ is a cardinal of M, this makes sense, $\sup\{\mathrm{lh}(E^{\mathcal{U}^*}_\alpha) : \alpha < \delta\} = \delta$, $i^{\mathcal{U}^*}_{0\delta}$ exists, and $i^{\mathcal{U}^*}_{0\delta}(\delta) = \delta$. It follows that $\mathcal{U}^*$ is in $\bar{A}$.

The following claim is a simpler version of Lemma 4.64 of [7].

Claim 5. $i^{\mathcal{U}}_{0\delta}(\xi) = i^{\mathcal{U}^*}_{0\delta}(\xi)$.

Proof. Let $\kappa < \delta$. It suffices to see that every function $f : \kappa \to \xi$ which is in M is also in $M|\xi + \omega$. To this end note that since $\rho^{M|\xi}_\omega = \delta$, there is a surjection $g : \delta \to \xi$ such that $g \in M|\xi + \omega = J(M|\xi)$. Define a function $h : \kappa \to \delta$ such that for $\alpha < \kappa$, $h(\alpha) = \min(g^{-1}(f(\alpha)))$. As δ is regular and $\kappa < \delta$, h is essentially a bounded subset of δ and thus, $h \in M|\xi$. But since f is definable from g and h this means that $f \in M|\xi + \omega$. ⊣

By Claims 4, 5, and the fact that $\mathcal{U}^* \in \bar{A}$, $i^{\mathcal{U}^*}_{0\delta}(\xi) + \omega = i^{\mathcal{U}}_{0\delta}(\xi) + \omega = \mathrm{OR}(\mathcal{M}^{\mathcal{U}^*}_\delta) < \omega_1^{\mathrm{CK}(x)}$. But x is in $M[g]$, so $\omega_1^{\mathrm{CK}(x)} < \omega_1^{M[g]} = \delta^{+M[g]}$. But we chose ξ with $\delta^{+M[g]} < i^{\mathcal{U}}_{0\delta}(\xi)$, a contradiction! ⊣

We need a second lemma with a similar flavor.

Lemma 25. *Let M be a premouse such that δ is a regular, uncountable cardinal of M and $M \models$ "δ^+ exists". Suppose that M is $(0, \delta + 1)$-iterable, and there is no initial segment of M which models KP + "there is a Woodin cardinal". Let $\mathcal{U}$ be a 0-maximal iteration tree on M such that $\mathrm{lh}(\mathcal{U}) = \delta + 1$ and $\sup\{\mathrm{lh}(E^{\mathcal{U}}_\alpha) : \alpha < \delta\} = \delta$. Let $b^{\mathcal{U}} := [0, \delta]^{\mathcal{U}}$. Suppose that there is $\eta \in b^{\mathcal{U}} \cap (\delta \setminus \sup(D^{\mathcal{U}} \cap b^{\mathcal{U}}))$ such that $i^{\mathcal{U}}_{\eta\delta}(\gamma) = \delta$ for some $\gamma < \delta$. Then $\delta^{+\mathcal{M}^{\mathcal{U}}_\delta} < \delta^{+M}$.*

Proof. The proof follows closely the proof of Lemma 24. Again we work in $V[g]$, where $g \subset \mathrm{Col}(\omega, \delta)$ is V-generic. We modify the definition of $\bar{A}$ by omitting 2 and modifying 4 to "there exists an $\eta \in b^{\mathcal{T}} \cap (\delta \setminus \sup(D^{\mathcal{T}} \cap b^{\mathcal{T}}))$ and a $\gamma < \delta$ such that $i^{\mathcal{T}}_{\eta\delta}(\gamma) = \delta$, where $b^{\mathcal{T}} := [0, \delta]^{\mathcal{T}}$". Moreover, we alter 1 by requiring that $\mathcal{T}$ is a putative iteration tree on $M|\delta$.

As before, for all $\mathcal{T} \in \bar{A}$, $\mathcal{T}$ is an iteration tree. Similar to before $\mathcal{U}^*$ will be the tree $\mathcal{U}$ considered on $M|\delta$. Again, $\mathcal{U}^* \in \bar{A}$ and the tree- and dropping-structure of $\mathcal{U}$ and $\mathcal{U}^*$ are the same. We have that $\mathcal{M}^{\mathcal{U}^*}_\alpha \trianglelefteq \mathcal{M}^{\mathcal{U}}_\alpha$ for all $\alpha \leq \delta$.

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By the same argument as in the proof of Lemma 24, we will have that $\mathrm{OR}^{\mathcal{M}_\delta^{\mathcal{U}^*}} < \omega_1^{\mathrm{CK}}(x) < \delta^{+M}$ where x codes $M|\delta$. We claim that this implies that $\delta^{+\mathcal{M}_\delta^{\mathcal{U}}} < \delta^{+M}$. If $D^{\mathcal{U}} \cap b^{\mathcal{U}} \neq \emptyset$, i.e. $\mathcal{U}$ is dropping on its main branch, $\mathcal{M}_\delta^{\mathcal{U}} = \mathcal{M}_\delta^{\mathcal{U}^*}$, so that $\delta^{+\mathcal{M}_\delta^{\mathcal{U}}} \leq \mathrm{OR}^{\mathcal{M}_\delta^{\mathcal{U}}} < \delta^{+M}$. So suppose that $D^{\mathcal{U}} \cap b^{\mathcal{U}} = \emptyset$, equivalently that $D^{\mathcal{U}^*} \cap b^{\mathcal{U}} = \emptyset$. Then $i^{\mathcal{U}} : M \to \mathcal{M}_\delta^{\mathcal{U}}$ and $i^{\mathcal{U}^*} : M|\delta \to \mathcal{M}_\delta^{\mathcal{U}^*}$ exist. Let $\eta < \delta$ be least such that there is $\bar{\delta} < \delta$ such that $i_{\eta\delta}^{\mathcal{U}^*}(\bar{\delta}) = \delta$. If $\eta = 0$, then since δ is regular in M and $\sup\{\mathrm{lh}(E_\alpha^{\mathcal{U}}) : \alpha < \delta\} = \delta$, $\delta^{+\mathcal{M}_\delta^{\mathcal{U}}} = i^{\mathcal{U}}(\bar{\delta}^{+M}) \leq i^{\mathcal{U}}(\delta) = \sup i^{\mathcal{U}^*}[\delta] < \omega_1^{\mathrm{CK}}(x) < \delta^{+M}$. So suppose that $\eta > 0$. Note that $\bar{\delta}^{+\mathcal{M}_\eta^{\mathcal{U}^*}}$ must exist, since otherwise $\bar{\delta}$ is the largest cardinal of $\mathcal{M}_\eta^{\mathcal{U}^*}$ which would imply that $\bar{\delta} \in \mathrm{ran}(i_{0\eta}^{\mathcal{U}^*})$ and so $\eta = 0$, contradicting our assumption on η. Moreover, since there is no dropping on the main branch, $i_{0\eta}^{\mathcal{U}^*}$ is cofinal in $\mathrm{OR}^{\mathcal{M}_\eta^{\mathcal{U}^*}}$. But then, $\delta^{+\mathcal{M}_\delta^{\mathcal{U}}} = i_{\eta\delta}^{\mathcal{U}}(\bar{\delta}^{+\mathcal{M}_\eta^{\mathcal{U}}}) = i_{\eta\delta}^{\mathcal{U}^*}(\bar{\delta}^{+\mathcal{M}_\eta^{\mathcal{U}^*}}) \leq i_{\eta\delta}^{\mathcal{U}^*}(i_{0\eta}^{\mathcal{U}^*}(\gamma)) = i^{\mathcal{U}^*}(\gamma) < \omega_1^{\mathrm{CK}}(x)$ for some $\gamma < \delta$ where it is not the case that $\delta = \gamma^{+\mathcal{M}}$ and $i_{0,\eta}^{\mathcal{U}^*}(\gamma) = \bar{\delta}$, since $\eta > 0$. ⊣

We are now ready to prove the main theorem of this section.

THEOREM 26. *Let Ω be a measurable cardinal. Let $\mathcal{W}$ and $\mathcal{R}$ be premice such that $\mathrm{OR}^{\mathcal{W}} = \mathrm{OR}^{\mathcal{R}} = \Omega$ and suppose that both are $(0, \Omega + 1)$-iterable. Suppose that neither $\mathcal{W}$ nor $\mathcal{R}$ have an initial segment which models KP $\wedge\, \exists\delta$("δ is Woodin"). Then the following are equivalent:*

1. *$\mathcal{W} \leq^* \mathcal{R}$,*
2. *there is a club $C \subset \Omega$ such that for all $\alpha \in C$, if α is regular, then $(\alpha^+)^{\mathcal{W}} \leq (\alpha^+)^{\mathcal{R}}$, and*
3. *there is a stationary set $S \subset \Omega$ such that for all $\alpha \in S$, α is regular and $(\alpha^+)^{\mathcal{W}} \leq (\alpha^+)^{\mathcal{R}}$.*

PROOF. Let us first show that clause 1 implies clause 2, i.e. suppose that $\mathcal{W} \leq^* \mathcal{R}$. We aim to show that there is a club $C \subset \Omega$ such that for all $\alpha \in C$, if α is regular, then $(\alpha^+)^{\mathcal{W}} \leq (\alpha^+)^{\mathcal{R}}$. Let $(\mathcal{T}, \mathcal{U})$ be the coiteration of $(\mathcal{W}, \mathcal{R})$. By Theorem 21, the coiteration is successful. Since $\mathcal{W} \leq^* \mathcal{R}$, $\mathcal{M}_\Omega^{\mathcal{T}} \trianglelefteq \mathcal{M}_\Omega^{\mathcal{U}}$ and $\mathcal{T}$ does not drop on its main branch. In particular, $i^{\mathcal{T}} : \mathcal{W} \to \mathcal{M}_\Omega^{\mathcal{T}}$ exists. We may assume by padding the trees if necessary that $\mathrm{lh}(\mathcal{T}) = \mathrm{lh}(\mathcal{U}) = \Omega + 1$. Note that by Theorem 21, $\mathrm{OR}^{\mathcal{M}_\Omega^{\mathcal{T}}} = \Omega$. Thus, by Lemma 22 (1), there is a club $C_{\mathcal{T}} \subset \Omega$ such that for all $\alpha \in C_{\mathcal{T}}$, $i_{0\alpha}^{\mathcal{T}}[\alpha] \subset \alpha$ and $b^{\mathcal{T}} \supset C_{\mathcal{T}}$. Let $\tilde{C}_{\mathcal{T}}$ be the club given by Lemma 22 (2) for $\mathcal{T}$. Set $B_{\mathcal{T}} := C_{\mathcal{T}} \cap \tilde{C}_{\mathcal{T}}$. Let $\alpha \in B_{\mathcal{T}}$ and suppose that it is regular. Then $i_{0\alpha}^{\mathcal{T}}(\alpha) = \alpha$. Thus, by Lemma 24, we have that $\alpha^{+\mathcal{W}} = \alpha^{+\mathcal{M}_\alpha^{\mathcal{T}}}$. But since $\alpha \in \tilde{C}_{\mathcal{T}}$, $\mathrm{crit}(i_{\alpha\Omega}^{\mathcal{T}}) \geq \alpha$. Therefore, $\alpha^{+\mathcal{M}_\alpha^{\mathcal{T}}} = \alpha^{+\mathcal{M}_\Omega^{\mathcal{T}}}$. Since $\mathcal{M}_\Omega^{\mathcal{T}} \trianglelefteq \mathcal{M}_\Omega^{\mathcal{U}}$ and Ω is a cardinal, we have that $\alpha^{+\mathcal{M}_\Omega^{\mathcal{T}}} = \alpha^{+\mathcal{M}_\Omega^{\mathcal{U}}}$ for all $\alpha \in B_{\mathcal{T}}$.

CASE 1. $\mathrm{OR}^{\mathcal{M}_\Omega^{\mathcal{U}}} > \Omega$.

Let $C_{\mathcal{U}} \subset \Omega$ be the club given by Lemma 22 (3) for $\mathcal{U}$ intersected with the club given by Lemma 22 (2) for $\mathcal{U}$. Then for all $\alpha, \beta \in C_{\mathcal{U}}$ such that $\alpha < \beta$, we have that $i_{\alpha\beta}^{\mathcal{U}}(\alpha) = \beta$. Suppose that $\alpha \in C'_{\mathcal{U}}$ is regular, where $C'_{\mathcal{U}}$ is the club of limit points of $C_{\mathcal{U}}$. We have by Lemma 25, that $\alpha^{+\mathcal{M}_\alpha^{\mathcal{U}}} < \alpha^{+\mathcal{R}}$. Moreover, since $\mathrm{crit}(i_{\alpha\Omega}^{\mathcal{U}}) = \alpha$ (note

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we are beyond the drops of $b^{\mathcal{U}}$), $\alpha^{+\mathcal{M}^{\mathcal{U}}_\alpha} = \alpha^{+\mathcal{M}^{\mathcal{U}}_\Omega}$. Thus, $C := C'_{\mathcal{U}} \cap B_{\mathcal{T}}$ witnesses clause 2 of the theorem in Case 1.

Case 2. $\mathrm{OR}^{\mathcal{M}^{\mathcal{U}}_\Omega} = \Omega$.

Then $i^{\mathcal{U}}_{0\Omega}$ exists. In this case we can construct a club $C_{\mathcal{U}}$ for $\mathcal{U}$ as $C_{\mathcal{T}}$ was constructed for $\mathcal{T}$ and set $C := C_{\mathcal{T}} \cap C_{\mathcal{U}}$. This C witnesses clause 2 of the theorem in Case 2.

The argument for showing that clause 2 implies clause 3 is standard. We now show that clause 3 implies clause 1. Suppose that there is a stationary set of regular cardinals $S \subset \Omega$ such that for all $\alpha \in S$, $\alpha^{+\mathcal{W}} \leq \alpha^{+\mathcal{R}}$. We aim to show that $\mathcal{W} \leq^* \mathcal{R}$. Let $(\mathcal{T}, \mathcal{U})$ be the coiteration of $(\mathcal{W}, \mathcal{R})$ which again by Theorem 21 is successful. Suppose for the sake of contradiction that $\mathcal{M}^{\mathcal{T}}_\infty \rhd \mathcal{M}^{\mathcal{U}}_\infty$. By the same arguments as before we can construct a club $D \subset \Omega$ such that for all regular $\alpha \in D$, $\alpha^{+\mathcal{W}} > \alpha^{+\mathcal{R}}$. But since S is stationary and consists of regular cardinals, $S \cap D \neq \emptyset$, contradiction! ⊣

Corollary 27. *Let Ω be a measurable cardinal. Let $\mathcal{W}$ and $\mathcal{R}$ be premice such that $\mathrm{OR}^{\mathcal{W}} = \mathrm{OR}^{\mathcal{R}} = \Omega$ and suppose that both are $(0, \Omega+1)$-iterable. Suppose that there is no transitive model of KP $\wedge\, \exists\delta$(“δ is Woodin”). Then the following are equivalent*:

- $\mathcal{W} \leq^* \mathcal{R}$,
- *there is a club $C \subset \Omega$ such that for all $\alpha \in C$, if α is regular, then $(\alpha^+)^{\mathcal{W}} \leq (\alpha^+)^{\mathcal{R}}$, and*
- *there is a stationary set $S \subset \Omega$ such that for all $\alpha \in S$, $\alpha \in S$ is regular and $(\alpha^+)^{\mathcal{W}} \leq (\alpha^+)^{\mathcal{R}}$.*

§4. The counterexample. In this section we construct a counterexample to Conjecture 1 assuming large cardinals. In order to construct the counterexample we need an iterable premouse M which has an initial segment $N \lhd M$ such that $N \models$ KP, N has a largest cardinal κ which is Woodin in N, κ is a measurable cardinal in the larger premouse M, and there exists some $\Omega \in (\kappa^{+M}, \mathrm{OR}^M)$ which is measurable in M.

We want to show that such N exists if we assume that M is the least iterable sound premouse, which models KP and has a largest cardinal δ which is $\underset{\sim}{\Sigma}_1$-Woodin in M.[9]

In Section 4.1 we will construct the above N. Note that since M is the least mouse with a $\underset{\sim}{\Sigma}_1$-Woodin, we will have that κ, the largest cardinal of N, is not $\underset{\sim}{\Sigma}_1$-Woodin in N. For the construction of the counterexample it will be important that if g codes a wellorder and is generic over N for the extender algebra, then the ordertype of g is less than OR^N. This will work since we showed in Section 2.1 that $N[g] \models$ KP if $g \subset \mathbb{P} \in N$ is N-generic, $N \models$ KP, and $N \models$ “$\mathbb{P}$ has the η-c.c.”, where η is the largest cardinal of N. In Section 4.2 we will construct the counterexample.

4.1. The construction of N.

Definition 28. Let M be a passive premouse and let $\delta < \mathrm{OR} \cap M$. We say that δ is a $\underset{\sim}{\Sigma}_1$-*Woodin cardinal (of M)* if for all $A \in \Sigma^M_1(M)$ there is some $\kappa < \delta$ which is $< \delta$, A-reflecting in M.

[9]See Definition 28 for the definition of $\underset{\sim}{\Sigma}_1$-Woodin cardinal.

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Definition 29. Let $M^{\mathrm{ad}}_{\utilde{\Sigma}_1}$ be the least $(0, \omega_1 + 1)$-iterable and sound premouse which models KP $\wedge \exists\delta$("δ is $\utilde{\Sigma}_1$-Woodin").

Let $M = M^{\mathrm{ad}}_{\utilde{\Sigma}_1}$. Note that M has a unique Woodin cardinal δ. Moreover, δ is acutally $\utilde{\Sigma}_1$-Woodin in M and is the largest cardinal of M. Furthermore, $\delta = \rho_1^M$, $p_1^M = \{\delta\}$, and $\rho_2^M = \omega$. Also, by the remarks in the proof of Lemma 15, M is actually $(1, \omega_1 + 1)$-iterable.

Definition 30. Let M be an admissible passive premouse such that $\rho_1^M < \mathrm{OR}^M$. We let $T^M \subset \rho_1^M$ be the set of ordinals coding $\mathrm{Th}^M_{\Sigma_1}(\rho_1^M \cup \{p_1^M\})$.

Note that by Lemma 4, ρ_1^M is the largest cardinal of M. Moreover, by definition $T^M \notin M$, but $T^M \in \Sigma_1^M(M)$. However, for every $\alpha < \rho_1^M$, $T^M \cap \alpha \in M$. In the proof of Lemma 31 below, we will consider degree 1 ultrapower embeddings $i_E : M \to U = \mathrm{Ult}_1(M, E)$, where U is wellfounded. Note that in this case, $T^U = i_E(T^M)$, where $i_E(T^M)$ denotes $\bigcup_{\alpha < \rho_1^M} i_E(T^M \cap \alpha)$.

Lemma 31. *Let δ be the largest cardinal of $M^{\mathrm{ad}}_{\utilde{\Sigma}_1}$. In $M^{\mathrm{ad}}_{\utilde{\Sigma}_1}$, let κ be $(<\delta, T^{M^{\mathrm{ad}}_{\utilde{\Sigma}_1}})$-reflecting and let $H := Hull_1^M(\kappa \cup \{\delta\})$. Then*

- *$H \cap \delta = \kappa$, and*
- *if $\pi : N \cong H \prec_{\Sigma_1} M^{\mathrm{ad}}_{\utilde{\Sigma}_1}$ is the transitive collapse of H, then $N \models KP$.*

Proof. Let $M = M^{\mathrm{ad}}_{\utilde{\Sigma}_1}$. Recall that $p_1^M = \{\delta\}$ and $\rho_1^M = \delta$. Let us first proof that $H \cap \delta = \kappa$. Suppose for the sake of contradiction that there is $\xi \in H \cap [\kappa, \delta)$. Let φ_ξ be a Σ_1 formula and $\alpha \in \kappa$ be such that ξ is the unique η such that $M \models \varphi_\xi(\eta, \alpha, p_1^M)$. Note that $\varphi_\xi(\xi, \alpha, p_1^M) \in \mathrm{Th}_1^M(\delta \cup \{p_1^M\})$. Thus, there is some $\lambda < \delta$ such that the code for $\varphi_\xi(\xi, \alpha, p_1^M)$ is below λ, i.e. $\varphi_\xi(\xi, \alpha, \dot{p}) \in T^M \cap \lambda$. Let $E \in \mathbb{E}^M$ be an extender witnessing that κ is (λ, T^M)-strong. Let $U := \mathrm{Ult}_1(M, E)$ and let $i_E : M \to U$ be the canonical ultrapower embedding. Note that we are taking a 1-ultrapower, which makes sense since $\kappa = \mathrm{crit}(E) < \delta = \rho_1^M$. This means that $i_E(T^M) = T^U$ (see the remarks following Definition 30). But since E is (λ, T^M)-strong, we have $T^U \cap \lambda = i_E(T^M) \cap \lambda = T^M \cap \lambda$. Thus, $\varphi_\xi(\xi, \alpha, \dot{p}) \in T^U \cap \lambda$, which means that $U \models \varphi_\xi(\xi, \alpha, p_1^U)$. Moreover, since $M \models \varphi_\xi(\xi, \alpha, p_1^M)$, $U \models \varphi_\xi(i_E(\xi), i_E(\alpha), i_E(p_1^M))$. But $i_E(\alpha) = \alpha < \kappa$ and $i_E(p_1^M) = p_1^U$,[10] so $U \models \varphi_\xi(i_E(\xi), \alpha, p_1^U)$. But since ξ was the unique witness for $\exists x \varphi_\xi(x, \alpha, p_1^M)$ in M, we have

$$M \models \forall x(x \neq \xi \to \neg\varphi_\xi(x, \alpha, p_1^M)).$$

However, this is $r\Pi_1$ and thus,

$$U \models \forall x(x \neq i_E(\xi) \to \neg\varphi_\xi(x, \alpha, p_1^U)).$$

This means $i_E(\xi) = \xi$. Contradiction!

[10]We actually have that $i_E(p_1^M) = p_1^U = \{\delta\} = p_1^M$.

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Let us now verify that $N \models \mathrm{KP}$. By what we have shown so far $\pi(\kappa) = \delta$. In particular, κ is the largest cardinal of N. So by Lemma 3 it suffices to check Δ_0-collection. Suppose that

$$N \models \forall \alpha < \kappa \exists y \varphi(\alpha, y, \bar{p}),$$

where φ is a Σ_1 formula and $\bar{p} \in N$. Note that since $H := \mathrm{Hull}_1^M(\kappa \cup \{\delta\})$, we may replace $\bar{p}$ with some tuple of ordinals $\vec{\alpha}_p \in [\kappa]^{<\omega}$ and $\{\kappa\}$. Letting $p = \pi(\bar{p})$, we have that $p = \vec{\alpha}_p \cup \{\{\delta\}\}$. Then

$$M \models \forall \alpha < \kappa \exists y \varphi(\alpha, y, \vec{\alpha}_p, \{\delta\}).$$

We aim to see that the bound κ can be replaced by δ here, giving that

$$M \models \forall \alpha < \delta \exists y \varphi(\alpha, y, \vec{\alpha}_p, \{\delta\}).$$

Suppose for the sake of contradiction that there is some $\xi \in [\kappa, \delta)$ such that $M \models \forall y \neg \varphi(\xi, y, \vec{\alpha}_p, \{\delta\})$. This means that $\exists y \varphi(\xi, y, \vec{\alpha}_p, \dot{p}) \notin T^M$. But then there is some $\lambda \in (\xi, \delta)$ such that this is witnessed by $T^M \cap \lambda$, i.e. the formula $\exists y \varphi(\xi, y, \vec{\alpha}_p, \dot{p}) \notin T^M \cap \lambda$ even though $\xi, \vec{\alpha}_p \in M|\lambda$. (Note that since we coded T^M as a theory with constant symbol $\dot{p}$ it suffices if we pick $\lambda > \sup\{\xi, \vec{\alpha}_p\}$.) Let $E \in \mathbb{E}^M$ be (λ, T^M)-strong. Then as in the previous argument, $\exists y \varphi(\xi, y, \vec{\alpha}_p, \dot{p}) \notin T^U \cap \lambda$. However, since i_E is $r\Sigma_2$-elementary, we have that

$$U \models \forall \alpha < i_E(\kappa) \exists y \varphi(\alpha, y, i_E(\vec{\alpha}_p, \{\delta\})).$$

Note that $i_E(\vec{\alpha}_p, \{\delta\}) = (\vec{\alpha}_p, \{\delta\})$, since $\vec{\alpha}_p \in [\kappa]^{<\omega}$ and $i_E(\{\delta\}) = p_1^U = p_1^M = \{\delta\}$. Thus, since $i_E(\kappa) > \xi$,

$$U \models \exists y \varphi(\xi, y, \vec{\alpha}_p, \{\delta\}).$$

Contradiction! ⊣

Lemma 32. *Let N be as in Lemma 31. Then $N \lhd M$ and N is an admissible passive premouse with largest cardinal κ which is Woodin in N.*

Proof. Note that a failure of κ being Woodin in N is an $r\Sigma_1$ fact about $N|\kappa$. Since $H \prec_{\Sigma_1} M$, this would imply by upwards-absoluteness that δ fails to be Woodin in M. Thus, κ is Woodin in N. Moreover, as already mentioned in the previous proof κ is the largest cardinal of N. But then as $N|\kappa = M|\kappa$ and so $N = J_\alpha(M|\kappa)$, we will have that $N \lhd M$, and we are done. ⊣

4.2. The construction of the counterexample. Let $M := M^{\mathrm{ad}}_{\underset{\sim}{\Sigma}_1}$ and let $N \lhd M$ be as given by Lemma 32, i.e. the following hold:

- M has a largest cardinal δ which is $\underset{\sim}{\Sigma}_1$-Woodin in M,
- N is an admissible mouse with a largest cardinal κ which is Woodin in N, but not $\underset{\sim}{\Sigma}_1$-Woodin in N,
- $\mathrm{OR}^N < \kappa^{+M}$, and
- κ is measurable in M, as witnessed by an extender E from the extender sequence of M.

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The construction of the counterexample is now as follows: Fix $\Omega \in (\kappa^{++M}, \delta)$, which is a regular cardinal in M. Working inside M we will linearly iterate $M|\operatorname{lh}(E)$ for Ω many times via E and its images. Let $\mathcal{T}$ be the corresponding iteration tree on $M|\operatorname{lh}(E)$ of length $\operatorname{lh}(\mathcal{T}) = \Omega + 1$. Note that $\mathcal{T} \in M$. Let $i^{\mathcal{T}} : M|\operatorname{lh}(E) \to \mathcal{M}^{\mathcal{T}}_{\Omega}$ be the iteration map. We will have the following:

- $i^{\mathcal{T}}(\kappa) = \Omega$,
- $\sup(i^{\mathcal{T}}[\kappa^{+M}]) = i^{\mathcal{T}}(\kappa^{+M}) = \Omega^{+\mathcal{M}^{\mathcal{T}}_{\Omega}} < \Omega^{+M}$,
- $W := i^{\mathcal{T}}(N) \models \mathrm{KP}$ and $\mathrm{OR}^W < \Omega^{+\mathcal{M}^{\mathcal{T}}_{\Omega}}$, and
- W is an admissible mouse with a largest cardinal Ω which is Woodin in W, but not $\utilde{\Sigma}_1$-Woodin in W.

Since $\Omega^{+\mathcal{M}^{\mathcal{T}}_{\Omega}} < \Omega^{+M}$, there is, inside M, some $A \subset \Omega$ which codes a well order $<_A$ such that $\operatorname{otp}(<_A) = \Omega^{+\mathcal{M}^{\mathcal{T}}_{\Omega}}$. Let $\mathcal{U}$ be the A-genericity iteration of W with respect to the Ω-generator extender algebra of W at Ω. Note that $\mathcal{U} \in M$, since by standard arguments there is for every limit $\alpha \leq \Omega$ a Q-structure for $\mathcal{U} \upharpoonright \alpha$ in M. Moreover, since Ω is inaccessible in W, and a cardinal in M, by the usual arguments $\mathcal{U}$ is non-dropping, $\operatorname{lh}(\mathcal{U}) = \Omega + 1$, and $i^{\mathcal{U}}_{0\Omega}(\Omega) = \Omega$.

Inside M we construct a club $C \subset \Omega$ such that for all $\alpha \in C$:

- there exists some $\pi_\alpha : M_\alpha \cong X_\alpha \prec_{1000} M||\Omega^{++}$ such that $M|(\operatorname{lh}(E) + \omega) \cup \{\Omega, A\} \subset \operatorname{ran}(\pi_\alpha)$, $\operatorname{crit}(\pi_\alpha) = \alpha$, and $\pi_\alpha(\alpha) = \Omega$,
- $\alpha = \sup\{\operatorname{lh}(E^{\mathcal{U}}_\beta) : \beta < \alpha\}$,
- $\alpha \in b := [0, \Omega]^{\mathcal{U}}$, and
- $\alpha = \operatorname{crit}(E^{\mathcal{T}}_\alpha)$.

The construction of such C is fairly standard, so we will omit it. Note that for $\alpha \in C$, $\operatorname{crit}(i^{\mathcal{U}}_{\alpha\Omega}) \geq \alpha$, since $\alpha = \sup\{\operatorname{lh}(E^{\mathcal{U}}_\beta) : \beta < \alpha\}$. Thus, by the usual argument $\alpha^{+\mathcal{M}^{\mathcal{U}}_{\alpha}} = \alpha^{+\mathcal{M}^{\mathcal{U}}_{\Omega}}$.

Claim. For $\alpha \in C$, $\alpha^{+\mathcal{M}^{\mathcal{U}}_{\alpha}} > \alpha^{+\mathcal{M}^{\mathcal{T}}_{\Omega}}$.

Proof. Fix $\alpha \in C$. Let $(\bar{\mathcal{T}}, \bar{\mathcal{U}}, \bar{A}, \bar{W}) \in M_\alpha$ such that $\pi_\alpha((\bar{\mathcal{T}}, \bar{\mathcal{U}}, \bar{A}, \bar{W})) = (\mathcal{T}, \mathcal{U}, A, W)$. Note that there are such $\bar{\mathcal{T}}, \bar{\mathcal{U}}, \bar{W} \in M_\alpha$, since $\mathcal{T}, \mathcal{U}, W$ are definable from $M|\operatorname{lh}(E)$, A, and Ω over $M||\Omega^{++}$.

By the properties of π_α, $\bar{A}$ is generic over $\mathcal{M}^{\bar{\mathcal{U}}}_\alpha$ and $\operatorname{otp}(<_{\bar{A}}) = \alpha^{+\mathcal{M}^{\bar{\mathcal{T}}}_{\alpha}} = \alpha^{+\mathcal{M}^{\mathcal{T}}_{\alpha}}$. By Corollary 11, $\mathcal{M}^{\bar{\mathcal{U}}}_\alpha[\bar{A}] \models \mathrm{KP}$. Since $<_{\bar{A}} \in \mathcal{M}^{\bar{\mathcal{U}}}_\alpha[\bar{A}]$ and the transitive collapse of $<_{\bar{A}}$ is a Σ_1-recursion, $\operatorname{otp}(<_{\bar{A}}) = \alpha^{+\mathcal{M}^{\mathcal{T}}_{\alpha}} \in \mathcal{M}^{\bar{\mathcal{U}}}_\alpha[\bar{A}]$. In particular, $\mathrm{OR}^{\mathcal{M}^{\bar{\mathcal{U}}}_\alpha[\bar{A}]} = \mathrm{OR}^{\mathcal{M}^{\bar{\mathcal{U}}}_\alpha} > \alpha^{+\mathcal{M}^{\mathcal{T}}_{\alpha}}$. Note that since $\operatorname{crit}(E^{\mathcal{T}}_\alpha) = \alpha$ it follows by the usual argument that $\alpha^{+\mathcal{M}^{\mathcal{T}}_{\alpha}} = \alpha^{+\mathcal{M}^{\mathcal{T}}_{\Omega}}$. So in order to finish the proof it suffices to see that $\mathrm{OR}^{\mathcal{M}^{\bar{\mathcal{U}}}_\alpha} \leq \alpha^{+\mathcal{M}^{\mathcal{U}}_{\alpha}}$.

Since $\operatorname{crit}(\pi_\alpha) = \alpha$, we have that $\mathcal{M}^{\bar{\mathcal{U}}}_\alpha|\alpha = \mathcal{M}^{\mathcal{U}}_\alpha|\alpha$. Moreover, since $i^{\mathcal{U}}_{0,\Omega}(\Omega) = \Omega$, $i^{\bar{\mathcal{U}}}_{0\alpha}(\alpha) = \alpha$ and so $\mathcal{M}^{\bar{\mathcal{U}}}_\alpha \models$ "α is Woodin", since $\pi_\alpha(\mathcal{M}^{\bar{\mathcal{U}}}_\alpha) = \mathcal{M}^{\mathcal{U}}_\Omega$ and $\mathcal{M}^{\mathcal{U}}_\Omega \models$ "Ω is Woodin". By the 1-smallness of $\mathcal{M}^{\mathcal{U}}_\Omega$ it follows that $\mathcal{M}^{\bar{\mathcal{U}}}_\alpha = J_\beta(\mathcal{M}^{\bar{\mathcal{U}}}_\alpha|\alpha)$ for some $\beta < \Omega$ and so $\mathcal{M}^{\bar{\mathcal{U}}}_\alpha \trianglelefteq \mathcal{Q}(\mathcal{U} \upharpoonright \alpha)$, so that $\mathcal{M}^{\bar{\mathcal{U}}}_\alpha \triangleleft \mathcal{M}^{\mathcal{U}}_\Omega$. Since $\mathcal{Q}(\mathcal{U} \upharpoonright \alpha) \trianglelefteq \mathcal{M}^{\mathcal{U}}_\Omega|\alpha^{+\mathcal{M}^{\mathcal{U}}_{\Omega}}$, we have $\mathrm{OR}^{\mathcal{M}^{\bar{\mathcal{U}}}_\alpha} \leq \alpha^{+\mathcal{M}^{\mathcal{U}}_{\alpha}}$. ⊣

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We have shown that for every $\alpha \in C$, $\alpha^{+\mathcal{M}^{\mathcal{T}}_{\Omega}} < \alpha^{+\mathcal{M}^{\mathcal{U}}_{\Omega}}$. If we consider the tree $\mathcal{U}^*$ which is just the tree $\mathcal{U}$ considered not on W but on $W|\Omega$, we will have that for all $\alpha \in C$, $\alpha^{+W|\Omega} < \alpha^{+\mathcal{M}^{\mathcal{U}^*}_{\Omega}}$. Moreover, since $i^{\mathcal{U}}(\Omega) = \Omega$, we have $\mathrm{OR}^{\mathcal{M}^{\mathcal{U}^*}_{\Omega}} = \Omega$, so $\mathcal{M}^{\mathcal{U}^*}_{\Omega}$ is a weasel in the sense of M.

However, if $(\mathcal{T}', \mathcal{U}')$ is the coiteration of $(W|\Omega, \mathcal{M}^{\mathcal{U}^*}_{\Omega})$, then $\mathcal{T}' = \mathcal{U}^*$ and $\mathcal{U}'$ is the trivial tree. Thus, $W|\Omega =^* \mathcal{M}^{\mathcal{U}^*}_{\Omega}$ and there exists a club $C \subset \Omega$ such that for all $\alpha \in C$, $\alpha^{+W|\Omega} < \alpha^{+\mathcal{M}^{\mathcal{U}^*}_{\Omega}}$. This contradicts Conjecture 1 inside M.

Remark. The construction above also works if we picked Ω to be δ. However, in this case, we have to modify the construction slightly: Let $\eta > \kappa$ be a measurable cardinal of M and let $F \in \mathbb{E}^M$ be the measure witnessing this. Let $\mathcal{T}'$ the linear iteration of $M|\eta^{++M}$ via F and its images of length $\delta + 1$. Note that by Δ_0-collection $\mathcal{T}' \in M$ and that δ^+ exists in $\mathcal{M}^{\mathcal{T}'}_{\delta}$. Now as before we let $\mathcal{T}$ be a linear iteration of $M|\mathrm{lh}(E)$ via E and its images of length $\delta + 1$. However, note that $\mathcal{T} \in \mathcal{M}^{\mathcal{T}'}_{\delta}$. Thus, there is $A \in \mathcal{M}^{\mathcal{T}'}_{\delta}$ such that $\mathrm{otp}(<_A) = \delta^{+\mathcal{M}^{\mathcal{T}}_{\delta}}$ and $A \subset \delta$. Since $\delta^{+\mathcal{M}^{\mathcal{T}'}_{\delta}}$ exists and there is no initial segment of $\mathcal{M}^{\mathcal{T}'}_{\delta}$ which models KP and has a $\underset{\sim}{\Sigma}_1$-Woodin cardinal, we see by the same argument as before, that Q-structures exist for $\mathcal{U}$, where $\mathcal{U}$ is the genericity iteration of $i^{\mathcal{T}}_{0\delta}(N)$ making A generic.

§5. On another question from CMIP. In this last section we discuss another question remarked about in [10, prior to Lemma 4.6] concerning the S-hull property. Throughout this section Ω is a fixed measurable cardinal and μ_0 is a fixed normal measure on Ω. We call a premouse of ordinal height Ω a *weasel*.

Note that our definitions of thickness and the Hull property are different from the ones in [10], yet equivalent. We chose these different definitions in order to emphasize that thickness is a property independent of a specific weasel.

Definition 33. Let $\mathcal{W}$ be a weasel and $S \subset \Omega$ be stationary. We say that *S is good for $\mathcal{W}$* iff there is a club $C \subset \Omega$ such that for all $\alpha \in C \cap S$

- α is inaccessible,
- $\alpha^+ = (\alpha^+)^{\mathcal{W}}$, and
- α is not the critical point of a total-on-$\mathcal{W}$ extender from the extender sequence of $\mathcal{W}$.

Definition 34. Let $\Gamma \subset \Omega$ and S be stationary. We say that Γ is *S-thick* iff there is a club $C \subset \Omega$ such that for all $\alpha \in C \cap S$, $\Gamma \cap \alpha^+$ contains an α-club and $\alpha \in \Gamma$.

Definition 35. Let $\mathcal{W}$ be a weasel and $S \subset \Omega$ such that S is good for $\mathcal{W}$. Let $\alpha < \Omega$. We say that $\mathcal{W}$ has the *S-hull property* at α iff for all $\Gamma \subset \Omega$ which are S-thick

$$\mathcal{P}(\alpha) \cap \mathcal{W} \subset \text{transitive collapse of } \mathrm{Hull}^{\mathcal{W}}_{\omega}(\alpha \cup \Gamma).$$

In Lemma 4.6 of [10, p. 32] it is proven that for an $\Omega + 1$-iterable weasel $\mathcal{W}$, if S is good for $\mathcal{W}$, then for μ_0-a.e. $\alpha < \Omega$ the S-hull property holds at α.

However, in [10, p. 32, prior to Lemma 4.6], Steel raises the issue whether the set $\mathrm{HP}^{\mathcal{W}}_S := \{\alpha < \Omega : \mathcal{W} \text{ has the } S\text{-hull property at } \alpha\}$ is closed. Note that $\mathrm{HP}^{\mathcal{W}}_S$ clearly cannot be closed in the usual sense, as the following example from [10, p. 29]

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shows: Suppose that M is an $\Omega+1$-iterable weasel which has the S-hull property at all $\alpha<\Omega$ and there is a total-on-$ME \in \mathbb{E}^M$ with at least two generators. Then $\mathrm{Ult}_0(M,E)$ has the S-hull property at all $\alpha < (\mathrm{crit}(E)^+)^M$ but not at $(\mathrm{crit}(E)^+)^M$. However, $\mathrm{Ult}_0(M,E)$ is still $\Omega+1$-iterable. Thus, $\mathrm{HP}_S^{\mathcal{W}}$ cannot be closed in the usual sense.

Definition 36. We say that $X \subset \Omega$ is *almost closed* if for every $\delta<\Omega$, if δ is the supremum of elements of X and elements of $\Omega \setminus X$, then $\delta \in X$.

Given the preceding remarks, we will substitute almost closure for closure in the issue raised in [10], and hence consider the following question: Let $\mathcal{W}$ be a $(0,\Omega+1)$-iterable weasel and $S\subset\Omega$ such that S is good for $\mathcal{W}$, is the set $\mathrm{HP}_S^{\mathcal{W}}$ almost closed?

This question is answered positively by the following theorem.

Theorem 37. *Let $\mathcal{W}$ be an $\Omega+1$-iterable weasel and $S\subset\Omega$ be such that S is good for $\mathcal{W}$. Then the set $\{\alpha<\Omega : \mathcal{W}$ has the S-hull property at $\alpha\}$ is almost closed.*

Proof. By Lemma 4.5 of [10] there is an $\Omega+1$-iterable weasel M and an elementary embedding $\pi : M\to\mathcal{W}$ such that $\mathrm{ran}(\pi)$ is S-thick and M has the S-hull property at all $\alpha<\Omega$, i.e. $\mathrm{HP}_S^M=\Omega$. Let $(\mathcal{T},\mathcal{U})$ be the coiteration of $\mathcal{W}$ and M. We will prove the theorem assuming that $\mathrm{lh}(\mathcal{T})=\mathrm{lh}(\mathcal{U})=\Omega+1$ and leave the remaining cases as an exercise to the reader. Since $\mathcal{W}$ and M are universal, $i^{\mathcal{T}}:\mathcal{W}\to\mathcal{M}_\Omega^{\mathcal{T}}$ and $i^{\mathcal{U}}:M\to\mathcal{M}_\Omega^{\mathcal{U}}$ exist and $\mathcal{M}_\Omega^{\mathcal{T}}=\mathcal{M}_\Omega^{\mathcal{U}}=: M_\infty$.

Note that by the remark following Example 4.3 in [10, p. 29], M_∞ has the S-hull property at ξ iff for no $\beta+1\in[0,\Omega]^{\mathcal{U}}$, $\xi\in[\mathrm{crit}(E_\beta^{\mathcal{U}})^{+\mathcal{M}_\eta^{\mathcal{U}}},\nu(E_\beta^{\mathcal{U}}))$, where $\eta=\mathrm{pred}^{\mathcal{U}}(\beta+1)$. Thus, the set $\mathrm{HP}_S^{M_\infty}$ is almost closed. Moreover, by arguments from the proof of Lemma 4.6 in [10] (using that the set of fixed points of $i^{\mathcal{T}}$ is S-thick) we have that for all $\alpha<\Omega$, $\mathcal{W}$ has the S-hull property at α iff M_∞ has the S-hull property at $i^{\mathcal{T}}(\alpha)$.

Suppose for the sake of contradiction that $\mathrm{HP}_S^{\mathcal{W}}$ is not almost closed. Let $\delta<\Omega$ be a witness for this. Since $\mathrm{HP}_S^{M_\infty}$ is almost closed this means that $i^{\mathcal{T}}(\delta)>\sup(i^{\mathcal{T}}[\delta])$. There are two ways this can happen.

Case 1. *There is $\beta+1\in[0,\Omega]^{\mathcal{T}}$ such that $\mathrm{crit}(E_\beta^{\mathcal{T}})=i_{0\gamma}^{\mathcal{T}}(\delta)$, where $\gamma=\mathrm{pred}_{\mathcal{T}}(\beta+1)$.*

Let us assume that β is least such. Note that in this case we must have that $\sup(i_{0\gamma}^{\mathcal{T}}[\delta])=i_{0\gamma}^{\mathcal{T}}(\delta)$ and $\delta':=i_{0\gamma}^{\mathcal{T}}(\delta)\in\mathrm{HP}_S^{M_\infty}$. Since $\mathrm{crit}(i_{\beta+1,\Omega}^{\mathcal{T}})>\delta'$, $\mathcal{M}_{\beta+1}^{\mathcal{T}}$ has the S-hull property at δ'. But then, since $\mathcal{M}_{\beta+1}^{\mathcal{T}}$ has the S-hull property at δ' and $\mathcal{P}(\delta')\cap\mathcal{M}_\gamma^{\mathcal{T}}=\mathcal{P}(\delta')\cap\mathcal{M}_{\beta+1}^{\mathcal{T}}$, $\mathcal{M}_\gamma^{\mathcal{T}}$ has the S-hull property at δ'. Contradiction!

Case 2. $W\models$ "δ is singular with cofinality μ" *and there is a minimal $\alpha\in[0,\Omega]^{\mathcal{T}}$ such that $i_{0\alpha}^{\mathcal{T}}(\mu)=\mathrm{crit}(i_{\alpha\Omega}^{\mathcal{T}})$.*

Let $\gamma+1\in[0,\Omega]^{\mathcal{T}}$ such that $\mathrm{pred}_{\mathcal{T}}(\gamma+1)=\alpha$ and set $W':=\mathcal{M}_\alpha^{\mathcal{T}}$, $W'':=\mathcal{M}_{\gamma+1}^{\mathcal{T}}$, $\delta':=i_{0\alpha}^{\mathcal{T}}(\delta)$, $j:=i_{\alpha,\gamma+1}^{\mathcal{T}}$, and $\delta'':=\sup(j[\delta'])$. Note that $\delta''<j(\delta')$. Moreover, letting $i:=i_{0\alpha}^{\mathcal{T}}$, we have that $\delta'=i(\delta)=\sup(i[\delta])$.

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CLAIM 1. *The S-hull property holds at δ'' in W''.*

PROOF. Note that δ'' is a limit of $\mathrm{HP}_S^{W''}$ and $\Omega \setminus \mathrm{HP}_S^{W''}$. So in order to see that the S-hull property holds at δ'' in W'' it suffices to see that $i^{\mathcal{T}}_{\gamma+1,\Omega}$ is continuous at δ''. To this end note that $\mathrm{cof}(\delta'')^{W''} = \kappa$ and that all extenders used along $[0,\Omega]^{\mathcal{T}}$ after $E^{\mathcal{T}}_\gamma$ have critical points greater than κ. Thus, $i^{\mathcal{T}}_{\gamma+1,\Omega}$ is continuous at δ'' from which the claim follows. ⊣

We claim that this implies that the S-hull property holds at δ' in W', which would be a contradiction. Let $A \subset \delta'$ such that $A \in W'$ and Γ be an S-thick set. We need to show that there is a term τ, $\vec{\xi} \in [\Gamma]^{<\omega}$, and $\vec{\beta} \in [\delta']^{<\omega}$ such that

$$A = \tau^{W'}[\vec{\xi}, \vec{\beta}] \cap \delta'.$$

Note that $j(A) \cap \delta'' \in W''$. Thus, there is a term σ, $\vec{\zeta} \in [\Gamma]^{<\omega}$, and $\vec{\gamma} \in \delta''$ such that

$$j(A) \cap \delta'' = \sigma^{W''}[\vec{\zeta}, \vec{\gamma}] \cap \delta''.$$

We may assume that j fixes $\vec{\zeta}$. Let $\vec{\gamma} = ([a_0, f_0]^{W'}_E, \dots, [a_n, f_n]^{W'}_E)$, where for $k \leq n$, $a_k \in [\nu(E)]^{<\omega}$ and $f_k \in {}^{[\kappa]^{<\omega}} W' \cap W'$. Note that by Łoś's Theorem,

$$\begin{aligned} &j(\xi) \in \sigma^{W''}[\vec{\zeta}, \vec{\gamma}] \iff \\ &\{b \in [\kappa]^{<\omega} : W' \models \xi \in \sigma[\vec{\zeta}, f_0^{a_0,a}(b), \dots, f_n^{a_n,a}(b)]\} \in E_a, \end{aligned} \tag{5.1}$$

where $a = \bigcup_{i \leq n} a_n$. Furthermore, for $\xi < \delta'$,

$$j(\xi) \in \sigma^{W''}[\vec{\zeta}, \vec{\gamma}] \iff j(\xi) \in j(A) \iff \xi \in A. \tag{5.2}$$

Note that for every $c \in [\nu(E)]^{<\omega}$, E_c is close to W'. In particular, since $W' \models \mathrm{ZFC}$, $E_c \in W'$ for every $c \in [\nu(E)]^{<\omega}$. Moreover, δ' is a limit cardinal in W' and GCH holds in W'. Thus, for every $c \in [\nu(E)]^{<\omega}$ the ordinal of the measure E_c in the W'-order $<_{W'}$ is an ordinal less than δ'. Furthermore, as the ordinals below δ'' might be represented via functions bounded in δ', we may assume that for $k \leq n$, f_k is bounded in δ' and thus also their ordinals in the W'-order are less than δ'. But this means that 5.1 and 5.2 give us a term τ and $\vec{\beta} \in [\delta']^{<\omega}$ such that

$$A = \tau^{W'}[\vec{\zeta}, \vec{\beta}] \cap \delta'.$$

This finishes the proof of the theorem. ⊣

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INSTITUT FÜR DISKRETE MATHEMATIK UND GEOMETRIE
TECHNISCHE UNIVERSITÄT WIEN
WIEDNER HAUPTSTRASSE 8-10/104
1040 WIEN, AUSTRIA

E-mail: jan.kruschewski@tuwien.ac.at
E-mail: farmer.schlutzenberg@tuwien.ac.at

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