AN EXPRESSION FOR BERNOULLI NUMBERS

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In Muir's *Theory of Determinants*, Vol. III, pp. 232–237, there will be found accounts of papers by H. Nägelsbach, J. Hammond and J. W. L. Glaisher, in which expressions for the Bernoulli numbers are obtained in terms of determinants. In the present paper an expression for B_n will be derived which appears to be new, but which is very like some of those mentioned by Muir.

When each side of the identity

$$\sum_{r=1}^{n} \cos 2(2r-1)\theta = \frac{\sin 4n\theta}{2\sin 2\theta}$$

is integrated from 0 to θ, we get, after letting $n \to \infty$,

$$\sum_{r=1}^{\infty} \frac{\sin 2(2r-1)\theta}{2r-1} = \frac{\pi}{4}.$$

We now integrate in succession $2n-1$ times from 0 to θ and obtain the $2n$ relations, in which for convenience we have written $z_m = \sum_{r=1}^{\infty} \frac{1}{(2r-1)^m}$:

$$\sum_{1}^{\infty} \frac{\sin 2(2r-1)\theta}{2r-1} = \frac{\pi}{4},$$

$$-\frac{1}{2}\sum_{1}^{\infty} \frac{\cos 2(2r-1)\theta}{(2r-1)^2} = \frac{\pi}{4}\theta - \tfrac{1}{2}z_2,$$

$$-\frac{1}{2^2}\sum_{1}^{\infty} \frac{\sin 2(2r-1)\theta}{(2r-1)^3} = \frac{\pi}{4}\frac{\theta^2}{2!} - \tfrac{1}{2}\theta z_2,$$

$$\frac{1}{2^3}\sum_{1}^{\infty} \frac{\cos 2(2r-1)\theta}{(2r-1)^4} = \frac{\pi}{4}\frac{\theta^3}{3!} - \frac{1}{2}\frac{\theta^2}{2!}z_2 + \frac{1}{2^3}z_4,$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$\frac{(-1)^n}{2^{2n-1}}\sum_{1}^{\infty} \frac{\cos 2(2r-1)\theta}{(2r-1)^{2n}} = \frac{\pi}{4}\frac{\theta^{2n-1}}{(2n-1)!} - \frac{1}{2}\frac{\theta^{2n-2}}{(2n-2)!}z_2 + \frac{1}{2^3}\frac{\theta^{2n-4}}{(2n-4)!}z_4 - \dots + \frac{(-1)^{n-1}}{2^{2n-3}}\cdot\frac{\theta^2}{2!}z_{2n-2} + \frac{(-1)^n}{2^{2n-1}}z_{2n}.$$

Of these $2n$ relations we use only those n which involve cosine series, and in these we put $\theta = \frac{\pi}{2}$ to obtain the following n relations, when we have reversed the order and have written ω_r for $\left(\frac{\pi}{2}\right)^r \frac{1}{r!}$ and ξ_r for $\frac{(-1)^r}{2^{2r-1}} z_{2r}$:

$$\begin{array}{llllllll}
2\xi_n + \omega_2\xi_{n-1} & + \omega_4\xi_{n-2} & + \dots + \omega_{2n-6}\,\xi_3 & + \omega_{2n-4}\xi_2 & + \omega_{2n-2}\xi_1 & + n\omega_{2n} & = 0, \\
0 + 2\xi_{n-1} & + \omega_2\xi_{n-2} & + \dots + \omega_{2n-8}\,\xi_3 & + \omega_{2n-6}\xi_2 & + \omega_{2n-4}\xi_1 & + (n-1)\,\omega_{2n-2} & = 0, \\
0 + 0 & + 2\xi_{n-2} & + \dots + \omega_{2n-10}\xi_3 & + \omega_{2n-8}\xi_2 & + \omega_{2n-6}\xi_1 & + (n-2)\,\omega_{2n-4} & = 0, \\
\dots\dots & \dots\dots & \dots\dots & \dots\dots & \dots\dots & \dots\dots & \\
0 + 0 & + 0 & + \dots + 2\xi_3 & + \omega_2\xi_2 & + \omega_4\xi_1 & + 3\omega_6 & = 0, \\
0 + 0 & + 0 & + \dots + 0 & + 2\xi_2 & + \omega_2\xi_1 & + 2\omega_4 & = 0, \\
0 + 0 & + 0 & + \dots + 0 & + 0 & + 2\xi_1 & + 1\omega_2 & = 0.
\end{array}$$

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When these n equations are solved for ξ_n we get

$$\xi_n = \frac{(-1)^n}{2^n} \begin{vmatrix} \omega_2 & \omega_4 & \omega_6 & \cdots & \omega_{2n-2} & n\omega_{2n} \\ 2 & \omega_2 & \omega_4 & \cdots & \omega_{2n-4} & (n-1)\,\omega_{2n-2} \\ 0 & 2 & \omega_2 & \cdots & \omega_{2n-6} & (n-2)\,\omega_{2n-4} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & \omega_4 & 3\omega_6 \\ 0 & 0 & 0 & \cdots & \omega_2 & 2\omega_4 \\ 0 & 0 & 0 & \cdots & 2 & 1\omega_2 \end{vmatrix}$$

Now each term in the expansion of this determinant contains the factor $\left(\frac{\pi}{2}\right)^{2n}$, and when this is taken out we get

$$\xi_n = (-1)^n \frac{\pi^{2n}}{2^{3n}} \begin{vmatrix} \frac{1}{2!} & \frac{1}{4!} & \frac{1}{6!} & \cdots & \frac{1}{(2n-2)!} & \frac{n}{(2n)!} \\ 2 & \frac{1}{2!} & \frac{1}{4!} & \cdots & \frac{1}{(2n-4)!} & \frac{(n-1)}{(2n-2)!} \\ 0 & 2 & \frac{1}{2!} & \cdots & \frac{1}{(2n-6)!} & \frac{(n-2)}{(2n-4)!} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & \frac{1}{4!} & \frac{3}{6!} \\ 0 & 0 & 0 & \cdots & \frac{1}{2!} & \frac{2}{4!} \\ 0 & 0 & 0 & \cdots & 2 & \frac{1}{2!} \end{vmatrix}$$

The nth Bernoulli number B_n is given by

$$B_n = \frac{2\,.\,(2n)\,!}{(2\pi)^{2n}} S_{2n}, \quad \text{where } S_{2n} = \sum_{r=1}^{\infty} \frac{1}{r^{2n}}.$$

Now

$$S_{2n} = z_{2n} + \frac{1}{2^{2n}} S_{2n},$$

whence

$$S_{2n} = \frac{2^{2n}}{2^{2n}-1} z_{2n},$$

and so

$$B_n = \frac{2^{2n+1}\,(2n)!}{(2\pi)^{2n}\,(2^{2n}-1)} z_{2n}$$

$$= \frac{(-1)^n\, 2^{2n}\,(2n)!}{\pi^{2n}\,(2^{2n}-1)} \xi_n.$$

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Hence we have

$$B_n=\frac{(2n)!}{2^n\,(2^{2n}-1)}\begin{vmatrix} \frac{1}{2!} & \frac{1}{4!} & \frac{1}{6!} & \cdots & \frac{1}{(2n-4)!} & \frac{1}{(2n-2)!} & \frac{n}{(2n)!} \\ 2 & \frac{1}{2!} & \frac{1}{4!} & \cdots & \frac{1}{(2n-6)!} & \frac{1}{(2n-4)!} & \frac{n-1}{(2n-2)!} \\ 0 & 2 & \frac{1}{2!} & \cdots & \frac{1}{(2n-8)!} & \frac{1}{(2n-6)!} & \frac{n-2}{(2n-4)!} \\ 0 & 0 & 2 & \cdots & \frac{1}{(2n-10)!} & \frac{1}{(2n-8)!} & \frac{n-3}{(2n-6)!} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & \frac{1}{2!} & \frac{1}{4!} & \frac{3}{6!} \\ 0 & 0 & 0 & \cdots & 2 & \frac{1}{2!} & \frac{2}{4!} \\ 0 & 0 & 0 & \cdots & 0 & 2 & \frac{1}{2!} \end{vmatrix}$$

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