Appendix 2

Homogeneous Lorentz transformations and their representations

We present here a brief discussion of the homogeneous Lorentz transformations and some of their finite dimensional representation matrices.

A2.1 The finite-dimensional representations

The generators of rotations $\hat{J}_i$ and boosts $\hat{K}_i$, introduced in (1.2.1), can be shown (see, for example, Gasiorowicz, 1967) to satisfy the following commutation relations:

$$\begin{aligned} \left[\hat{J}_j, \hat{J}_k\right] &= i\epsilon_{jkl}\hat{J}_l \\ \left[\hat{J}_j, \hat{K}_k\right] &= i\epsilon_{jkl}\hat{K}_l \\ \left[\hat{K}_j, \hat{K}_l\right] &= -i\epsilon_{jkl}\hat{J}_l. \end{aligned} \tag{A2.1}$$

If we now define

$$\hat{\mathbf{A}} \equiv \tfrac{1}{2}\left(\hat{\mathbf{J}} + i\hat{\mathbf{K}}\right) \qquad \hat{\mathbf{B}} \equiv \tfrac{1}{2}\left(\hat{\mathbf{J}} - i\hat{\mathbf{K}}\right) \tag{A2.2}$$

then $\hat{\mathbf{A}}, \hat{\mathbf{B}}$ behave like angular momentum operators, and they commute with each other:

$$\begin{aligned} \left[\hat{A}_j, \hat{A}_k\right] &= i\epsilon_{jkl}\hat{A}_l \\ \left[\hat{B}_j, \hat{B}_k\right] &= i\epsilon_{jkl}\hat{B}_l \\ \left[\hat{A}_j, \hat{B}_k\right] &= 0. \end{aligned} \tag{A2.3}$$

The most general Lorentz transformation is of the form

$$U(\boldsymbol{\vartheta}, \boldsymbol{\alpha}) = \exp\left(-i\boldsymbol{\vartheta}\cdot\hat{\mathbf{J}} - i\boldsymbol{\alpha}\cdot\hat{\mathbf{K}}\right). \tag{A2.4}$$

The vector $\boldsymbol{\vartheta}$ specifies a positive rotation through angle θ about an axis along $\boldsymbol{\vartheta}$. The vector $\boldsymbol{\alpha}$ specifies a pure boost of speed $\beta = \tanh\alpha$ along the direction of $\boldsymbol{\alpha}$.

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Equation (A2.4) can be rewritten as

$$U(\vartheta, \boldsymbol{\alpha}) = \exp\left[-\hat{\mathbf{A}} \cdot (\boldsymbol{\alpha} + i\boldsymbol{\vartheta}) + \hat{\mathbf{B}} \cdot (\boldsymbol{\alpha} - i\boldsymbol{\vartheta})\right]$$
$$= \exp\left[-i\hat{\mathbf{A}} \cdot (\boldsymbol{\vartheta} - i\boldsymbol{\alpha})\right] \exp\left[-i\hat{\mathbf{B}} \cdot (\boldsymbol{\vartheta} + i\boldsymbol{\alpha})\right] \qquad \text{(A2.5)}$$

the last step following because the $\hat{A}_j$ commute with the $\hat{B}_j$.

As discussed in Appendix 1, the $(2l + 1)$-dimensional representation matrices of the rotation group are the matrix elements of the rotation operator. Here we are not using the Euler angles to specify the rotation, but that is irrelevant. The matrix $\mathscr{D}^{(l)}_{m'm}(\boldsymbol{\vartheta})$ representing the rotation operator

$$U[(r(\boldsymbol{\vartheta})] \equiv e^{-i\hat{\mathbf{J}}\cdot\boldsymbol{\vartheta}}$$

is given by

$$\mathscr{D}^{(l)}_{m'm}(\boldsymbol{\vartheta}) = \left\langle j, m' | e^{-i\hat{\mathbf{J}}\cdot\boldsymbol{\vartheta}} | j, m \right\rangle \qquad \text{(A2.6)}$$

with $-j \leq m, m' \leq j$ and $j =$ integer or half-integer.

From (A2.5) and (A2.6) we see that we can represent the Lorentz transformation $U(\boldsymbol{\vartheta}, \boldsymbol{\alpha})$ by the $(2A + 1)(2B + 1)$-dimensional matrix

$$\mathscr{D}^{(A,B)}_{a'b',ab}(\boldsymbol{\vartheta}, \boldsymbol{\alpha}) \equiv \mathscr{D}^{(A)}_{a'a}(\boldsymbol{\vartheta} - i\boldsymbol{\alpha})\mathscr{D}^{(B)}_{b'b}(\boldsymbol{\vartheta} + i\boldsymbol{\alpha}) \qquad \text{(A2.7)}$$

where A, B are integer or half-integer, $-A \leq a, a' \leq A$, $-B \leq b, b' \leq B$. Note that the operators $\hat{\mathbf{A}}$, $\hat{\mathbf{B}}$ are here represented by hermitian matrices $\mathscr{D}^{(A,B)}$, so these matrices are only unitary if $\boldsymbol{\beta} = 0$, i.e. for pure rotations. Generally they are not irreducible for pure rotations; they behave like the product of representations of spin $A \otimes$ spin B.

It is clear from the product structure of (A2.7) and from the theory of addition of angular momentum that if we take the direct product of two representations (A_1, B_1) and (A_2, B_2) then the Clebsch–Gordan decomposition will be of the general form

$$\begin{aligned}(A_1, B_1) \otimes (A_2, B_2) = {} & (A_1 + A_2, B_1 + B_2) \oplus (A_1 + A_2 - 1, B_1 + B_2) \oplus \\ & \cdots \oplus (|A_1 - A_2|, B_1 + B_2) \oplus (A_1 + A_2, B_1 + B_2 - 1) \\ & \cdots \oplus (A_1 + A_2, |B_1 - B_2|) \\ & \cdots \oplus (|A_1 - A_2|, |B_1 - B_2|).\end{aligned} \qquad \text{(A2.8)}$$

Perhaps the simplest representations are $(s, 0)$ and $(0, s)$, where

$$\mathscr{D}^{(s,0)}_{a'b',ab}(\boldsymbol{\vartheta}, \boldsymbol{\alpha}) = \delta_{b'b}\mathscr{D}^{(s)}_{a'a}(\boldsymbol{\vartheta} - i\boldsymbol{\alpha}). \qquad \text{(A2.9)}$$

Clearly the b, b' labels are irrelevant and we may use

$$\mathscr{D}^{(s,0)}_{a'a}(\boldsymbol{\vartheta}, \boldsymbol{\alpha}) = \mathscr{D}^{(s)}_{a'a}(\boldsymbol{\vartheta} - i\boldsymbol{\alpha}). \qquad \text{(A2.10)}$$

Similarly we may take

$$\mathscr{D}^{(0,s)}_{b'b}(\boldsymbol{\vartheta},\boldsymbol{\alpha}) = \mathscr{D}^{(s)}_{b'b}(\boldsymbol{\vartheta} + i\boldsymbol{\alpha}). \tag{A2.11}$$

Note that from (A2.6) and (A2.7) that

$$\mathscr{D}^{(0,s)}(l) = \mathscr{D}^{(s,0)}(l^{-1})^{\dagger} = \left[\mathscr{D}^{(s,0)}(l)^{\dagger}\right]^{-1} \tag{A2.12}$$

for an arbitrary Lorentz transformation l.

Now, as mentioned in subsection 2.4.2, for a pure rotation the complex conjugate representation $\mathscr{D}^{(s)^*}$ is equivalent to $\mathscr{D}^{(s)}$ ($\mathscr{D}^{(s)^*} \approx \mathscr{D}^{(s)}$), i.e. there exists a unitary matrix C, which depends on s but not upon the parameters of the rotation, such that

$$\mathscr{D}^{(s)^*}(\boldsymbol{\vartheta}) = C\mathscr{D}^{(s)}(\boldsymbol{\vartheta})C^{-1} \tag{A2.13}$$

with $C^*C = (-1)^{2s}$ and $C^{\dagger}C = 1$. Conventionally one takes

$$C_{\lambda\lambda'} = (-1)^{s-\lambda}\delta_{\lambda,-\lambda'} \tag{A2.14}$$

Then from (A2.7) one can see that

$$\mathscr{D}^{(A,B)^*}(\boldsymbol{\vartheta},\boldsymbol{\alpha}) = C\mathscr{D}^{(B,A)}(\boldsymbol{\vartheta},\boldsymbol{\alpha})C^{-1} \tag{A2.15}$$

where here C is a direct product of the matrices in (A2.14):

$$C_{a'b',ab} = C_{a'a}C_{b'b}. \tag{A2.16}$$

In particular $\mathscr{D}^{(0,s)^*}$ is equivalent to $\mathscr{D}^{(s,0)}$.

A2.2 Spinors

The case of $s = 1/2$ is especially important, because of its relevance to the Dirac equation and the spinor calculus. There are four sets of 2×2 representation matrices of interest: $\mathscr{D}^{(1/2,0)}$; $\mathscr{D}^{(0,1/2)^*}$, which is equivalent to $\mathscr{D}^{(1/2,0)}$; $\mathscr{D}^{(0,1/2)}$; and $\mathscr{D}^{(1/2,0)^*}$ which is equivalent to $\mathscr{D}^{(0,1/2)}$. It is easy to check that (A2.13) and (A2.14) correspond to

$$(i\sigma_2)\mathscr{D}^{(1/2,0)}(i\sigma_2)^{-1} = \mathscr{D}^{(0,1/2)^*} \tag{A2.17}$$

Since we shall only discuss $s = 1/2$ it is conventional to define

$$\mathscr{D} \equiv \mathscr{D}^{(1/2,0)} \tag{A2.18}$$

and then to introduce

$$D_a{}^b \equiv \mathscr{D}_{ab} \equiv \mathscr{D}^{(1/2,0)}_{ab} \tag{A2.19}$$

$$D_{\dot a}{}^{\dot b} \equiv \mathscr{D}^{(1/2,0)^*}_{ab} = \mathscr{D}^*_{ab} \tag{A2.20}$$

i.e. a 'dot' on a row or column label signifies use of the complex conjugate representation.

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We can then define two kinds of two-component spinors χ_a and $\chi_{\dot{a}}$ such that if the reference frame undergoes some Lorentz transformation, then the components of the spinors in the transformed frame are, analogously to (1.1.15),

$$\chi'_a = D_a{}^b \chi_b \tag{A2.21}$$

$$\chi'_{\dot{a}} = D_{\dot{a}}{}^{\dot{b}} \chi_{\dot{b}} \tag{A2.22}$$

where we have used the shorthand notation χ'_a for $(\chi_a)_{S'}$ used in Chapter 1.

One can introduce a kind of 'metric spinor'

$$\epsilon^{ab} = \epsilon_{ab} = (i\sigma_2)_{ab}$$

$$= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{A2.23}$$

$$= \epsilon^{\dot{a}\dot{b}} = \epsilon_{\dot{a}\dot{b}} \tag{A2.24}$$

and then define the 'contravariant' spinors

$$\chi^a = \epsilon^{ab} \chi_b \tag{A2.25}$$

and

$$\chi^{\dot{a}} = \epsilon^{\dot{a}\dot{b}} \chi_{\dot{b}}. \tag{A2.26}$$

Note that the inverse of (A2.25), for example, is

$$\chi_a = -\epsilon_{ab} \chi^b = \epsilon_{ba} \chi^b \tag{A2.27}$$

since

$$\epsilon_{ab} \epsilon^{bc} = -\delta^c_a. \tag{A2.28}$$

The minus sign in (A2.28) has the peculiar effect that if $\boldsymbol{\chi}$ and $\boldsymbol{\eta}$ are two spinors then

$$\chi^\alpha \eta_\alpha = -\chi_\alpha \eta^\alpha. \tag{A2.29}$$

Now using (A2.21) and (A2.25) one finds

$$\chi^{a'} = \mathscr{D}^{(0,1/2)^*}_{ab} \chi^b. \tag{A2.30}$$

Conventionally one defines

$$D^a{}_b \equiv \mathscr{D}^{(0,1/2)^*}_{ab} \tag{A2.31}$$

so that (A2.30) becomes

$$\chi^{a'} = D^a{}_b \chi^b \tag{A2.32}$$

and from (A2.12)

$$D^a{}_b = \left[\left(\mathscr{D}^{-1} \right)^T \right]_{ab}. \tag{A2.33}$$

Similarly, for (A2.26), under transformation of the reference frame

$$\chi^{\dot{a}'} = \mathscr{D}^{(0,1/2)}_{ac}\chi^{\dot{c}}. \tag{A2.34}$$

One defines

$$D^{\dot{a}}{}_{\dot{b}} \equiv \mathscr{D}^{(0,1/2)}_{ab} \tag{A2.35}$$

so that (A2.34) reads

$$\chi^{\dot{a}'} = D^{\dot{a}}{}_{\dot{b}}\chi^{\dot{b}} \tag{A2.36}$$

and by (A2.12)

$$D^{\dot{a}}{}_{\dot{b}} = \left[\left(\mathscr{D}^{-1}\right)^{\dagger}\right]_{ab}. \tag{A2.37}$$

Let us summarize the transformation laws for the various two-component spinors introduced:

$$\begin{aligned} \chi_a &: \quad \mathscr{D}^{(1/2,0)} \\ \chi_{\dot{a}} &: \quad \mathscr{D}^{(1/2,0)^*} \approx \mathscr{D}^{(0,1/2)} \\ \chi^a &: \quad \mathscr{D}^{(0,1/2)^*} \approx \mathscr{D}^{(1/2,0)} \\ \chi^{\dot{a}} &: \quad \mathscr{D}^{(0,1/2)}. \end{aligned} \tag{A2.38}$$

An important question is how to form invariants from these. The Clebsch–Gordan decomposition (A2.8) tells us that both $(1/2, 0) \otimes (1/2, 0)$ and $(0, 1/2) \otimes (0, 1/2)$ will contain the invariant representation $(0, 0)$.

Hence if χ_a and η_b are spinors of type $(1/2, 0)$ then we expect some linear combination $f^{ab}\chi_a\eta_b$ to be invariant. In fact the combination is just

$$\epsilon^{ab}\chi_a\eta_b = \chi_a\eta^a \tag{A2.39}$$

since

$$\begin{aligned} \chi'_a\eta^{a'} &= D_a{}^b D^a{}_c \chi_b\eta^c \\ &= \mathscr{D}_{ab}\left[\left(\mathscr{D}^{-1}\right)^T\right]_{ac}\chi_b\eta^c = \chi_a\eta^a \end{aligned} \tag{A2.40}$$

i.e. it is indeed invariant.

Similarly

$$\epsilon^{\dot{a}\dot{b}}\chi_{\dot{a}}\eta_{\dot{b}} = \chi_{\dot{a}}\eta^{\dot{a}} \tag{A2.41}$$

is invariant.

Finally, by using complex conjugation, we can build up an invariant out of spinors ξ_a of type $(1/2, 0)$ and $\zeta^{\dot{a}}$ of type $(0, 1/2)$. Namely, under transformation of the reference frame, writing (A2.21), (A2.36) and (A2.37) in matrix form, the spinors transform as

$$\xi' = \mathscr{D}\xi \qquad \text{and} \qquad \zeta' = (\mathscr{D}^{-1})^{\dagger}\zeta$$

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so that

$$\zeta'^{\dagger}\xi' = \zeta^{\dagger}\xi \tag{A2.42}$$

i.e. is invariant.

A2.3 Connection between spinor and vector representations

Let A^{μ} be a 4-vector. Under a Lorentz transformation l applied to the reference frame, the components of A^{μ} in the transformed frame are (see (1.2.14))

$$A^{\mu\prime} = \Lambda^{\mu}{}_{\nu}(l^{-1})A^{\nu} \tag{A2.43}$$

where $A^{\mu\prime}$ is short for $(A^{\mu})_{S_l}$.

The $\Lambda^{\mu}{}_{\nu}$ are the transformation matrices for the *vector representation* and are the basic blocks for building up tensor representations, the latter being generally reducible.

We shall now demonstrate that the representation $\mathscr{D}^{(1/2,1/2)}$ is equivalent to the vector representation. This is a result of great importance since it gives a fundamental connection between spinors and 4-vectors.

Firstly, from the form of Clebsch–Gordan decomposition (A2.8) we have that

$$\mathscr{D}^{(1/2,0)} \otimes \mathscr{D}^{(0,1/2)} = \mathscr{D}^{(1/2,1/2)}. \tag{A2.44}$$

But from (A2.15), $\mathscr{D}^{(0,1/2)}$ is equivalent to $\mathscr{D}^{(1/2,0)^*}$. Hence

$$\mathscr{D}^{(1/2,0)^*} \otimes \mathscr{D}^{(1/2,0)} \approx \mathscr{D}^{(1/2,1/2)}. \tag{A2.45}$$

We thus need to show that transformation under the left-hand side of (A2.45) is equivalent to the vector transformation. Hence if ξ is a two-component spinor of type $(1/2,0)$ we need to prove the existence of a set of coefficients $C^{\mu ab}$ such that $C^{\mu ab}\xi_a^*\xi_b$ transforms like a vector. But it is well known that if one adds the two-dimensional unit matrix to a set of Pauli matrices to form

$$\sigma^{\mu} = (I, \boldsymbol{\sigma}) \tag{A2.46}$$

then

$$V^{\mu} \equiv \xi^{\dagger}\sigma^{\mu}\xi \tag{A2.47}$$

transforms as a 4-vector, i.e.

$$V^{\mu\prime} = \xi'^{\dagger}\sigma^{\mu}\xi' = \Lambda^{\mu}{}_{\nu}V^{\nu}. \tag{A2.48}$$

This is easily shown for rotations or pure boosts upon using

$$e^{i\boldsymbol{\vartheta}\cdot\boldsymbol{\sigma}/2} = \cos\theta/2 + i\hat{\boldsymbol{\vartheta}}\cdot\boldsymbol{\sigma}\sin\theta/2 \tag{A2.49}$$

and

$$e^{\boldsymbol{\alpha}\cdot\boldsymbol{\sigma}/2} = \cosh\alpha/2 + \hat{\boldsymbol{\alpha}}\cdot\boldsymbol{\sigma}\sinh\alpha/2. \qquad \text{(A2.50)}$$

Of course we can write (A2.47) in the form

$$V^{\mu} = (\sigma^{\mu})_{ab}\,\xi_a^*\xi_b, \qquad \text{(A2.51)}$$

which casts an interesting new light on the matrices σ^{μ}. The elements $(\sigma^{\mu})_{ab}$ are the elements of the transformation matrix from the $(1/2,0)^*\otimes(1/2,0) \approx (1/2,1/2)$ representation to the equivalent usual 4-vector representation.

Note that we have been a little cavalier with the group-theoretical aspects. Strictly speaking, the representations (A,B) with which we have been dealing are representations of the group $SL(2,c)$, whereas the 4-vector representation is the vector representation of the group $O(1,3)$.

For a detailed discussion of the spinor calculus and its use in constructing relativistic wave equations the reader is referred to Carruthers (1971), where there is also a treatment of the *unitary* (hence, infinite-dimensional) representations of the homogeneous Lorentz group. For applications to supersymmetry see Sohnius (1985).