# Mesoturbulence+)

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+) In favour of a concise discussion of the basic concept of mesoturbulence we have refrained from presenting a comprehensive review of the work done in this field. Completeness, however, was aimed at in the references at the end of the contribution by E. Sedlmayr.

## Abstract

The influence of a stochastic velocity field with a finite scale length l on the transfer of line radiation is described by means of a generalization of the transfer equation. Micro- and macroturbulence are contained in this mesoturbulence approach as limiting cases $l \to 0$ and $l \to \infty$ respectively.

## Introductory Remarks

In hydrodynamics the term "turbulence" describes velocity fields which are dominated by inertial forces and of which only the statistical properties are controlled by initial- and boundary conditions. The meaning of the same word as used by spectroscopists is quite different. It encompasses any flow of unresolved pattern which - in addition to thermal motion - contributes to the Dopplerbroadening of spectral lines. So this term describes a situation which is characterized by a lack of information concerning the underlying velocity field. It is for this reason that the spectroscopist takes recourse to a description in statistical terms.

First of all, the basic information is contained in the mean square velocity $\langle v^2 \rangle = \sigma^2$, where v is the velocity component parallel to the ray. The second important parameter, the scale length l, is more difficult to determine.

Struve and Elvey (1934) discussed the limiting case $\kappa_{line} \cdot l \ll 1$ - with $\kappa_{line}[cm^{-1}]$ being the line absorption coefficient - where the hydrodynamic flow simply acts as an additional thermal broadening of the atomic absorption profile. In this case the saturation in the line decreases with the consequence that the curve of growth (say of stellar

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absorption lines) changes due to increasing equivalent widths. This is the microturbulence limit. Macroturbulence, on the other hand, is defined by $\kappa_{line} \cdot l >> 1$. Then there is no velocity gradient along the ray and the radiative transfer is not affected. The profile in the radiation leaving the source has to be convoluted with the velocity distribution, a procedure which does not alter the equivalent widths.

The need for an approach based on the assumption of a finite l, which bridges the gap between these two limits, becomes obvious if one realizes that:

a) There is no doubt that "microturbulence" is a well established phenomenon in stellar spectroscopy. So l cannot have been large compared to $\kappa_{line}^{-1}$.

b) One has to exclude very small values of l since they would imply strong velocity gradients and hence excessive dissipation of kinetic energy.

Indeed, we note that under very general conditions the energy dissipation in a turbulent hydrodynamic flow is

$$\frac{dE}{dt} = 15\ \nu\ \langle v^2 \rangle\ \lambda^{-2}\ [\mathrm{erg\ g^{-1}\ sec^{-1}}] \tag{1}$$

if $\nu$ is the kinematic viscosity, which is roughly given by

$$\nu = v_{thermal} \cdot l_{free\ path} \tag{2}$$

and $\lambda$ the microscale of the flow defined by

$$\langle v^2 \rangle / \lambda^2 = \langle (\frac{dv}{ds})^2 \rangle . \tag{3}$$

If the flow is stationary $\frac{dE}{dt}$ must be equal to $\langle v^2 \rangle/2$ divided by a time which is characteristic for the renewal of the hydrodynamic energy. Let this time be the ratio of the equivalent height H of the atmosphere to the velocity of the flow; an assumption which seems to be reasonable either in case of buoyancy forces driving the turbulence or of convective transport of kinetic energy. If all velocities are of the same order of magnitude one finds that the smallest possible scale for hydrodynamic fields (in stellar atmospheres) is of the order of

$$\lambda = (30\ H\ l_{free\ path})^{1/2} . \tag{4}$$

Inserting data for the solar photosphere yields $\lambda \sim 10^4 .. 10^5$ [cm] which is of the same order of magnitude as the mean free path of a photon in an absorption line of medium strength.

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If the effects of microturbulence are caused by sound waves of sawtooth form, Hearn (1974) has shown that the observed microturbulence velocities require a flux of mechanical energy which is about 100times the acoustic energy generated by the convection zones.

Hence, one cannot escape the conclusion that the naive interpretation of line broadening by small scale hydrodynamic flow as microturbulence has to be abandoned since it interfers with basic laws of hydrodynamics.

## The Microturbulence Criterion Revisited

In the following we consider in more detail the condition for the validity of the microturbulence approach. Let

$$\frac{dI}{ds} + \kappa(v)\ I = 0 \tag{5}$$

be the transfer equation for the monochromatic intensity I in case of pure absorption. The inclusion of a non-zero source function S would make the formulae more involved without altering our conclusions. Note that I(s) depends on all values of v(s') for s'<s so that I(s) is not a function but a functional of v.

The microturbulent solution $I_{mic}$ satisfies the equation

$$\frac{dI_{mic}}{ds} + \langle\kappa\rangle\ I_{mic} = 0 \tag{6}$$

with

$$\langle\kappa\rangle = \int_{-\infty}^{+\infty} \kappa(v)\ \mathbb{P}_1(v)\,dv \tag{7}$$

where $\mathbb{P}_1(v)$ is the one-point distribution function for the velocities along the ray. Defining u(s) by

$$I(s) = u(s)\cdot I_{mic}(s) \tag{8}$$

we obtain for u(s) the differential equation

$$\frac{du}{ds} + \Delta\kappa(v)\cdot u = 0 \tag{9}$$

with

$$\Delta\kappa(v) = \kappa(v) - \langle\kappa\rangle\ . \tag{10}$$

Solving eq. (9) with the initial value u(s=0)=1 by means of Picard's iteration one obtains

$$u(s)=1-\int_o^s ds_1\Delta\kappa_1+\int_o^s ds_1\int_o^{s_1} ds_2\Delta\kappa_1\Delta\kappa_2-\int_o^s ds_1\int_o^{s_1} ds_2\int_o^{s_2} ds_3\Delta\kappa_1\Delta\kappa_2\Delta\kappa_3\ldots \tag{11}$$

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where

$$\Delta\kappa_i = \Delta\kappa(s_i, v(s_i)) \quad . \tag{12}$$

Recalling that by means of the definition eq. (10) we have

$$\langle\Delta\kappa\rangle = 0 \tag{13}$$

we obtain for the lowest order deviation from unity of the expectation value of u(s)

$$\langle u(s)\rangle - 1 \approx \int_o^s ds_1 \int_o^{s_1} ds_2 \, \langle\Delta\kappa_1\Delta\kappa_2\rangle \quad . \tag{14}$$

Thus the lowest order deviation from the microturbulent case is determined by the two-point correlation of $\Delta\kappa$ which is calculated by means of the two-point velocity distribution $\mathbb{P}_2(v_1,s_1,v_2,s_2)$ according to

$$\langle\Delta\kappa_1\Delta\kappa_2\rangle = \int_{v_1} dv_1 \int_{v_2} dv_2 \, \Delta\kappa_1\Delta\kappa_2 \, \mathbb{P}_2(v_1,s_1,v_2,s_2) . \tag{15}$$

If $s_1-s_2>l$ the two point velocity distribution factorizes into two one-point distributions so that due to eq. (13) the contribution to the integral is zero. Taking this into account the rhs. of eq. (14) can be estimated as

$$\langle u(s)\rangle-1 \approx \frac{l\cdot s}{2} \left(\langle\kappa^2\rangle - \langle\kappa\rangle^2\right) \quad . \tag{16}$$

So we obtain the following condition for the validity of the microturbulence approach

$$\frac{\tau_l \, \tau_s}{2} \left( \frac{\langle\kappa^2\rangle}{\langle\kappa\rangle^2} - 1\right) \ll 1 \tag{17}$$

where the optical depths $\tau_l = l\cdot\langle\kappa\rangle$ and $\tau_s = s\langle\kappa\rangle$ have been introduced. Since in all cases of interest $\tau_s$ will be of the order one, we see that microturbulence is a good approach if $\tau_l \ll 1$ (the usual assumption) <u>and/or if $\langle\kappa^2\rangle/\langle\kappa\rangle^2 - 1$ approaches zero</u>. This is a condition which limits the amplitude of the velocity distribution. It should be mentioned that $\langle\kappa^2\rangle/\langle\kappa\rangle^2-1$ is zero at the line center for small $\langle v^2\rangle$ if the distribution functions are symmetric.

## Approach to Mesoturbulence

The foregoing discusssion provides a first step towards a more general description of the radiative transfer which incorporates the to parameters $\langle v^2\rangle$ and $l$ from the very beginning. Indeed, eq. (11) can be considered as the formal solution of such a transfer problem. One clearly sees that all higher order correlations of the velocity field enter.

An approach by means of a perturbation expansion - which bears at least some relation to the above formalism - has been formulated by Rybicki (1975). He developed all relevant quantities in orders of the perturbation by the velocity field.

$$\begin{aligned} \kappa &= \kappa^{(0)} + \kappa^{(1)} + \kappa^{(2)} + \dots \\ I &= I^{(0)} + I^{(1)} + I^{(2)} + \dots \\ S &= S^{(0)} + S^{(1)} + S^{(2)} + \dots \quad . \end{aligned} \tag{18}$$

S is the monochromatic source function which in case of scattering or non-LTE may depend on the velocity. Inserting these expansions into the transfer equation and equating terms of different order individually one obtains the perturbation expansion of the transfer equation. It turns out that the formal solution of the order (n-1) can be inserted as inhomogeneous term into the rhs. of the equation of order n. So there exists no closure problem since there is no dependence of the low order equations on the higher order solutions. It is by means of these rhs. terms that the correlations of the velocity field enter.

Apparently no attempts have been made to work out in more detail this formalism, into which correlations of all orders enter, but which - for practical reasons - is restricted to weak turbulence.

Obviously the problem is to determine the order of correlations which have to be taken into account. Apart from all theoretical considerations the answer to this question depends on the amount of information concerning the pattern of the velocity field which can be derived from the observed profiles. Apparently at present we can hardly expect to determine by such an analysis more than the influence of the lowest order correlations on the radiative transfer. Thus the assumption that all higher order correlations factorize into two-point correlations seems to be adequate.Approaches of this type which lead to simple formulae and which do not impose any constraints to $\langle v^2 \rangle$ have been followed independently by Auvergne et al. (1973) and by Gail et al. (1974).

The essentials common to the work of both groups can be presented in a very simple way: Let us assume LTE and a source function S which does not depend on the velocities. Then the transfer equation can be written as

$$dI = -\kappa(v)(I-S)ds \tag{19}$$

If the velocities v(s') for s'<1 are deterministic also I(s) is deterministic, but in case of random velocities I(s) will be random. Consider all I(s) which are compatible with the constraint that at s the velocity

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is in the interval $v...v+dv$, the probability of which is $\mathbb{P}_1(v,s)dv$. Let $q(v,s)$ be the mean value of the intensities subjected to this constraint. We now define

$$Q(v,s) = \mathbb{P}_1(v,s) \cdot q(v,s) \tag{20}$$

and obtain for the expectation value of the intensity

$$\langle I(s)\rangle = \int_v Q(v,s)dv \, . \tag{21}$$

We want to emphasize that in contrast to $I(s)$, which is a functional of $v$, the quantities $q(v,s)$ and $Q(v,s)$ are functions of $v$.

Aiming now at the transfer equations for $q(v,s)$ or $Q(v,s)$ we first restrict ourselves to the case of macroturbulence and assume constant $v$ for all $s$. Then $q(v,s)$ and $I(s)$ have to comply with the same transfer equation since for $v$ = const. both quantities are identical. Hence

$$dq(v,s) = -\kappa(v)(q(v,s)-S)ds \tag{22}$$

and by multiplication with $\mathbb{P}_1(v)$ - which we assume to be independent of $s$ -

$$dQ(v,s) = -\kappa(v)(Q(v,s) - \mathbb{P}_1(v)S)ds \tag{23}$$

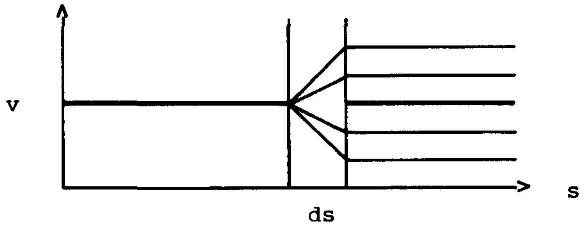
We now relax the macroturbulence condition ($v$=const.) by assuming a random field, the structure of which is dominated by two-point correlations or by the corresponding two-point distribution functions $\mathbb{P}_2(v_1,s_1,v_2,s_2)$, respectively. These can be written as the product of a one-point distribution function times a transition probability

$$\mathbb{P}_2(v_1,s_1,v_2,s_2) = \mathbb{P}_1(v_1,s_1) \cdot W(v_2|v_1,s_1,s_2-s_1) \tag{24}$$

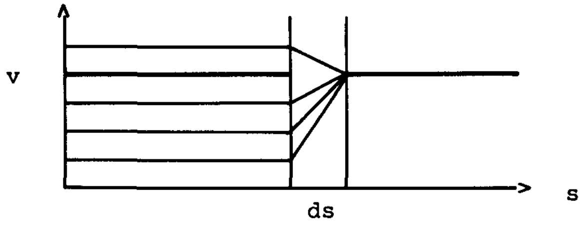
where $W(v_2|v_1,s_1,s_2-s_1)dv_2$ is the probability of finding at $s_2$ the velocity $v_2$ in the interval $dv_2$ provided that at $s_1$ the velocity is $v_1$.

Any change of the velocities with the step $ds$ will clearly affect the transfer equation (23). There will be an additional sink term for $Q(v,s)$ which is $Q(v,s)$ times the probability that with the step $ds$ the velocity changes to any other velocity $v'$. Also an additional source term occurs, given by $Q(v",s)$ times the probability of a transition from any $v"$ to $v$. It is by means of these transition probabilities that the scale length $l$ is introduced.

Transitions of v which lead to




sink terms
in Q(v,s)




source terms
in Q(v,s)

So, with W independent of s, eq. (23) turns into

$$dQ(v,s) = -\kappa(v)(Q(V,s) - \mathbb{P}_1(v)\cdot S)ds - \int_{-\infty}^{+\infty} Q(v,s)W(v'|v,ds)dv' + \int_{-\infty}^{+\infty} Q(v'',s)W(v|v'',ds)dv'' \; . \tag{25}$$

This is the central equation of the two-point correlation approach. It has some resemblance with the transfer equation in case of scattering of line radiation. Indeed, if one relates the transition probability W with the redistribution function in case of scattering one can interpret eq. (25) as describing the transfer of monochromatic line radiation subjected to a scattering process in velocity space.

One can look at the problem from a different point of view. Let P(I,v,s)dIdv be the joint probability of finding at s the intensity I in dI and the velocity v in dv. Then with the assumption of a velocity field governed by two-point correlations only, the transition probability for the velocity depends only on v at s and on no other data. Since the transfer equation is a first order differential equation, the change of I depends also only on the values of I and v at s, so that one can consider I and v as being subjected to a Markovian process in s. In this case the smooth change of P(I,v,s) with s can be described by an equation of the Einstein-Smoluchowski type:

$$P(I,v,s)= \int_{v'} dv' \int_{I'} dI' \delta(I-I'+\kappa(v')(I'-S)\Delta s)\cdot W(v|v',\Delta s)P(I',v',s'). \quad (26)$$

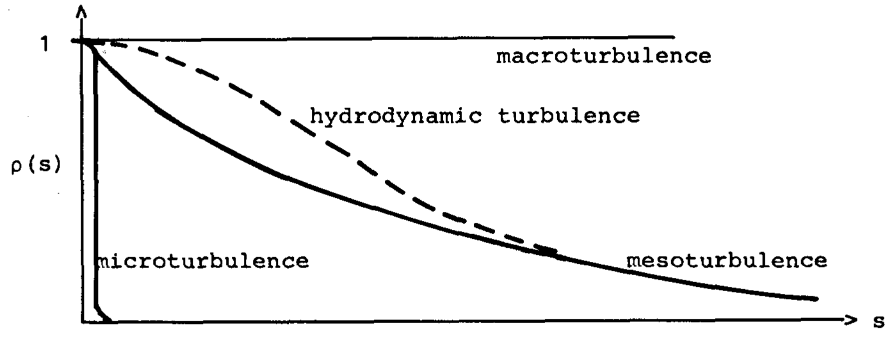
In order to establish the relation to eq. (25) one has to make use of the fact that $Q(v,s)$ is the first moment of $P(I,v,s)$ with respect to I

$$Q(v,s)= \int_{I} P(I,v,s)I\, dI. \quad (27)$$

The relations of this mesoturbulence approach to the micro- and macroturbulence limits and to hydrodynamic turbulence can best be illustrated by means of the corresponding two-point velocity correlations

$$\rho(s)= \langle v(s')v(s'+s)\rangle/\langle v^2\rangle \ . \quad (28)$$

Qualitatively the graphs of $\rho(s)$ are:




In order to work out eq. (25) in more detail the transition probability for the velocities has to be specified. This is the point where different approaches have been followed.

Auvergne et al. (1973) conceived a process (the Kubo-Anderson process) in which the two-point velocity distribution is a linear combination of a) a completely uncorrelated part consisting of the product of two one-point distribution functions $\mathbb{P}_1(v,1)\cdot \mathbb{P}_1(v',s')$ and b) a part which is completely correlated $\mathbb{P}_1(v',s')\cdot\delta(v-v')$. Then the transition probability is

$$W(v|v',\Delta s)=(1-\rho(\Delta s))\ \mathbb{P}_1(v')+\rho(\Delta s)\delta(v-v'). \quad (29)$$

Then with

$$\rho(ds) = 1 - \frac{ds}{l} \quad (30)$$

eq. (25) turns into

$$\frac{\partial Q}{\partial s}=-\kappa(Q-\mathbb{P}_1 S)-\frac{1}{l}\{Q-\mathbb{P}_1\int_{-\infty}^{+\infty} Q(v')dv'\}=-(\kappa+\frac{1}{l})Q+(\kappa S+\frac{1}{l}\langle I\rangle)\mathbb{P}_1 \quad . \quad (31)$$

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The sink and source terms in the curly bracket can easily be interpreted as being due to "complete redistribution in velocity space"

Gail et al. (1974), on the other hand, preferred Gaussian one- and two-point velocity distributions with the consequence that

$$\mathbb{P}_1 = (2\pi\sigma^2)^{-1/2} \exp\left(-\frac{v^2}{2\sigma^2}\right) \tag{32}$$

and

$$W(v|v',\Delta s) = (2\pi\sigma^2(1-\rho^2))^{-1/2} \exp\left(-\frac{(\rho v'-v)^2}{2\sigma^2(1-\rho^2)}\right) . \tag{33}$$

With these assumptions which define an Uhlenbeck-Ornstein process (Wang and Uhlenbeck, 1945) eq. (25) can be written as

$$\frac{\partial Q}{\partial s} = -\kappa(Q - \mathbb{P}_1 S) + \frac{1}{l}\frac{\partial}{\partial v}\left(v + \sigma^2 \frac{\partial}{\partial v}\right) Q . \tag{34}$$

The differential operator on the rhs. of eq. (34) is consistent with the assumption of a continuous velocity field, hence only infinitesimal changes of v within ds have non zero probabilities; the "scattering" is almost "coherent". Eq. (34) is a partial differential equation of parabolic type which is easily solved by standard numerical techniques.

The central eq. (25) and hence also eq. (31) and eq. (34) have been derived starting from the macroturbulence limit ($l \to \infty$). We use eq. (34) in order to show that they contain also the microturbulence limit ($l \to \infty$).

Noting that the Hermite functions

$$\phi_n\left(\frac{v}{\sigma}\right) = \phi_n(\hat{v}) \tag{35}$$

are the eigenfunctions of the differential operator occurring in eq. (34)

$$\frac{\partial}{\partial \hat{v}}\left(\hat{v} + \frac{\partial}{\partial \hat{v}}\right) \phi_n = - n\, \phi_n \tag{36}$$

we use the expansion

$$Q(s,\hat{v}) = \sum_{n=0}^{\infty} T_n(s)\, \phi_n(\hat{v}) \tag{37}$$

and obtain due to the orthonormality relations of the Hermite functions the following system of ordinary differential equations

$$\frac{dT_m}{ds} = - \sum_{n=0}^{\infty} \kappa_{mn} T_n + \kappa_{mo} S - \frac{m}{l} T_m , \quad m = 0,1,2.... \tag{38}$$

with

$$\kappa_{mn} = \int_{-\infty}^{+\infty} \phi_o^{-1} \phi_m \phi_n \kappa(\hat{v}) d\hat{v} . \tag{39}$$

It is obvious that in the limit $l \to 0$ all modes with $m \neq 0$ will have zero amplitude due to the last term on the rhs. of eq. (38). Hence in this limit eq. (38) reduces to

$$\frac{dT_o}{ds} = - \kappa_{oo} (T_o - S) \tag{40}$$

This is the microturbulence equation since

$$\phi_o(\hat{v}) = \mathbb{P}_1 (\hat{v}) \tag{41}$$

and

$$\langle I(s) \rangle = \int_{-\infty}^{+\infty} Q(s,\hat{v}) d\hat{v} = \sum_{n=0}^{\infty} T_n(s) \int_{-\infty}^{+\infty} \phi_o^{-1} \phi_o \phi_n d\hat{v} = T_o . \tag{42}$$

## Conclusion

1) The word "turbulence" denotes a velocity field which can be described only in statistical terms. However, the reasons for such a description may be different. For a hydrodynamicist it is the very nature of the flow, for a spectroscopist, however, it is lack of information concerning the flow pattern.

2) The basic parameter of such a velocity field is the mean square velocity $\langle v^2 \rangle$. The scale length $l$ is a further relevant information. Micro- and macroturbulence are limiting cases $l \to 0$ and $l \to \infty$ respectively.

3) If the limit $l \to 0$ is taken literally the spectroscopists concept of turbulence would be in conflict with basic principles of hydrodynamics. Hence an approach with finite $l$ -Mesoturbulence- is needed.

4) Whereas in principle the radiative transfer depends on all correlations of the velocity field, the two-point correlation, which contains most of the coherence properties of the field, seems to provide a sensible first order approximation.

5) The formalism of the mesoturbulent radiative transfer in this approximation is simple. It may be interpreted correctly in terms of scattering in velocity space.

6) Efficient numerical methods exist for the solution of the resulting equations.

7) The possibility to extend the mesoturbulence formalism to non-LTE problems has not been touched upon (see Gail et al. 1975, Frisch and Frisch 1976, Traving 1976).

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