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ON THE DEGREE OF APPROXIMATION BY SZÁSZ OPERATORS

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The aim of the present note is to give the degree of approximation by Szász operators.

1. Introduction

The linear positive operators (S_nf) defined as

$$(1.1) \qquad (S_nf)(x) = \left(S_n(f(t); x)\right) = e^{-nx} \sum_{k=0}^{\infty} \left((nx)^k/\underline{|k}\right) f(k/n)$$

were introduced by Szász [3] to approximate $f \in C[0, \infty)$. Stancu [2] has given the following result in uniform norm.

THEOREM 1. *Let* $f \in C^1[0, a]$, $a > 0$, *and let* $\omega(f'; \cdot)$ *be its modulus of continuity. Then, for* $n \in N$,

$$(1.2) \qquad \|S_nf-f\| \le (a+\sqrt{a}) \cdot 1/\sqrt{n} \cdot \omega(f'; 1/\sqrt{n}) .$$

Recently we [4] have obtained the estimate

$$(1.3) \qquad \|S_nf-f\| \le \left(\sqrt{a}+(a/2)\right) \cdot 1/\sqrt{n} \cdot \omega(f'; 1/\sqrt{n})$$

which is sharper than the corresponding estimate (1.2).

The object of the present note is to extend the result (1.3) for

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$f \in C^{r+1}[0, a]$, $r = 0, 1, 2, \ldots$.

We prove the following.

THEOREM 2. *Let* $f \in C^{r+1}[0, a]$, $a > 0$, *and let* $\omega(f^{(r+1)}; \cdot)$ *be its modulus of continuity. Then, for* $n \in N$,

(1.4) $$\left\|S_n^{(r)}f - f^{(r)}\right\| \le r/n \cdot \|f^{(r+1)}\| + K_{n,r} \cdot 1/\sqrt{n} \cdot \omega\left(f^{(r+1)}; 1/\sqrt{n}\right)$$

where

$$K_{n,r} = \left[(a/2)+(r/2\sqrt{n})+\left(r^2/4n\right)+\left(\left(r^2/4n\right)+a\right)^{\frac{1}{2}} \cdot \left(1+(r/2\sqrt{n})\right)\right] .$$

2. Proof

We use the following results [1],

(2.1) $$\left(S_n 1\right)(x) = 1 ,$$

(2.2) $$\left(S_n(t-x)\right)(x) = 0 ,$$

(2.3) $$\left(S_n(t-x)^2\right)(x) = x/n .$$

After differentiating (1.1) r times with respect to x , we get

(2.4) $$\left(S_n^{(r)}f\right)(x) = n^r e^{-nx} \sum_{k=0}^{\infty} \left((nx)^k/\underline{|k}\right)\Delta^r_{n^{-1}} f(k/n) \quad (r \le n) ,$$

where $\Delta^r_{n^{-1}} f(k/n)$ represents the difference of order r of the function f with step $1/n$ starting from value k/n . This difference of order r is defined by

$$\Delta^1_{n^{-1}} f(k/n) = \Delta_{n^{-1}} f(k/n) = f\left((k+1)/n\right) - f(k/n)$$

and

$$\Delta^{r+1}_{n^{-1}} f(k/n) = \Delta_{n^{-1}}\left(\Delta^r_{n^{-1}} f(k/n)\right) , \quad r = 1, 2, \ldots .$$

By using the mean value theorem,

(2.5) $$\Delta^r_{n^{-1}} f(k/n) = \left(1/n^r\right) f^{(r)}\left(\left(k+r\theta_k\right)/n\right) , \quad \theta_k \in (0, 1) .$$

With the help of (2.4) and (2.5),

$$(2.6) \qquad \left(S_n^{(r)} f(x)\right) = e^{-nx} \sum_{k=0}^{\infty} \left((nx)^k / \underline{|k}\right) f^{(r)}\left(\left(k+r\theta_k\right)/n\right) .$$

We know that

$$(2.7) \quad f^{(r)}(x) - f^{(r)}\left(\left(k+r\theta_k\right)/n\right)$$
$$= \left(x-\left(\left(k+r\theta_k\right)/n\right)\right) f^{(r+1)}(x) + \int_x^{\left(k+r\theta_k\right)/n} \left[f^{(r+1)}(x)-f^{(r+1)}(\eta)\right] d\eta .$$

Using (2.6), (2.7) and the inequality

$$\left|f^{(r+1)}(x)-f^{(r+1)}(\eta)\right| \le [1+(|\eta-x|/\delta)] \cdot \omega\left(f^{(r+1)};\ \delta\right) \quad (\delta > 0) ,$$

we obtain that

$$\left|f^{(r)}(x)-\left(S_n^{(r)} f\right)(x)\right|$$
$$\le \left|f^{(r+1)}(x)\right| \cdot \left|e^{-nx} \sum_{k=0}^{\infty} \left((nx)^k / \underline{|k}\right) \left(x-\left(\left(k+r\theta_k\right)/n\right)\right)\right|$$
$$+ \omega\left(f^{(r+1)};\ \delta\right) e^{-nx} \sum_{k=0}^{\infty} \left((nx)^k / \underline{|k}\right) \left|\int_x^{\left(k+r\theta_k\right)/n} [1+(|\eta-x|/\delta)] d\eta\right|$$
$$= S_1 + S_2 \quad \text{(say)}.$$

Clearly

$$S_1 = \left|f^{(r+1)}(x)\right| \cdot \left|e^{-nx} \sum_{k=0}^{\infty} \left((nx)^k / \underline{|k}\right) \left[x-(k/n)-\left(r\theta_k/n\right)\right]\right|$$
$$\le \left|f^{(r+1)}(x)\right| \cdot r/n \cdot e^{-nx} \sum_{k=0}^{\infty} \left((nx)^k / \underline{|k}\right) \left|\theta_k\right|$$
$$\le r/n \cdot \left\|f^{(r+1)}\right\| ,$$

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$$S_2 = \omega\left(f^{(r+1)};\ \delta\right)e^{-nx}\sum_{k=0}^{\infty}\left((nx)^k/\lfloor k\right) \times \left[\left|\left(\left(k+r\theta_k\right)/n\right)-x\right|+(1/2\delta)\left(\left(\left(k+r\theta_k\right)/n\right)-x\right)^2\right]$$

$$\le \omega\left(f^{(r+1)};\ \delta\right)e^{-nx}\sum_{k=0}^{\infty}\left((nx)^k/\lfloor k\right) \times \left[|x-(k/n)-(r/2n)|+(r/2n)+(1/2\delta)\left(|x-(k/n)-(r/2n)|+(r/2n)\right)^2\right]$$

$$\le \omega\left(f^{(r+1)};\ \delta\right)\left[\left((r/2n)+\left(r^2/8n^2\delta\right)\right)+(1/2\delta)e^{-nx}\sum_{k=0}^{\infty}\left((nx)^k/\lfloor k\right) \times \left(x-(k/n)-(r/2n)\right)^2+\left(1+(r/2n\delta)\right)e^{-nx}\sum_{k=0}^{\infty}\left((nx)^k/\lfloor k\right)|x-(k/n)-(r/2n)|\right].$$

From (2.1), (2.2), (2.3) we know that

$$e^{-nx}\sum_{k=0}^{\infty}\left((nx)^k/\lfloor k\right)\left(x-(k/n)-(r/2n)\right)^2 = x/n + r^2/4n^2$$

and

$$e^{-nx}\sum_{k=0}^{\infty}\left((nx)^k/\lfloor k\right)|x-(k/n)-(r/2n)| \le \sqrt{e^{-nx}\sum_{k=0}^{\infty}\left((nx)^k/\lfloor k\right)\left(x-(k/n)-(r/2n)\right)^2\cdot\left(S_n 1\right)(x)} = \left(\left(r^2/4n^2\right)+(x/n)\right)^{\frac{1}{2}}.$$

By choosing $\delta = 1/\sqrt{n}$, we finally get, for all $x \in [0, a]$, that

$$S_2 \le K_{n,r}\cdot 1/\sqrt{n}\cdot\omega\left(f^{(r+1)};\ 1/\sqrt{n}\right).$$

This completes the proof.

COROLLARY 1. *If* $f^{(r+1)} \in \mathrm{Lip}_M\,\alpha$, $0 < \alpha \le 1$, $M > 0$, *then, for* $n \in N$,

$$\left\|S_n^{(r)}f-f^{(r)}\right\| \le r/n\cdot\left\|f^{(r+1)}\right\| + M\cdot K_{n,r}\cdot n^{-(\alpha+1)/2}.$$

COROLLARY 2. *If, in addition to the hypotheses of Theorem 2,* $f \in C^{r+2}[0, a]$, *then, for* $n \in N$,

$$\left\|S_n^{(r)}f-f^{(r)}\right\| \le r/n \cdot \|f^{(r+1)}\| + K_{n,r}/n \cdot \|f^{(r+2)}\| .$$

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