# 2 Small Data Are Beautiful

The Introduction made a number of claims about the relevance of time-frequency approaches in signal processing, sketching some kind of a program for the present book. Let us start with three examples supporting those claims: one in physics, one in bioacoustics, and one in mathematics.

## 2.1 Gravitational Waves

The first direct observation of a gravitational wave was reported in early 2016 [5]. Since their prediction by Albert Einstein as a consequence of his theory of general relativity, proof of their existence had long been awaiting direct evidence, mostly because the extremely tiny effects they induce on matter make their detection a formidable challenge. The search for gravitational waves has therefore led to ambitious research programs based on the development of giant interferometers. The rationale is that the propagation of a gravitational wave essentially modifies the local structure of space-time, with the consequence that its impinging on an interferometer produces a differential effect on the length of its arms, and hence an oscillation in the interference pattern. Similarly to electromagnetic waves that result from accelerated charges, gravitational waves result from accelerated masses and, to be detectable, only extreme astrophysical events can be considered as candidates for producing gravitational waves. The preferred scenarios are mergers of compact binaries made of neutron stars or black holes. Within this picture of two very massive objects revolving around each other, the loss of energy due to the hypothesized radiation of a gravitational wave is expected to make them get closer and closer, hence causing them to revolve around each other at a faster and faster pace for the sake of conservation of the angular momentum. The overall result is that the signature of such a gravitational wave in the interferometric data takes the form of a "chirp" (i.e., a transient waveform modulated in both amplitude and frequency), with an increase in both amplitude and instantaneous frequency during the inspiral part that precedes the coalescence.

The event corresponding to the first detection (referred to as GW150914) was precisely of this type. It consisted of the merger of two black holes (each of about 30 solar masses), with an observation made of two transient signals of short duration (a fraction of a second), which were detected by the two LIGO interferometers located in Hanford, WA and Livingston, LA, respectively.



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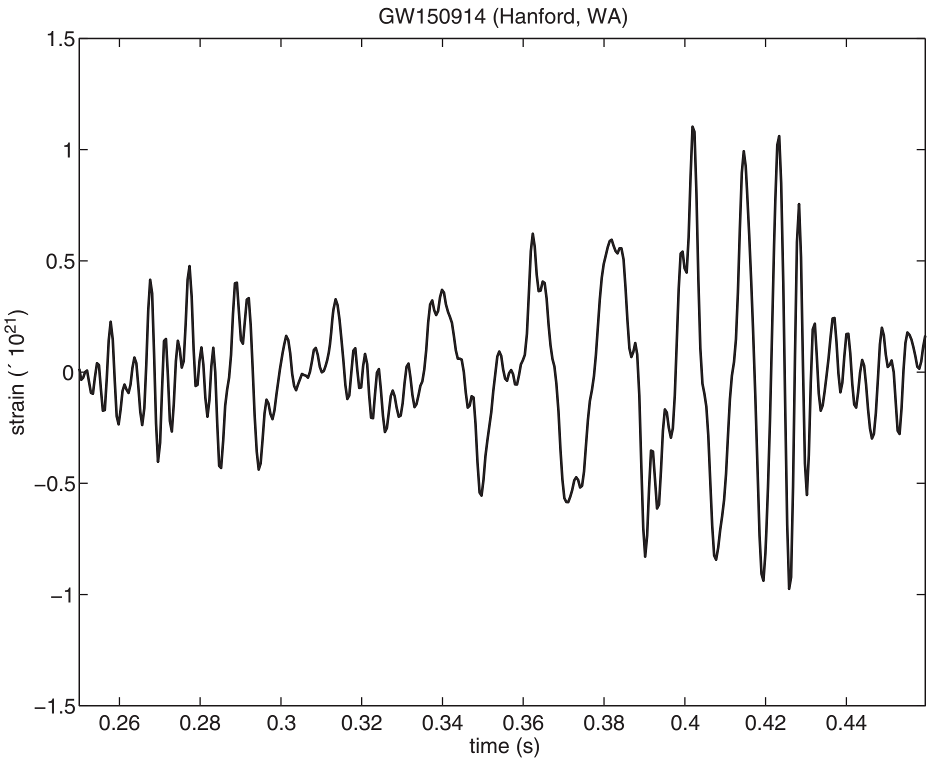


**Figure 2.1** Gravitational wave chirp 1. This figure plots the temporal signature of GW150914, as it was recorded by the LIGO interferometer in Hanford, WA, and pre-processed for getting rid of known perturbations due to the measurement system.

The temporal signature of GW150914, as it was recorded by the LIGO interferometer in Hanford, WA, and pre-processed for getting rid of known perturbations due to the measurement system, is plotted in Figure 2.1. This plot gives us an idea of the chirping nature of the (noisy) waveform, but a much clearer picture of what happens is obtained when we turn to the time-frequency plane, as shown in Figure 2.2.[1]

> Gravitational waves offer an example of "small" signals (a few thousand samples at most), with characteristics that convey physical information about the system from which they originate.

*Remark.* Although "small," gravitational wave signals result from a prototypical example of "big science": 45 years of efforts at the forefront of technology for developing giant interferometers with arms 4 km long and a sensitivity of $10^{-21}$, thousands of researchers and engineers, more than 1,000 coauthors in the landmark publication [5] . . . Each data point, therefore, has an immense value, calling again for accurate methods of

[1] In this figure, as in most figures throughout the book, time is horizontal, frequency vertical, and the energy content is coded in gray tones, ranging from white for the lower values to black for the maximum.

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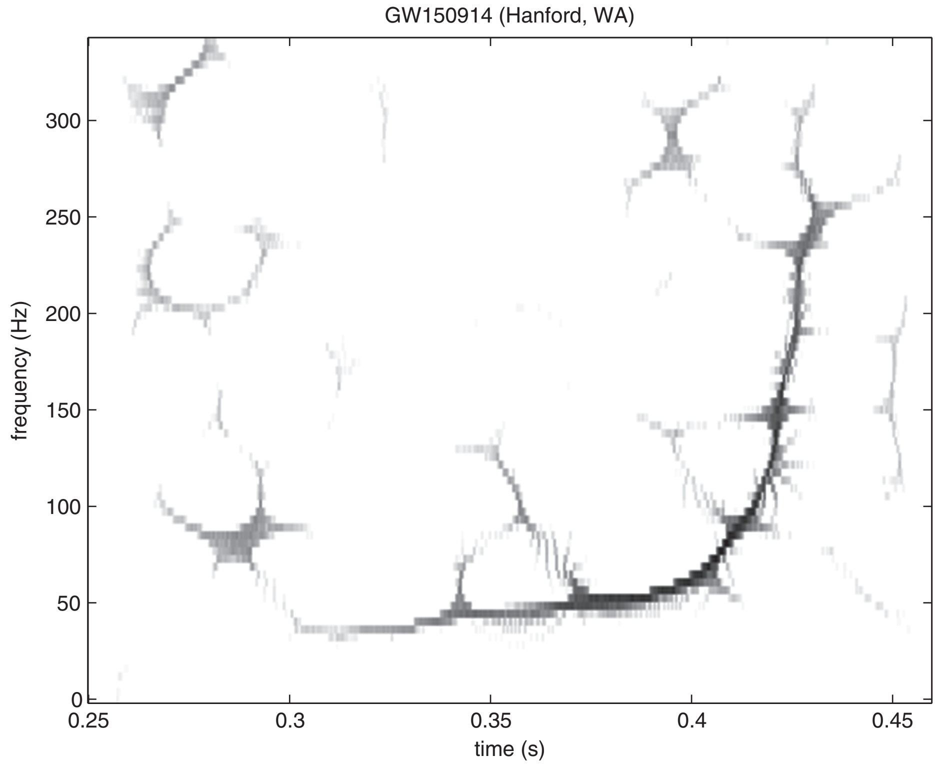


**Figure 2.2** Gravitational wave chirp 2. The waveform plotted in Figure 2.1 is displayed here as an energy distribution in the time-frequency plane (namely, a "reassigned spectrogram," which will be introduced in Chapter 10). As compared to the time plot of the signal, this "musical score" reveals in a much clearer way the inner structure of the waveform, namely the frequency evolution of an ascending chirp. The energy scale is logarithmic, with a dynamic range of 24 dB.

analysis, in particular for comparing observation to theoretical models and confirming the amazing agreement that has been obtained so far [5].

Detecting gravitational waves, de-noising the corresponding chirps, and extracting physical information from them can take advantage of time-frequency approaches. We will come back to this in Chapter 16.

## 2.2 Bats

By following the musical score analogy outlined previously, we can switch from music to singing voice, and from singing voice to speech. All of these instances of audio signals offer countless opportunities for a time-frequency analysis aimed at displaying inner structures of sounds in a graphic, easily understandable way that matches perception. And indeed, it is not by chance that one of the first books ever published on time-frequency analysis [4] had the title *Visible Speech*!

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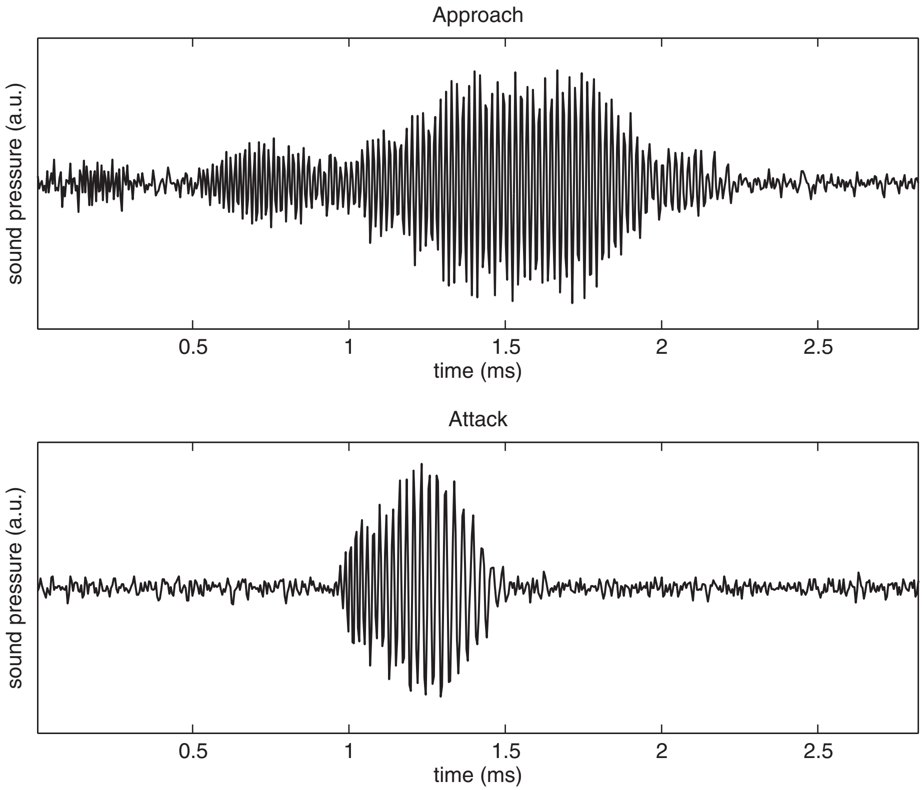


**Figure 2.3** Two examples of bat echolocation calls 1. The two waveforms displayed in this figure are part of a sequence that lasts for about 1 s and contains a few dozens of such calls, with a structure (duration, spectrum, and modulations) that varies in between the beginning of the active part (“approach”) and the end of the sequence (“attack”).

Those common situations enter what is essentially a framework of *communication*, in which some “message” is sent by somebody, somewhere, to be received by somebody else, elsewhere. We could comment further on specific features attached to such audio signals but we will not here. We will, rather, choose as examples other types of waveforms that share much with conventional audio signals, but which differ from speech or music in at least two respects. First, whereas the transmission of a speech message can be viewed as “active” by the speaker and “passive” by the listener, there exist other situations where the system is doubly “active” in the sense that the emitter is at the same time the receiver, and where the received information is not so much the message itself as it is the modifications it may have experienced during its propagation. Second, although they are acoustic, the transmitted signals can have a frequency content that lies outside of the audio range. These two ingredients are typical of the *echolocation* system used by bats and, more generally, by other mammals such as dolphins, or even by humans in detection systems such as radar, sonar, or nondestructive evaluation.

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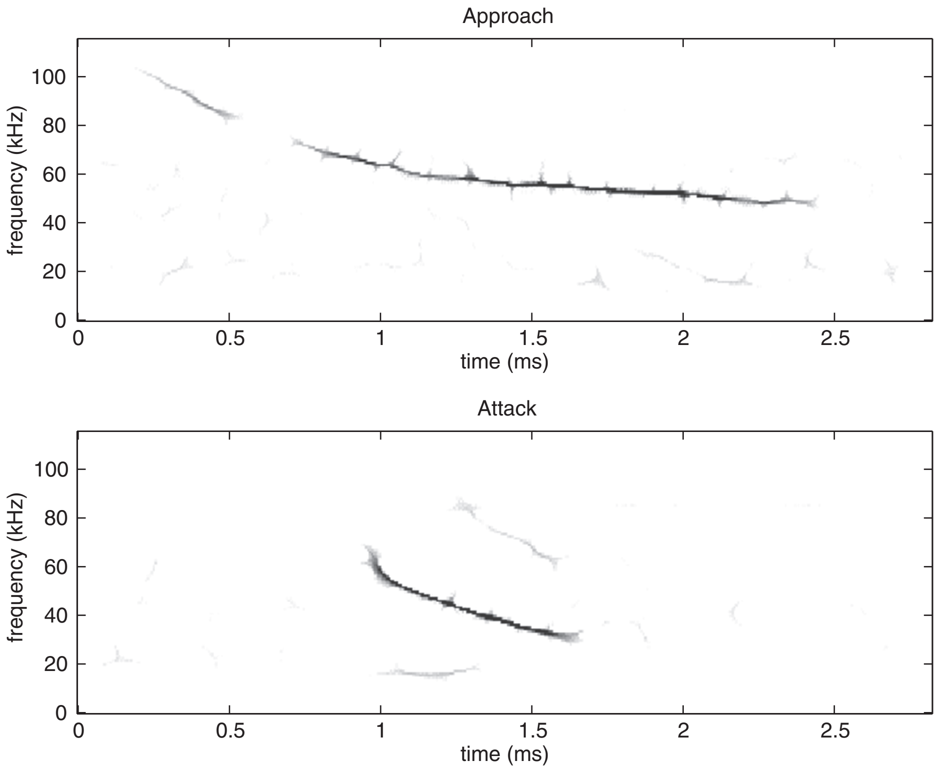


**Figure 2.4** Two examples of bat echolocation calls 2. The two waveforms plotted in Figure 2.3 are displayed here as energy distributions in the time-frequency plane. As compared to the time plots of the signals, those "musical scores" reveal in a much clearer way the inner structure of the waveforms, e.g., the nature of their frequency modulations. In both diagrams, the time span is as in Figure 2.3, and the energy scale is logarithmic, with a dynamic range of 30 dB.

If we confine ourselves to bats, the story begins in 1794 when Lazzaro Spallanzani first suggested – on the basis of crucial, yet cruel experiments – that bats should have some specific sensorial capability for navigating in the dark [24]. It seemed to be related to hearing rather than to sight, since it was altered when making the animal mute and/or deaf, while making it blind was of no consequence on the flight. This question puzzled physiologists for almost two centuries, until the zoologist Donald W. Griffin reopened this mysterious case in 1938 together with the engineer George W. Pierce, who had just developed a new kind of microphone that was sensitive to ultrasounds, i.e., sounds whose frequency is above the upper limit of perception of the human ear (∼ 20 kHz). In this way they were able to prove that bats were indeed emitting ultrasounds [25], and their study launched a fascinating area of research [26], with implications in both biology and engineering.

Two typical bat echolocation calls, emitted by *Myotis mystacinus* when hunting and recorded in the field, are plotted in Figure 2.3. The two waveforms are part of a sequence that lasts for about 1 s and contains a few dozen such calls, with a structure (duration, spectrum, and modulations) that varies in between the beginning of the active part (the so-called "approach" phase, during which the bat manages to get closer to the target it has identified as a potential prey item) and the end of the sequence (the "attack" phase, which terminates with the actual catch). Thanks to the time-frequency methods that will be described later in this book, we can get a much clearer picture of the inner structure of those waveforms by drawing their "musical score" as shown in Figure 2.4 (in this figure, as in Figure 2.2, we made use of a "reassigned spectrogram"). As for gravitational waves, they happen to be "chirps," with characteristics that vary within a sequence. From those diagrams, we can expect to extract more easily, and in a more directly interpretable way, the necessary information about the why and how of the observed signals in relation with a given task.

> Bat echolocation calls are an example of "small" signals (a few hundred samples at most), with a well-defined time-frequency structure whose fine characterization calls for precise analysis tools.

As for gravitational waves, we will come back to this in Chapter 16.

## 2.3 Riemann-Like Special Functions

The third family of "small" signals we will mention as an example is somewhat different since it concerns mathematics and, more precisely, some special functions.

Interest in the complementary descriptions of special functions beyond the mere inspection of their analytical formulation has been raised, e.g., by Michael V. Berry, who has suggested transforming such functions into sounds and listening to them [27]. The examples he chose are related to Riemann's zeta function, an analytic function of the complex variable $z \in \mathbb{C}$ which reads

$$\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z}, \tag{2.1}$$

and admits the equivalent representation (Euler product):

$$\zeta(z) = \prod_{p \in \mathcal{P}} \frac{1}{1 - p^{-z}}, \tag{2.2}$$

where $\mathcal{P}$ stands for the set of all prime numbers. The distribution of primes turns out to be connected with that of the zeros of the zeta function and, since a *spectrum* can be attached to this distribution [28], this paves the way for giving a real existence to a "music of primes" that had previously been evoked as an allegory.

> Of course, in parallel to the *hearing* of this music, we can naturally think of *seeing* it by writing down its "musical score" thanks again to some time-frequency analysis.

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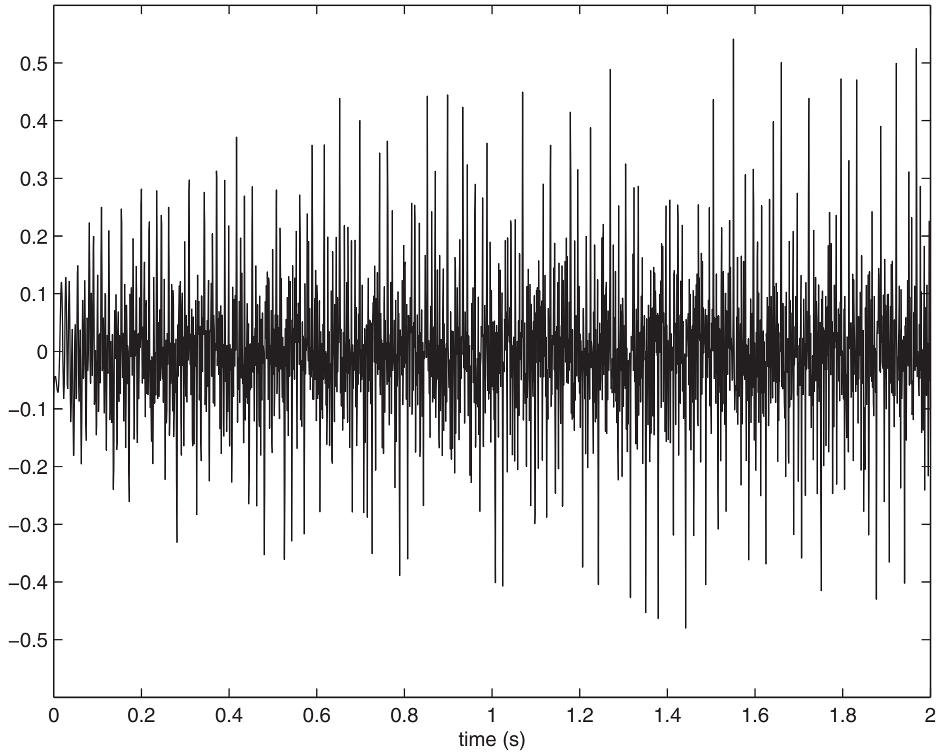


**Figure 2.5** Zeta function 1. For a sampling rate fixed to 32 kHz, this figure plots the first 2 seconds of the (real part of) the waveform $Z(t)$ defined in (2.3), which essentially describes $\zeta(z)$ as a function of its imaginary part $\mathrm{Im}\{z\} = t$ for the fixed value of its real part $\mathrm{Re}\{z\} = 1/2$.

Following [27], we can consider the function

$$Z(t) = \zeta\left(\tfrac{1}{2} + \mathrm{i}t\right) \exp\{\mathrm{i}\,\theta(t)\}, \tag{2.3}$$

with

$$\theta(t) = \mathrm{Im}\left\{\Gamma\left(\tfrac{1}{2}\left(\tfrac{1}{2} + \mathrm{i}t\right)\right)\right\} - (t \log \pi)/2. \tag{2.4}$$

This function is of special importance with respect to the so-called *Riemann hypothesis*, which stipulates that all zeros of the zeta function that fall within the strip $0 < \mathrm{Re}\{z\} < 1$ are aligned along the only line given by $\mathrm{Re}(z) = 1/2$. A plot of the first 2 seconds of the function $Z(t)$, sampled at 32 kHz [27], is given in Figure 2.5. It rather looks like noise, and gives few insights into the spectral structure, if any. In contrast, the time-frequency image given in Figure 2.6 evidences a fairly well-structured organization in terms of up-going chirps, that calls for explanations. In this case, the chosen time-frequency representation is an "ordinary" spectrogram, so as to put emphasis on the crystal-like structure of zeros in the time-frequency plane.

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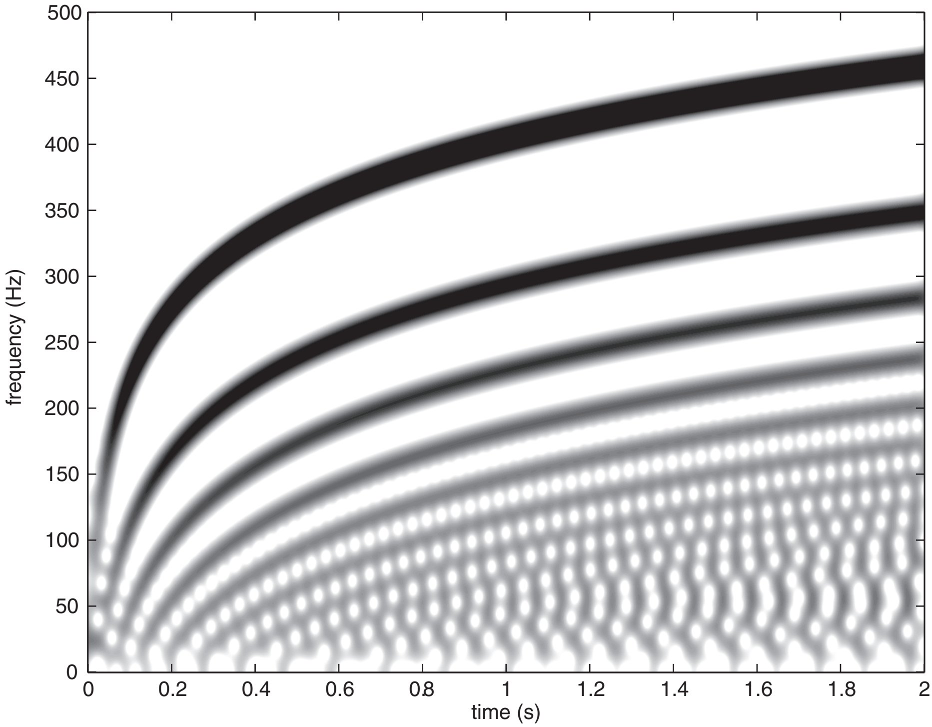


**Figure 2.6** Zeta function 2. The waveform plotted in Figure 2.5 is displayed here as an energy distribution in the time-frequency plane (spectrogram). As compared to the time history of the signal, this "musical score" reveals in a much clearer way the inner structure of the waveform, namely the existence of ascending chirps. The energy scale is logarithmic, with a dynamic range of 15 dB.

As for bats and gravitational waves, we will come back to this in Chapter 16, together with some other examples of special functions (like Weierstrass's function) that admit natural time-frequency interpretations.

## 2.4 Chirps (Everywhere)

The three (families of) signals considered so far are just examples, and we are far from exhausting the number of situations where waveforms of a similar nature are encountered. If we consider their structure, they all share the common property of being (possibly multicomponent) "chirps." In other words, they are basically characterized by a well-defined structure that takes the form of time-frequency *trajectories* which reflect the existence of *sweeping* frequencies. This phenomenological description calls for considering what being a "chirp" means, in more mathematical terms.

The concept of frequency is indeed closely related to notions such as oscillations or cycles – which are ubiquitous in nature and technology, from the motion of celestial objects to atomic descriptions – via biological rhythms or rotating machinery. Therefore:

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Chirps appear essentially as transient time-dependent variations on sustained harmonic oscillations, whence their ubiquity.

In order to make this point clearer, consider a simple pendulum of length $L_0$ in the usual gravity field. In the approximation of small oscillations, it is well-known that the angle $\theta(t)$ is governed by the equation

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2}(t) + \frac{g}{L_0}\,\theta(t) = 0, \tag{2.5}$$

where $g$ stands for the acceleration of gravity. Up to some pure phase term, the solution of (2.5) is simply

$$\theta(t) = \theta_0 \cos \omega_0 t, \tag{2.6}$$

with $\theta_0$ the maximum amplitude of the oscillation, $\omega_0 = \sqrt{g/L_0}$ its (angular) frequency, and $T = 2\pi/\omega_0$ its period.

Let us then make the pendulum length become a slowly varying function of time (by imposing, e.g., $L_0$ to be transformed into $L(t) = L_0\,(1 + \varepsilon\, t)$, with $\varepsilon > 0$ small enough to keep the length variation small at the scale of one oscillation). This results in oscillations that become time-dependent, with an "instantaneous" frequency that is almost "frozen" on a short-term basis, yet in the longer term undergoes the evolution $\omega(t) \approx \sqrt{g/L(t)}$, which is progressively slowed down as the pendulum length is increased. If we further include viscous damping, the actual amplitude of the oscillations becomes time-varying, with an exponential decay. To summarize, combining both effects transforms a sine wave into a chirp!

This very simple example of a damped pendulum with time-varying length illustrates what we understand when adding "instantaneous" to words such as amplitude or frequency. In the nominal situation of a pure harmonic oscillation $y(t)$, there is no ambiguity in writing

$$y(t) = a \cos \omega t \tag{2.7}$$

and in considering that $a$ is its amplitude and $\omega$ its frequency. When accepting some possible time dependencies, it is tempting to generalize (2.7) by writing

$$x(t) = a(t) \cos \varphi(t), \tag{2.8}$$

letting $a$ become a (nonnegative) function of time $a(t)$, and replacing $\omega t$ with a phase term $\varphi(t)$ undergoing some possibly nonlinear evolution.

Unfortunately, there is no unique way of expressing a given observation $x(t) \in \mathbb{R}$ in a form such as (2.8). The usual way out is to come back to (2.7) and write $x(t) = \mathrm{Re}\{z_x(t)\}$, i.e., to consider that $y(t)$ is the real part of some complex-valued signal $z_y(t)$, thus calling for a decision on what the imaginary part should be. A "natural" choice is to complement the *cosinusoidal* real part with a *sinusoidal* imaginary part, so that

$$z_y(t) = a \exp\{\mathrm{i}\omega t\}. \tag{2.9}$$

This is the classic "Fresnel" (or "Argand") representation of a monochromatic wave, whose interpretation in the complex plane is particularly appealing. Indeed, as parameterized by time $t$, the complex-valued signal $z_y(t)$ in (2.9) can be seen as a rotating vector – with real and imaginary parts as coordinates – whose extremity describes a *circle* with a *constant* angular speed. The amplitude $a$ is precisely the modulus $|z_y(t)|$, whereas the angular speed, which is the time derivative of the phase $\omega t$, is identified with the frequency.

Given this interpretation, extending to time-varying situations is fairly obvious. It amounts to adding an imaginary part to the real-valued observation $x(t) \in \mathbb{R}$. Since there is no unicity for such an extension, the most "natural" one consists in mimicking the relationship that exists between a cosine and a sine in a complex exponential. As it can be easily established, the linear filter which turns a cosine into a sine (with the exact same amplitude and frequency) has for transfer function $H(\omega) = -\mathrm{i}\,\mathrm{sgn}\,\omega$. In the time domain, this corresponds to the *Hilbert transform* **H** such that:

$$(\mathbf{H}x)(t) = \frac{1}{\pi}\,\mathrm{p.v.}\int_{-\infty}^{\infty} \frac{x(s)}{t-s}\,\mathrm{d}s, \tag{2.10}$$

where "p.v." indicates that the integral has to be computed as a "principal value" in Cauchy's sense, i.e., as

$$\mathrm{p.v.}\int_{-\infty}^{\infty} = \lim_{\epsilon\to 0}\left[\int_{-\infty}^{-\epsilon} + \int_{\epsilon}^{+\infty}\right]. \tag{2.11}$$

Applying the above recipe for the complexification of the real-valued signal $x(t)$ thus leads to the well-known solution of the so-called *analytic signal* which reads [29, 30]:

$$z_x(t) = x(t) + \mathrm{i}\,(\mathbf{H}x)(t). \tag{2.12}$$

The "real/imaginary" expression (2.12) admits the equivalent "modulus/phase" representation:

$$z_x(t) = a(t)\,\exp\{\mathrm{i}\varphi(t)\} \tag{2.13}$$

which is now unique and from which an "instantaneous amplitude" $a(t) \geq 0$ and an "instantaneous frequency" $\omega(t)$ can be derived as in the purely harmonic case, *mutatis mutandis*:

$$a(t) = |z_x(t)| \quad ; \quad \omega(t) = \frac{\mathrm{d}}{\mathrm{d}t}\arg\{z_x(t)\}. \tag{2.14}$$

Generalizing upon the pendulum example, "slowly-varying" conditions are often assumed for chirps. Usual heuristic conditions assume that $|\dot{a}(t)/a(t)| \ll |\dot{\varphi}(t)|$, i.e., that the amplitude is *almost constant* at the scale of one pseudo-period $T(t) = 2\pi/|\dot{\varphi}(t)|$, and that $|\ddot{\varphi}(t)/\dot{\varphi}^2(t)| \ll 1$, i.e., that the pseudo-period $T(t)$ is itself *slowly varying* from one oscillation to the next [31, 32].

*Remark.* Although "classic" and the most used one, the definition of "instantaneous frequency" on the preceding page may lead to some paradoxes and present difficulties in its physical interpretation. It can, for instance, attain negative values, or have excursions outside of a frequency band in which the signal spectrum is supposed to be limited [22].

One can argue, however, that such unexpected behaviors apply to situations that can be considered as departing from the assumed model. This is especially the case when more than one chirp component is present at a given time, a situation in which one would normally expect two values; of course, this is impossible with the definition of a mono-valued function. Without having recourse to time-frequency analysis (which will prove to be a better approach), alternative definitions – with their own pros and cons – have been proposed. These will not be discussed here; for more information, refer to [33].

In retrospect, it is clear that the examples considered in Sections 2.1–2.3 of this chapter can be reasonably considered as *multicomponent chirps* (also known as *multicomponent AM-FM (Amplitude Modulated – Frequency Modulated) signals*), all of which accept a model of the form

$$x(t) = \sum_{k=1}^{K} a_k(t) \cos \varphi_k(t). \tag{2.15}$$

We confine ourselves here to (and will discuss further in Chapter 16) a few such signals, but it is worth stressing that waveforms of a very similar nature can be found in many domains. The following list shows a few of the possibilities one can mention:

- *Birdsongs* – This is of course the first instance, and the one from which the name "chirp" comes from since, according to Webster's 1913 Dictionary, a chirp is "a sharp sound made by small birds or insects."
- *Animal vocalizations* – Besides birds (and bats), many animals make use of transient chirps or longer AM-FM signals, mostly for communication purposes: short chirps by frogs, longer vocalizations by whales (listen, e.g., to the many sound files available at http://cis.whoi.edu/science/B/whalesounds/index.cfm).
- *Audio, speech, and music* – Audio is a rich source of chirping waveforms: singing voice, vibrato, glissando . . . Several examples have been exhibited since the early days of time-frequency analysis [4].
- *Dispersive media* – A brief pulse can be idealized as the coherent superposition of "all" possible frequencies. If such a waveform is sent through a dispersive medium for which the group velocity is frequency-dependent, the different frequency components travel at different speeds, resulting in a distortion that, in time, spreads the highly localized pulse and transforms it into a chirp. This can be observed, e.g., in the backscattering from simply shaped elastic objects in underwater acoustics [34].
- *Whistling atmospherics* – A companion example where chirps are due to dispersion is to found in geophysics, with low-frequency "whistlers" that can follow impulsive atmospherics (such as lightning strokes) after propagation in the outer ionosphere [35].
- *Turbulence* – Turbulent flows can be given both statistical and geometrical descriptions. In 2D, disorder can be modeled *via* some random distribution of spiraling coherent structures, namely vortices (e.g., the swirls that can be observed in rivers downstream from a bridge). Intersecting such a 2D vortex

results in a 1D profile taking the form of a “singular” core surrounded by a chirp [36].

- *Warping* – In a natural generalization of the pendulum example, oscillations of moving sources lead to chirping behaviors. This was noticed for gravitational waves, but this also applies to Doppler effect, where a pure tone emitted by a moving source in its own referential ends up with a modulated wave that is perceived by a receiver as compressed or stretched when it passes by. Similarly, acceleration warps the characteristic rotation frequencies of an engine.
- *Electrophysiological signals* – When recording electroencephalograpic (EEG) signals to monitor brain activity, it turns out that the abnormal neural synchrony attached to epilepsy has a chirp signature [37]. In a different context, uterine electromyographic (EMG) signals do chirp too during contraction episodes, with a bell-shaped ascending/descending narrowband frequency component [38].
- *Critical phenomena* – Oscillations that accelerate when approaching a singularity are in some sense “universal” [39]. This has been advocated for identifying precursors in domains as diverse as earthquakes or financial crashes.
- *Man-made sounders* – Mimicking the echolocation principle used by bats or dolphins, some human-made systems make use of chirp-like signals for sounding their environment. One can cite the field of “fisheries acoustics” where FM signals are used by broadband sounders [40]. Another key example is vibroseismics, where sweeping frequencies are sent through the ground in a particular area – for the sake of oil exploration – by means of specially equipped trucks [41].

We could provide many more such examples, but we will stop here. One more good reason for closing the list with vibroseismics is that it is emblematic of the way science can be driven by applied problems. Indeed, it is no exaggeration to say that the whole field of wavelet analysis – which can be viewed as a twin sister to the time-frequency analysis on which this book is primarily focused – originated from Morlet’s concerns to improve upon “classic” Fourier-based vibroseismic signal analysis. When based on solid mathematical grounds and equipped with efficient algorithms, the roads Morlet started to explore led to unexpected “hot spots” with new perspectives far beyond the initial problem and its original time-frequency flavor.