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The Sebastiani–Thom Isomorphism in the Derived Category

DAVID B. MASSEY
Department of Mathematics, Northeastern University, Boston, MA, 02115, U.S.A.
e-mail: dmassey@neu.edu

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Abstract. We prove that the Sebastiani–Thom isomorphism for Milnor fibres and their monodromies exists as a natural isomorphism between vanishing cycles in the derived category.

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Key words. Milnor fibre, vanishing cycles, derived category, Sebastiani–Thom, perverse sheaves

0. Introduction

Let $f\colon X \to \mathbb{C}$ and $g\colon Y \to \mathbb{C}$ be complex analytic functions. Let π_1 and π_2 denote the projections of $X \times Y$ onto X and Y, respectively. In [S-T], Sebastiani and Thom prove that the cohomology of the Milnor fibre of $f \circ \pi_1 + g \circ \pi_2$ is isomorphic to the tensor product of the cohomologies of the Milnor fibres of f and g (with a shift in degrees); they prove this in the case where X and Y are smooth and f and g have isolated critical points. In addition, they prove that the monodromy isomorphism induced by $f \circ \pi_1 + g \circ \pi_2$ is the tensor product of those induced by f and g. The point, of course, is to break up the complicated critical activity of $f \circ \pi_1 + g \circ \pi_2$ into more manageable pieces.

Sebastiani–Thom-type results have been proved on smooth spaces with constant coefficients by Némethi [N1], [N2], Oka [O], and Sakamoto [S]. The work of Némethi includes global results on affine space for polynomial functions. There are also recent Sebastiani–Thom results when the underlying spaces have isolated singularities and/or when there is more structure present than merely the cohomology; see [E-S], [N-Sa], and [N-St].

In this paper, we prove that this Sebastiani–Thom isomorphism exists regardless of how singular the spaces X and Y may be, regardless of how bad the critical loci of f and g are, and regardless of what coefficients one uses. Moreover, we prove that the Sebastiani–Thom isomorphism is actually a consequence of a natural isomorphism in the derived category of bounded, constructible complexes of sheaves on $X \times Y$.

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To state our result precisely, we must introduce more notation – all of which can be found in [K-S].

Let R be a regular Noetherian ring with finite Krull dimension (e.g., $\mathbb{Z}$, $\mathbb{Q}$, or $\mathbb{C}$). Let $\mathbf{A}^\bullet$ and $\mathbf{B}^\bullet$ be bounded, constructible complexes of sheaves of R-modules on X and Y, respectively. Recall that, in this situation,

$$\mathbf{A}^\bullet \overset{L}{\boxtimes} \mathbf{B}^\bullet := \pi_1^*\mathbf{A}^\bullet \overset{L}{\otimes} \pi_2^*\mathbf{B}^\bullet.$$

Let us adopt the similar notation $f \boxplus g := f \circ \pi_1 + g \circ \pi_2$.

Let p_1 and p_2 denote the projections of $V(f) \times V(g)$ onto $V(f)$ and $V(g)$, respectively, and let k denote the inclusion of $V(f) \times V(g)$ into $V(f \boxplus g)$.

If $h\colon Z \to \mathbb{C}$ is an analytic function, and $\mathbf{F}^\bullet$ is a complex on Z, then $\phi_h\mathbf{F}^\bullet$ denotes the sheaf of vanishing cycles of $\mathbf{F}^\bullet$ along h. Here, $\phi_h\mathbf{F}^\bullet$ is defined as in 8.6.2 of [K-S] and, hence, is shifted by 1 from the more traditional definitions, i.e., in this paper, the stalk cohomology of $\phi_h\mathbf{F}^\bullet$ in degree i is the degree i relative hypercohomology of a small ball modulo the Milnor fibre with coefficients in $\mathbf{F}^\bullet$. Thus, in the constant $\mathbb{Z}$-coefficient case, $H^i(\phi_h\mathbb{Z}^\bullet_X)_\mathbf{x} \cong \widetilde{H}^{i-1}(F_{h,\mathbf{x}})$, where $\widetilde{H}$ denotes reduced cohomology and $F_{h,\mathbf{x}}$ denotes the Milnor fibre of h at $\mathbf{x}$.

We prove the following theorem:

THEOREM (Sebastiani–Thom Isomorphism). *There is a natural isomorphism*

$$k^*\phi_{f\boxplus g}\big(\mathbf{A}^\bullet \overset{L}{\boxtimes} \mathbf{B}^\bullet\big) \;\cong\; \phi_f\mathbf{A}^\bullet \overset{L}{\boxtimes} \phi_g\mathbf{B}^\bullet,$$

and this isomorphism commutes with the corresponding monodromies.

Moreover, if we let $\mathbf{p} := (\mathbf{x}, \mathbf{y}) \in X \times Y$ *be such that* $f(\mathbf{x}) = 0$ *and* $g(\mathbf{y}) = 0$*, then, in an open neighborhood of* $\mathbf{p}$*, the complex* $\phi_{f\boxplus g}\big(\mathbf{A}^\bullet \overset{L}{\boxtimes} \mathbf{B}^\bullet\big)$ *has support contained in* $V(f) \times V(g)$*, and, in any open set in which we have this containment, there are natural isomorphisms*

$$\phi_{f\boxplus g}\big(\mathbf{A}^\bullet \overset{L}{\boxtimes} \mathbf{B}^\bullet\big) \;\cong\; k_!(\phi_f\mathbf{A}^\bullet \overset{L}{\boxtimes} \phi_g\mathbf{B}^\bullet) \;\cong\; k_*(\phi_f\mathbf{A}^\bullet \overset{L}{\boxtimes} \phi_g\mathbf{B}^\bullet).$$

1. Proof of the General Sebastiani–Thom Isomorphism

The proof of the Sebastiani–Thom isomorphism uses two Morse-theoretic lemmas and two formal, derived category propositions. First, however, we need to discuss the definition of the vanishing cycles that we shall use.

Kashiwara and Schapira define the vanishing cycles in 8.6.2 of [K-S]. However, we shall use the more down-to-earth characterization (via natural equivalence) given in Exercise VIII.13 of [K-S]. Hence, we use as our definition: $\phi_h\mathbf{F}^\bullet = \big(R\Gamma_{\{\mathrm{Re}\,h \leqslant 0\}}\mathbf{F}^\bullet\big)_{|V(h)}$. (We have reversed the inequality used in [K-S]. We do this for aesthetics only – we prefer to think of the vanishing cycles in terms of a ball modulo the Milnor fibre over a small *positive* value of the function.) The monodromy isomorphism is

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easy to describe: for all θ, there is an isomorphism $\left(R\Gamma_{\{\mathrm{Re}\,h\leqslant 0\}}\mathbf{F}^\bullet\right)_{|_{V(h)}} \cong \left(R\Gamma_{\{\mathrm{Re}\,e^{-i\theta}h\leqslant 0\}}\mathbf{F}^\bullet\right)_{|_{V(h)}}$, and the monodromy isomorphism results when $\theta = 2\pi$.

We continue with our notation from the introduction.

LEMMA 1.1. *Let $\mathcal{S}$ (resp. $\mathcal{S}'$) denote a complex Whitney stratification of X (resp. Y) with respect to which $\mathbf{A}^\bullet$ (resp. $\mathbf{B}^\bullet$) is constructible. Let $\Sigma_{\mathcal{S}}f$ and $\Sigma_{\mathcal{S}'}g$ denote the stratified critical loci. Then,*

$$\mathrm{supp}\phi_{f\boxplus g}\left(\mathbf{A}^\bullet \overset{L}{\boxtimes} \mathbf{B}^\bullet\right) \subseteq \left(\Sigma_{\mathcal{S}}f \times \Sigma_{\mathcal{S}'}g\right) \cap V(f \boxplus g);$$

in particular, if $\Sigma_{\mathcal{S}'}g \subseteq V(g)$, then

$$\mathrm{supp}\phi_{f\boxplus g}\left(\mathbf{A}^\bullet \overset{L}{\boxtimes} \mathbf{B}^\bullet\right) \subseteq V(f) \times V(g).$$

Moreover, if $\mathbf{p} := (\mathbf{x}, \mathbf{y}) \in X \times Y$ is such that $f(\mathbf{x}) = 0$ and $g(\mathbf{y}) = 0$, then, near $\mathbf{p}$,

$$\mathrm{supp}\phi_{f\boxplus g}\left(\mathbf{A}^\bullet \overset{L}{\boxtimes} \mathbf{B}^\bullet\right) \subseteq V(f) \times V(g).$$

Proof. As the complexes $\mathbf{A}^\bullet$ and $\mathbf{B}^\bullet$ are constructible with respect to $\mathcal{S}$ and $\mathcal{S}'$, $\mathbf{A}^\bullet \overset{L}{\boxtimes} \mathbf{B}^\bullet$ is constructible with respect to the product stratification, $\mathcal{S} \times \mathcal{S}'$. The support of $\phi_{f\boxplus g}\left(\mathbf{A}^\bullet \overset{L}{\boxtimes} \mathbf{B}^\bullet\right)$ is contained in the stratified critical locus of $f \boxplus g$, which is trivially seen to be equal to the product of the stratified critical loci of f and g. Finally, near $\mathbf{x}$ and $\mathbf{y}$, these stratified critical loci are contained in $V(f)$ and $V(g)$, respectively. □

Recall that $k\colon V(f) \times V(g) \hookrightarrow V(f \boxplus g)$ denotes the inclusion. Let $q\colon V(f \boxplus g) \hookrightarrow X \times Y$ and $m\colon V(f) \times V(g) \hookrightarrow X \times Y$ also denote the inclusions, so that $m = q \circ k$.

LEMMA 1.2. *Let $P := \{\mathbf{x} \in X \mid \mathrm{Re}\,f(\mathbf{x}) \leqslant 0\}$, let $Q := \{\mathbf{y} \in Y \mid \mathrm{Re}\,g(\mathbf{y}) \leqslant 0\}$, and let $Z := \{(\mathbf{x}, \mathbf{y}) \in X \times Y \mid \mathrm{Re}\,(f(\mathbf{x}) + g(\mathbf{y})) \leqslant 0\}$. Note that $P \times Q \subseteq Z$.*

The natural map

$$R\Gamma_{P\times Q}(\mathbf{A}^\bullet \overset{L}{\boxtimes} \mathbf{B}^\bullet) \to R\Gamma_Z(\mathbf{A}^\bullet \overset{L}{\boxtimes} \mathbf{B}^\bullet)$$

induces a natural isomorphism

$$m^*R\Gamma_{P\times Q}(\mathbf{A}^\bullet \overset{L}{\boxtimes} \mathbf{B}^\bullet) \cong m^*R\Gamma_Z(\mathbf{A}^\bullet \overset{L}{\boxtimes} \mathbf{B}^\bullet) \cong k^*\phi_{f\boxplus g}\left(\mathbf{A}^\bullet \overset{L}{\boxtimes} \mathbf{B}^\bullet\right).$$

Proof. As in the first lemma, we use nothing about $\mathbf{A}^\bullet \overset{L}{\boxtimes} \mathbf{B}^\bullet$ other than the fact that it is constructible with respect to the product stratification; let us use $\mathbf{F}^\bullet$ to denote $\mathbf{A}^\bullet \overset{L}{\boxtimes} \mathbf{B}^\bullet$.

From the definition of the vanishing cycles, we have

$$m^*R\Gamma_Z(\mathbf{F}^\bullet) = (q \circ k)^*R\Gamma_Z(\mathbf{F}^\bullet) \cong k^*q^*R\Gamma_Z(\mathbf{F}^\bullet) \cong k^*\phi_{f\boxplus g}(\mathbf{F}^\bullet).$$

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Let $\mathbf{p} \in V(f) \times V(g)$. We need to show that we have the isomorphism

$$H^*(R\Gamma_{P\times Q}(\mathbf{F}^\bullet))_{\mathbf{p}} \xrightarrow{\cong} H^*(R\Gamma_Z(\mathbf{F}^\bullet))_{\mathbf{p}}.$$

Let $\Theta\colon X \times Y \to \mathbb{R}^2$ be given by $\Theta(\mathbf{x}, \mathbf{y}) := (\mathrm{Re} f(\mathbf{x}), \mathrm{Re}\, g(\mathbf{y}))$. Use u and v for the coordinates in $\mathbb{R}^2$. Let $C := \{(u, v) \mid u \leqslant 0, v \leqslant 0\}$, and $D := \{(u, v) \mid u + v \leqslant 0\}$. What we need to show is that we have an isomorphism

$$H^*(R\Gamma_{\Theta^{-1}(C)}(\mathbf{F}^\bullet))_{\mathbf{p}} \xrightarrow{\cong} H^*(R\Gamma_{\Theta^{-1}(D)}(\mathbf{F}^\bullet))_{\mathbf{p}}. \qquad (\dagger)$$

In a small neighborhood of $\mathbf{p}$, the map Θ is a stratified submersion over the complement of $\{(u, v) \mid uv = 0\}$ (the coordinate-axes); for a critical point of $\mathrm{Re} f$ (resp. $\mathrm{Re}\, g$) on a stratum occurs at a critical point of f (resp. g) on the stratum. The desired result will follow by *moving the wall* (see [G-M2]); essentially, one deforms the region D to the region C by the obvious homotopy, and lifts this deformation up to $X \times Y$ via Θ.

To avoid the critical values along the u and v axes, it is slightly easier to work with the complements of C and D. Note that, since we have the morphism of distinguished triangles

$$\begin{array}{ccccccc}
\to R\Gamma_{\Theta^{-1}(C)}(\mathbf{F}^\bullet) & \to & \mathbf{F}^\bullet & \to & R\Gamma_{\Theta^{-1}(\mathbb{R}^2 - C)}(\mathbf{F}^\bullet) & \xrightarrow{[1]} \\
\downarrow & & \downarrow & & \downarrow & \\
\to R\Gamma_{\Theta^{-1}(D)}(\mathbf{F}^\bullet) & \to & \mathbf{F}^\bullet & \to & R\Gamma_{\Theta^{-1}(\mathbb{R}^2 - D)}(\mathbf{F}^\bullet) & \xrightarrow{[1]}
\end{array}$$

proving $(\dagger)$ is equivalent to proving the isomorphism

$$H^*(R\Gamma_{\Theta^{-1}(\mathbb{R}^2 - C)}(\mathbf{F}^\bullet))_{\mathbf{p}} \xrightarrow{\cong} H^*(R\Gamma_{\Theta^{-1}(\mathbb{R}^2 - D)}(\mathbf{F}^\bullet))_{\mathbf{p}}.$$

Therefore, it suffices to produce a fundamental system $\{N_i\}$ of open neighborhoods of $\mathbf{p}$ such that we have induced isomorphisms

$$\mathbb{H}^*(N_i - \Theta^{-1}(C);\ \mathbf{F}^\bullet) \xrightarrow{\cong} \mathbb{H}^*(N_i - \Theta^{-1}(D);\ \mathbf{F}^\bullet). \qquad (*)$$

Let $E := \{u \geqslant 0, v \geqslant 0\}$. To show $(*)$ via moving the wall, we will produce N_i such that $\Theta_{|N_i}$ is a stratum-preserving, locally trivial fibration over $\mathbb{R}^2 - C - E$; as $\mathbb{R}^2 - C - E$ consists of two contractible pieces, this will imply that the obvious homotopy from $\mathbb{R}^2 - C$ to $\mathbb{R}^2 - D$ lifts to give us $(*)$.

Take local embeddings of X and Y into affine spaces. Let B_ε denote a closed ball of radius ε centered at $\pi_1(\mathbf{p})$ intersected with X, and let B_δ denote a closed ball of radius δ centered at $\pi_2(\mathbf{p})$ intersected with Y. Let $\overset{\circ}{B}_\varepsilon$ and $\overset{\circ}{B}_\delta$ denote the intersections of the associated open balls with X and Y, respectively. For positive η, let T_η denote the open square in $\mathbb{C}$ given by $T_\eta = \{z \mid |\mathrm{Re}\, z| < \eta,\ |\mathrm{Im}\, z| < \eta\}$. We claim that

$$N_i := \left(\overset{\circ}{B}_{\varepsilon_i} \times \overset{\circ}{B}_{\delta_i} \right) \cap (f \circ \pi_1)^{-1}(T_{\alpha_i}) \cap (g \circ \pi_2)^{-1}(T_{\beta_i}),$$

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where $\alpha_i \ll \varepsilon_i$ and $\beta_i \ll \delta_i$ is a fundamental system of open neighborhoods for which $(*)$ holds.

To see this, endow $B_{\varepsilon_i} \times B_{\delta_i}$ with the obvious Whitney stratification, and consider the map $\Omega_i\colon B_{\varepsilon_i} \times B_{\delta_i} \to \mathbb{C}^2$ given by $\Omega_i := (f \circ \pi_1, g \circ \pi_2)$. Let a and b denote the coordinates in $\mathbb{C}^2$. The stratified critical points of Ω_i occur at points $(\mathbf{x}, \mathbf{y})$ where either $\mathbf{x}$ is in the stratified critical locus of $f_{|_{B_{\varepsilon_i}}}$ or $\mathbf{y}$ is in the stratified critical locus of $g_{|_{B_{\delta_i}}}$; the standard Milnor fibration argument guarantees that, near $\mathbf{p}$, in a small neighborhood of $0 \in \mathbb{C}^2$, the stratified critical values occur only along $V(ab)$. Therefore, for α_i and β_i sufficiently small, Ω_i is a proper stratified submersion over $T_{\alpha_i} \times T_{\beta_i} - V(ab)$. As $\Theta_{|_{N_i}} = \operatorname{Re}(\Omega_i)$, it follows immediately that $\Theta_{|_{N_i}}$ is a stratum-preserving, locally trivial fibration over $\mathbb{R}^2 - C - E$. □

In the next proposition, we refer to P and Q and, of course, later we will apply this proposition to the P and Q given in Lemma 1.2. However, in Proposition 1.3, P and Q are completely general.

PROPOSITION 1.3. *Let r_1 and r_2 denote the projections of $P \times Q$ onto P and Q, respectively. Let $l\colon P \times Q \to X \times Y$, $i\colon P \to X$, and $j\colon Q \to Y$ be such that $i \circ r_1 = \pi_1 \circ l$ and $i \circ r_2 = \pi_2 \circ l$, i.e., we have a commutative diagram*

$$\begin{array}{ccccc}
P & \xleftarrow{r_1} & P \times Q & \xrightarrow{r_2} & Q \\
\downarrow i & & \downarrow l & & \downarrow j \\
X & \xleftarrow{\pi_1} & X \times Y & \xrightarrow{\pi_2} & Y.
\end{array}$$

Then, there is a natural isomorphism

$$i^!\mathbf{A}^\bullet \overset{L}{\boxtimes} j^!\mathbf{B}^\bullet \;\cong\; l^!(\mathbf{A}^\bullet \overset{L}{\boxtimes} \mathbf{B}^\bullet).$$

Proof. We use 3.4.4 and 3.1.13 of [K-S].

$$i^!\mathbf{A}^\bullet \overset{L}{\boxtimes} j^!\mathbf{B}^\bullet \;\cong\; \mathcal{D}i^*\mathcal{D}\mathbf{A}^\bullet \overset{L}{\boxtimes} j^!\mathbf{B}^\bullet \;\cong\; R\mathrm{Hom}^\bullet(r_1^*i^*\mathcal{D}\mathbf{A}^\bullet, r_2^!j^!\mathbf{B}^\bullet) \;\cong$$
$$R\mathrm{Hom}^\bullet(l^*\pi_1^*\mathcal{D}\mathbf{A}^\bullet, l^!\pi_2^!\mathbf{B}^\bullet) \;\cong\; l^!R\mathrm{Hom}^\bullet(\pi_1^*\mathcal{D}\mathbf{A}^\bullet, \pi_2^!\mathbf{B}^\bullet) \;\cong\; l^!(\mathcal{D}\mathcal{D}\mathbf{A}^\bullet \overset{L}{\boxtimes} \mathbf{B}^\bullet)$$
$$\cong\; l^!(\mathbf{A}^\bullet \overset{L}{\boxtimes} \mathbf{B}^\bullet). \qquad \square$$

In the next proposition, we use repeatedly that if $t\colon T \hookrightarrow W$ is the inclusion of a closed subset and $\mathbf{F}^\bullet$ is a complex on W, then $R\Gamma_T(\mathbf{F}^\bullet)$ is naturally isomorphic to $t_!t^!\mathbf{F}^\bullet$; this follows from 3.1.12 of [K-S].

PROPOSITION 1.4. *We continue with the notation from the previous proposition, but we now assume that l, i, and j are inclusions of closed subsets. Then, there is*

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a natural isomorphism

$$R\Gamma_{P}(\mathbf{A}^\bullet)\overset{L}{\boxtimes} R\Gamma_{Q}(\mathbf{B}^\bullet) \;\cong\; R\Gamma_{P\times Q}(\mathbf{A}^\bullet \overset{L}{\boxtimes} \mathbf{B}^\bullet).$$

Proof. Let $\check{i}\colon P\times Y\hookrightarrow X\times Y$ denote the inclusion and let $\check{\pi}_1\colon P\times Y\to P$ denote the projection. Analogously, let $\check{j}\colon X\times Q\hookrightarrow X\times Y$ denote the inclusion and let $\check{\pi}_2\colon X\times Q\to Q$ denote the projection.

We have a natural isomorphism

$$R\Gamma_{P}(\mathbf{A}^\bullet)\overset{L}{\boxtimes} R\Gamma_{Q}(\mathbf{B}^\bullet) \;\cong\; \pi_1^* i_! i^! \mathbf{A}^\bullet \overset{L}{\otimes} \pi_2^* j_! j^! \mathbf{B}^\bullet.$$

Using the dual of 3.1.9 of [K-S], $\pi_1^* i_! \cong \check{i}_!\check{\pi}_1^*$ and $\pi_2^* j_! \cong \check{j}_!\check{\pi}_2^*$. Thus, we have

$$R\Gamma_{P}(\mathbf{A}^\bullet)\overset{L}{\boxtimes} R\Gamma_{Q}(\mathbf{B}^\bullet) \;\cong\; \check{i}_!\check{\pi}_1^* i^! \mathbf{A}^\bullet \overset{L}{\otimes} \check{j}_!\check{\pi}_2^* j^! \mathbf{B}^\bullet,$$

and by the Künneth formula (Exercise II.18.i of [K-S]), this last expression is naturally isomorphic to $l_!(i^!\mathbf{A}^\bullet \overset{L}{\boxtimes} j^!\mathbf{B}^\bullet)$. Apply Proposition 1.3. □

1.5 *Proof of the Sebastiani–Thom Isomorphism*. We will use all of our previous notation and results. Let $s_1\colon V(f)\hookrightarrow X$ and $s_2\colon V(g)\hookrightarrow Y$ denote the inclusions. Then,

$$\begin{aligned}
\phi_f\mathbf{A}^\bullet \overset{L}{\boxtimes} \phi_g\mathbf{B}^\bullet \;&=\; p_1^*\phi_f\mathbf{A}^\bullet \overset{L}{\otimes} p_2^*\phi_g\mathbf{B}^\bullet \;\cong\; p_1^*s_1^*R\Gamma_{P}(\mathbf{A}^\bullet)\overset{L}{\otimes} p_2^*s_2^*R\Gamma_{Q}(\mathbf{B}^\bullet)\\
&\cong m^*\pi_1^*R\Gamma_{P}(\mathbf{A}^\bullet)\overset{L}{\otimes} m^*\pi_2^*R\Gamma_{Q}(\mathbf{B}^\bullet) \cong\; m^*(R\Gamma_{P}(\mathbf{A}^\bullet)\overset{L}{\boxtimes} R\Gamma_{Q}(\mathbf{B}^\bullet))\\
&\cong m^*R\Gamma_{P\times Q}(\mathbf{A}^\bullet \overset{L}{\boxtimes} \mathbf{B}^\bullet) \;\cong\; k^*\phi_{f\boxplus g}\big(\mathbf{A}^\bullet \overset{L}{\boxtimes} \mathbf{B}^\bullet\big).
\end{aligned}$$

The remaining statements of the theorem – other than the monodromy statement follow from Lemma 1.1.

The monodromy statement follows at once from the proof of the Sebastiani–Thom isomorphism, for the monodromy of $f\boxplus g$ results from how the set $\{(\mathbf{x},\mathbf{y})\in X\times Y \mid \mathrm{Re}\,(\mathrm{e}^{-i\theta}(f(\mathbf{x})+g(\mathbf{y})))\leqslant 0\}$ moves as θ goes from 0 to 2π. The isomorphism in Lemma 1.2 identifies this with the movement of $\{\mathrm{e}^{-i\theta}\mathrm{Re}\,f(\mathbf{x})\leqslant 0\}\times\{\mathrm{e}^{-i\theta}\mathrm{Re}\,g(\mathbf{y})\leqslant 0\}$, which describes the product of the two monodromies of f and g. □

2. Consequences

2.1. The local situation

Certainly there is some satisfaction in knowing that the Sebastiani–Thom isomorphism holds for general spaces, even with constant $\mathbb{Z}$-coefficients and only

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on the level of cohomology; in this case, the isomorphism reduces to

$$\begin{aligned}
&\widetilde{H}^{i-1}(F_{f\boxplus g,\mathbf{p}})\\
&\cong \left[\bigoplus_{a+b=i}\left(\widetilde{H}^{a-1}(F_{f,\pi_1(\mathbf{p})})\otimes \widetilde{H}^{b-1}(F_{g,\pi_2(\mathbf{p})})\right)\right]\oplus\\
&\oplus\left[\bigoplus_{c+d=i+1}\operatorname{Tor}\left(\widetilde{H}^{c-1}(F_{f,\pi_1(\mathbf{p})}),\ \widetilde{H}^{d-1}(F_{g,\pi_2(\mathbf{p})})\right)\right],
\end{aligned}$$

where, as in the introduction, $\widetilde{H}$ denotes reduced, integral cohomology and $F_{h,\mathbf{x}}$ denotes the Milnor fibre of h at $\mathbf{x}$.

2.2. Problems with the global situation

There are, however, a number of difficulties in passing to global information, as is done in [N1] and [N2]. As we have proved an isomorphism of complexes of sheaves, applying hypercohomology certainly yields *some* global result, but is it what one wants? Then, there is the separate issue of considering the monodromy as one moves around *all* of the critical values, instead of merely moving around 0.

If we have a *proper* function $h\colon Z\to\mathbb{C}$ and $F^\bullet$ is a complex on Z, then, by patching together local data, one obtains that $\mathbb{H}^*(V(h);\ \phi_h F^\bullet)\cong\mathbb{H}^*(h^{-1}(\overset{\circ}{\mathbb{D}}),\ h^{-1}(v);\ F^\bullet)$, where $\overset{\circ}{\mathbb{D}}$ is a small disk around the origin in $\mathbb{C}$ and $v\in\overset{\circ}{\mathbb{D}}-\{0\}$. Moreover, if $\mathcal{S}$ is a Whitney stratification with respect to which $F^\bullet$ is constructible, and W is a union of strata of $\mathcal{S}$, then the same isomorphism of hypercohomologies holds for $h_{|_W}$.

Thus, if f and g are proper in sections 0 and 1, then by combining the Sebastiani–Thom isomorphism with the Künneth formula (see [K-S], Ex. II.18), we obtain a global Sebastiani–Thom isomorphism on hypercohomology. Moreover, even if f and g are not proper, but are obtained, along with $\mathbf{A}^\bullet$ and $\mathbf{B}^\bullet$, by restricting the proper situation to unions of strata, then we once again obtain the global isomorphism on hypercohomology.

Such proper extensions automatically exist in the case of polynomials on affine space, using the constant sheaf for coefficients; this is the setting of [N1] and [N2].

However, even if we assume that the entire situation compactifies nicely, we see no easy way of obtaining a result analogous to those of [N1] and [N2] about the monodromy around the entire set of critical values. There are (at least) two problems: the vanishing cycles, by definition, lie above a single critical value, and there is a substantial difference between the vanishing cycles and the nearby cycles when using non-constant coefficients. Némethi's monodromy results actually deal with the nearby cycles; however, a Sebastiani–Thom isomorphism where the vanishing cycles are replaced with the nearby cycles does not exist.

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2.3. Iterated suspensions

If we restrict ourselves to using coefficients in a field, say $\mathbb{Q}$ or $\mathbb{C}$. Then, when the external tensor product $\overset{L}{\boxtimes}$ is applied to two perverse sheaves, it returns a perverse sheaf. In addition, the vanishing cycle functor ϕ_h takes perverse sheaves to perverse sheaves. Therefore, the Sebastiani–Thom isomorphism yields an isomorphism in the Abelian category of perverse sheaves on $X \times Y$ and, consequently the isomorphism preserves much more subtle data than that provided by the stalk cohomology.

T. Braden works in this context in [B], and it was his Lemma 3.16 which motivated the writing of this paper. In [B], the base ring R is the field $\mathbb{C}$, $Y := \mathbb{C}^m$, $g\colon \mathbb{C}^m \to \mathbb{C}$ is the ordinary quadratic function $g := y_1^2 + \cdots + y_m^2$, and $\mathbf{B}^\bullet := \mathbb{C}_Y^\bullet$, the constant sheaf on Y. Hence, $\phi_g \mathbf{B}^\bullet$ is the constant sheaf on the origin, shifted by $-m$, and extended by zero onto $V(g)$, i.e., if α denotes the inclusion of 0 into $\mathbb{C}^m$, $\phi_g \mathbf{B}^\bullet \cong \alpha_! \mathbb{C}_0^\bullet[-m]$. Let $\tau\colon V(f) \hookrightarrow V(f \boxplus g)$ be the inclusion given by $\tau(\mathbf{x}) := (\mathbf{x}, 0)$.

If X, f, and $\mathbf{A}^\bullet$ are still arbitrary, we wish to show

COROLLARY. *There is a natural isomorphism*

$$\phi_{f \boxplus g}(\pi_1^* \mathbf{A}^\bullet) \cong \tau_*(\phi_f \mathbf{A}^\bullet)[-m].$$

Proof. As the critical locus of g is simply the origin, it follows from Lemma 1.1 and the Sebastiani–Thom isomorphism that

$$\phi_{f \boxplus g}(\pi_1^* \mathbf{A}^\bullet) \;\cong\; k_!(p_1^* \phi_f \mathbf{A}^\bullet \overset{L}{\otimes} p_2^* \alpha_! \mathbb{C}_0^\bullet[-m]).$$

Consider the pull-back diagram

$$\begin{array}{ccc} V(f) \times 0 & \xrightarrow{\hat{\alpha}} & V(f) \times V(g) \\ {\scriptstyle \hat{p}_2}\downarrow & & \downarrow{\scriptstyle p_2} \\ 0 & \xrightarrow{\quad \alpha \quad} & V(g). \end{array}$$

Then, we have $p_2^* \alpha_! \cong \hat{\alpha}_! \hat{p}_2^*$, and so

$$\phi_{f \boxplus g}(\pi_1^* \mathbf{A}^\bullet) \;\cong\; k_!(p_1^* \phi_f \mathbf{A}^\bullet \overset{L}{\otimes} \hat{\alpha}_! \hat{p}_2^* \mathbb{C}_0^\bullet[-m]) \;\cong\; k_!(p_1^* \phi_f \mathbf{A}^\bullet \overset{L}{\otimes} \hat{\alpha}_! \mathbb{C}_{V(f)\times 0}^\bullet[-m]).$$

Applying Proposition 2.6.6 of [K-S], we obtain that

$$\begin{aligned} \phi_{f \boxplus g}(\pi_1^* \mathbf{A}^\bullet) &\cong k_!(p_1^* \phi_f \mathbf{A}^\bullet \overset{L}{\otimes} \hat{\alpha}_! \mathbb{C}_{V(f)\times 0}^\bullet[-m]) \\ &\cong k_! \hat{\alpha}_!(\hat{\alpha}^* p_1^* \phi_f \mathbf{A}^\bullet \overset{L}{\otimes} \mathbb{C}_{V(f)\times 0}^\bullet[-m]) \\ &\cong k_! \hat{\alpha}_!(\hat{\alpha}^* p_1^* \phi_f \mathbf{A}^\bullet[-m]) \;\cong\; (k \circ \hat{\alpha})_!(p_1 \circ \hat{\alpha})^* \phi_f \mathbf{A}^\bullet[-m]. \end{aligned}$$

As $p_1 \circ \hat{\alpha}$ is the isomorphism which identifies $V(f) \times 0$ and $V(f)$, and as $k \circ \hat{\alpha}$ is the closed inclusion of $V(f) \times 0$ into $V(f \boxplus g)$, we are finished. □

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2.4. Brieskorn varieties and intersection cohomology

As a final application of our result, we will consider Brieskorn varieties with twisted intersection cohomology coefficients.

Consider a local system, $\mathcal{L}$, of complex vector spaces of dimension d on the punctured complex plane, $\mathbb{C}^*$; such a local system is determined up to isomorphism by its own monodromy isomorphism $\mu_{\mathcal{L}}: \mathbb{C}^d \to \mathbb{C}^d$. We consider $\mathcal{L}$ as a complex $\mathcal{L}^\bullet$ by placing $\mathcal{L}$ in degree zero and placing zeroes in all other degrees.

The intersection cohomology complex with coefficients in $\mathcal{L}$, $\mathrm{IC}^\bullet_{\mathbb{C}}(\mathcal{L})$, agrees with $\mathcal{L}^\bullet[1]$ on $\mathbb{C}^*$ and has stalk cohomology, at the origin, which is zero outside of degree -1 and is isomorphic to $\ker(\mathrm{id} - \mu_{\mathcal{L}})$ in degree -1 (see the construction or axiomatic characterization in [G-M2]).

Now, consider a collection of local systems $\mathcal{L}_i$ of rank d_i on $\mathbb{C}^*$, with monodromy isomorphisms μ_i; denote the corresponding intersection cohomology complex by $\mathrm{IC}^\bullet_{\mathbb{C}}(\mathcal{L}_i)$, and consider the functions $f_i(z) := z^{a_i}$ on $\mathbb{C}$, where the a_i's are positive integers. The vanishing cycles $\phi_{f_i}\mathrm{IC}^\bullet_{\mathbb{C}}(\mathcal{L}_i)$ are a perverse sheaf which is supported only at 0; consequently, $\phi_{f_i}\mathrm{IC}^\bullet_{\mathbb{C}}(\mathcal{L}_i)$ is non-zero only in degree 0, where it has dimension equal to the dimension of the nearby cycles minus the dimension of the stalk at the origin (both in degree -1), i.e., $a_i d_i - \dim\big(\ker(\mathrm{id} - \mu_i)\big)$.

On the other hand, the external product of intersection cohomology complexes is an intersection cohomology complex. Thus,

$$\mathrm{IC}^\bullet_{\mathbb{C}}(\mathcal{L}_1) \overset{L}{\boxtimes} \cdots \overset{L}{\boxtimes} \mathrm{IC}^\bullet_{\mathbb{C}}(\mathcal{L}_n) \;\cong\; \mathrm{IC}^\bullet_{\mathbb{C}^n}(\mathcal{L}_1 \overset{L}{\boxtimes} \cdots \overset{L}{\boxtimes} \mathcal{L}_n),$$

where $\mathcal{L}_1 \overset{L}{\boxtimes} \cdots \overset{L}{\boxtimes} \mathcal{L}_n$ is the local system on $(\mathbb{C}^*)^n$ determined by the monodromy representation $\mu: \mathbb{Z}^n \cong \pi_1\big((\mathbb{C}^*)^n\big) \to \mathrm{Aut}\big(\mathbb{C}^{d_1} \times \cdots \times \mathbb{C}^{d_n}\big)$ given by

$$\mu(t_1, \ldots, t_n)(v_1, \ldots, v_n) = (\mu_1(v_1), \ldots, \mu_n(v_n)).$$

Therefore, by iterating the Sebastiani–Thom isomorphism, we find that

$$\phi_{z_1^{a_1} + \cdots + z_n^{a_n}}\big(\mathrm{IC}^\bullet_{\mathbb{C}^n}(\mathcal{L}_1 \overset{L}{\boxtimes} \cdots \overset{L}{\boxtimes} \mathcal{L}_n)\big)$$

is supported only at the origin, only in degree 0, and in degree 0 has dimension equal to

$$\prod_i \big(a_i d_i - \dim\big(\ker(\mathrm{id} - \mu_i)\big)\big).$$

In the case where $d_i = 1$ and $\mu_i = \mathrm{id}_{\mathbb{C}}$ for all i, we are back in the standard case of the constant sheaf, and we recover the well-known result of Brieskorn–Pham that the dimension of the vanishing cycles in degree $n-1$ is $\prod_i (a_i - 1)$.

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