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COMPACT ALMOST DISCRETE HYPERGROUPS

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ABSTRACT. A compact hypergroup is called *almost discrete* if it is homeomorphic to the one-point-compactification of a countably infinite discrete set. If the group U_p of all p-adic units acts multiplicatively on the p-adic integers, then the associated compact orbit hypergroup has this property. In this paper we start with an exact projective sequence of finite hypergroups and use successive substitution to construct a new surjective projective system of finite hypergroups whose limit is almost discrete. We prove that all compact almost discrete hypergroups appear in this way—up to isomorphism and up to a technical restriction. We also determine the duals of these hypergroups, and we present some examples coming from partitions of compact totally disconnected groups.

Introduction. If the group U_p of all p-adic units acts multiplicatively on the additive group $\mathbb{Z}_p$ of p-adic integers, then the associated compact orbit space $\mathbb{Z}_p^{U_p}$ is a commutative hypergroup which is almost discrete, *i.e.*, it is homeomorphic to the one-point-compactification of a countably infinite discrete set. These examples of almost discrete hypergroups were introduced by Dunkl and Ramirez [5] and studied in [6, 12, 13, 15, 21]. Vrem [21] pointed out that these hypergroups appear as projective limits of finite hypergroups formed by successive hypergroup joins; for hypergroup joins we refer to [8, 21, 24].

In this paper we present a construction which is more general than taking successive joins. It leads from a given exact projective sequence of finite hypergroups to compact almost discrete hypergroups. We prove that this construction gives all compact almost discrete hypergroups—up to isomorphism and up to a technical restriction. Our result can be regarded as a classification of compact almost discrete hypergroups in terms of exact projective sequence of finite hypergroups. Classifications of hypergroup structures on other topological spaces—at least under certain additional conditions—are given in Connett and Schwartz [3, 11], Zeuner [23, 24], and Voit [19].

This paper is organized as follows: In the first section we recapitulate some basic facts. In particular we there discuss projective limits of hypergroups (see Voit [18]) as well as the method of substituting open subhypergroups (see Voit [19]). Substitution is crucial for this paper; it generalizes the join of hypergroups as follows: If H is an open subhypergroup of a hypergroup K, and if π is an open and proper hypergroup homomorphism from a further hypergroup L onto H, then the disjoint union of the spaces $K - H$ and L carries a natural hypergroup structure. In Section 2 we shall start with an exact projective sequence of finite hypergroups and use successive substitutions to construct

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a new projective sequence of finite hypergroups whose connecting homomorphisms are surjective and whose limit is almost discrete. We shall in particular prove that all compact almost discrete hypergroups appear in this way, if the identity has a neighbourhood base consisting of normal subhypergroups. As the latter exists at least in the commutative case, it follows that we at least obtain all commutative compact almost discrete hypergroups by our construction. Section 3 is devoted to dual spaces of compact almost discrete hypergroups. We shall describe them in terms of the duals of the hypergroups in the exact sequences from which the compact almost discrete hypergroup is constructed. Finally, Section 4 contains some examples. We there in particular present a method to derive examples from certain group partitions. Our method generalizes a construction of Spector [12, 13].

1. **Preliminaries.** For an introduction to hypergroups we generally refer to Jewett [8] and to Bloom and Heyer [2]. All notation will be standard and is the same as in Voit [18, 19]. However, we here recapitulate some notations and facts for the convenience of the reader.

Let K be a locally compact (Hausdorff) space. Then by $M(K)$, $M_b(K)$, $M_b^+(K)$, and $M^1(K)$ we denote all Radon measures, the bounded ones, those that are bounded and nonnegative, and the probability measures on K respectively. The spaces $C_b(K)$, $C_0(K)$ and $C_c(K)$ consist of the $\mathbb{C}$-valued continuous bounded functions on K, those that are continuous and zero at infinity, and those that are continuous and compactly supported respectively. $\delta_x \in M^1(K)$ is the point measure at $x \in K$.

1.1. Hypergroup homomorphisms (Voit [16] and Zeuner [24]). Let $(K,*)$ and $(J,\bullet)$ be hypergroups with identities e_J and e_K. A continuous mapping $p\colon K \longrightarrow J$ is a hypergroup homomorphism if $\delta_{p(x)} \bullet \delta_{p(y)} = p(\delta_x * \delta_y)$ for all $x,y \in K$. It is easy to see that then $p(e_K) = e_J$ and, for $x \in K$, $e_J \in \{p(x)\} \bullet \{p(\bar{x})\}$ and thus $p(\bar{x}) = \overline{p(x)}$ holds.

p is a *hypergroup isomorphism* if it is also a homeomorphism.

Now let H be a compact normal subhypergroup of K with normalized Haar measure ω_H. Then the coset space K/H bears a natural hypergroup structure such that the quotient mapping $\pi\colon K \longrightarrow K/H$ is a hypergroup homomorphism. In this case,

$$\tilde{\pi}\colon M_b(K|H) := \{\mu \in M_b(K) : \omega_H * \mu = \mu\} \longrightarrow M_b(K/H), \quad \mu \longmapsto \pi(\mu)$$

is an isometric isomorphism of Banach algebras.

1.2. Orbital morphisms (see Jewett [8] and Voit [17]). Let J and K be hypergroups with identities e_J and e_K. A continuous, proper, surjective, and open mapping $\Phi\colon J \longrightarrow K$ is called an *orbital mapping*. Φ is said to be *unary* if $\Phi^{-1}(e_K) = \{e_J\}$.

A *recomposition of* Φ is a weakly continuous mapping $x \longmapsto q_x$ from K to $M^1(J)$ with $\operatorname{supp} q_x = \Phi^{-1}(x)$ for all $x \in K$. Φ is a *generalized orbital morphism associated with* $(q_x)_{x\in K}$ if in addition $q_{\bar{x}} = q_x^-$ and $\Phi(q_x * q_y) = \delta_x * \delta_y$ for all $x,y \in K$.

If there is a measure $l \in M^+(J)$ with $l = \int_J q_{\Phi(y)}\, dl(y)$, then $(q_x)_{x\in K}$ is called *consistent with* l. If $(q_x)_{x\in K}$ is consistent with the Haar measure ω on J, then the generalized orbital morphism Φ is called an *orbital morphism*. Let Φ be a generalized orbital morphism associated with $(q_x)_{x\in K}$. If $M := \{\mu \in M_b(J) : \mu = \int_K q_y\, d\nu(y), \nu \in M_b(K)\}$ is closed under convolution, then Φ is called *consistent*. Obviously, any injective consistent generalized orbital morphism is a hypergroup isomorphism.

1.3 Substitution (see Voit [19]). Let $(K,*)$ and $(L,\bullet)$ be hypergroups and $\pi: L \longrightarrow K$ a proper and open hypergroup homomorphism. Set $W := \operatorname{kern}\pi$. Then there is—up to isomorphism—a unique hypergroup $(M,\diamond)$ with the following properties:

(i) There is an injective and open homomorphism $\tau: L \longrightarrow M$ and a proper, surjective, and open homomorphism $p: M \longrightarrow K$ with $p \circ \tau = \pi$ and $\operatorname{kern} p = \tau(W)$.

(ii) If there exists a further hypergroup $\tilde{M}$, an injective and open hypergroup homomorphism $\tilde{\tau}: L \longrightarrow \tilde{M}$, and a proper, surjective hypergroup homomorphism $\tilde{p}: \tilde{M} \longrightarrow K$ with $\tilde{p} \circ \tilde{\tau} = \pi$ and $\operatorname{kern}\tilde{p} = \tilde{\tau}(W)$, then there exists a unary consistent generalized orbital morphism $\varphi: \tilde{M} \longrightarrow M$ with $\tilde{p} = p \circ \varphi$.

We say *M is obtained from K by substituting the open subhypergroup $H := \pi(L)$ of K by L via π*. M will be denoted by $S(K, H \xrightarrow{\pi} L)$ where the π will be often omitted. $(M,\diamond)$ can be realized as follows: Take $M := (K - H) \cup L$ as the disjoint union of $K - H$ and L where both sets are embedded into M as open sets. Then, $\diamond$ is given by

$$\delta_x \diamond \delta_y := \begin{cases} \delta_x \bullet \delta_y & \text{for } x, y \in L \subset M \\ \delta_{\pi(x)} * \delta_y & \text{for } x \in L \text{ and } y \in M - L = K - H \\ \delta_x * \delta_{\pi(y)} & \text{for } y \in L \text{ and } x \in M - L = K - H \\ (\delta_x * \delta_y)|_{M-L} + \tilde{\pi}^{-1}\big((\delta_x * \delta_y)|_H\big) & \text{for } x, y \in M - L, \end{cases}$$

$\tilde{\pi}: M_b(L|W) \longrightarrow M_b(H)$ being the isometric Banach-$*$-algebra isomorphism of Section 1.1. Identity and involution of M are taken from $K - H$ and L.

The following definitions and results are taken from Voit [18].

1.4. Projective systems. Let $(I, <)$ be a directed set and $(K_i)_{i\in I}$ a family of hypergroups. Let $(g_{i,j})_{j<i}$ be proper hypergroup homomorphisms $g_{i,j}: K_i \longrightarrow K_j$ such that $g_{i,i}$ is the identity and $g_{j,i} \circ g_{k,j} = g_{k,i}$ for $i < j < k$. Then $(K_i, g_{i,j}, I)$ is called a *projective system of hypergroups*. It is called *surjective* if all $g_{i,j}$ are surjective.

THEOREM 1.5. *For each projective system $(K_i, g_{i,j}, I)$ of hypergroups there exists—up to isomorphism—a unique hypergroup K with the following properties:*

(i) There is a family $(g_i)_{i\in I}$ of proper hypergroup homomorphisms $g_i: K \longrightarrow K_i$ such that $g_{j,i} \circ g_j = g_i$ and $g_i(K) = \bigcap_{l>i} g_{l,i}(K_l)$ for all $i < j$.

(ii) Let $\tilde{K}$ be a further hypergroup and $(\tilde{g}_i)_{i\in I}$ a family of proper homomorphisms $\tilde{g}_i: \tilde{K} \longrightarrow K_i$ with $g_{j,i} \circ \tilde{g}_j = \tilde{g}_i$ and $\tilde{g}_i(\tilde{K}) = \bigcap_{l>i} g_{l,i}(K_l)$ for $i < j$. Then there is a surjective and proper homomorphism $\tau: \tilde{K} \longrightarrow K$ with $\tilde{g}_i = g_i \circ \tau$ for $i \in I$.

K is called the projective limit of $(K_i, g_{i,j}, I)$.

2. **Construction of compact, almost discrete hypergroups.** In this paper the following definition will be convenient:

DEFINITION 2.1. A topological space K is called *almost discrete* if K is countably infinite and every element of K except for exactly one point is isolated.

As a hypergroup is discrete if and only if its identity element is isolated (Jewett [8], Theorem 7.1B), the cluster point of an almost discrete hypergroup has to be the identity element.

The purpose of this section is to discuss a general principle for constructing compact, almost discrete hypergroups. The method is based on successive substitution of finite subhypergroups and then taking a projective limit of hypergroups.

2.2. A construction. Let $(H_i, g_{i,j}, \mathbb{N})$ be a projective system of hypergroups with H as its projective limit according to Sections 1.4 and 1.5 (this in particular means that the homomorphisms $g_{i,j}: H_i \longrightarrow H_j$ are proper). Assume that the subhypergroups $L_i := g_{i+1,i}(H_{i+1})$ are open in H_i for all $i \in \mathbb{N}$. Then the homomorphisms $g_{i,j}: H_i \longrightarrow H_j$ are open by Proposition 1.7 of Voit [16]. We inductively construct a family $(K_i, \pi^i)_{i\in\mathbb{N}}$ of hypergroups K_i and open, proper, and surjective homomorphisms $\pi^i: K_{i+1} \longrightarrow K_i$ as follows:

(1) Put $K_1 := H_1$.

(2) Assume that K_i is constructed and that K_i contains H_i as an open subhypergroup in a natural way. Then L_i is an open subhypergroup of K_i. Using the surjective, open, and proper homomorphism $g_{i+1,i}: H_{i+1} \longrightarrow L_i$, we define

$$K_{i+1} := S(K_i, L_i \longrightarrow H_{i+1}).$$

Then H_{i+1} is an open subhypergroup of K_{i+1}, and we can iterate the process.

The substitution $S(K_i, L_i \longrightarrow H_{i+1})$ induces a natural surjective and proper homomorphism $\pi^i: K_{i+1} \longrightarrow K_i$ satisfying $\pi^i|_{H_{i+1}} = g_{i+1,i}$. By Proposition 1.7 of [16], π^i is also open. For $j < i$ we define $g^{i,j}: \pi^j \circ \pi^{j+1} \circ \cdots \pi^{i-1}: K_i \longrightarrow K_j$. Then $(K_i, g^{i,j}, \mathbb{N})$ is a surjective projective system of hypergroups. Let K be its projective limit.

H can be regarded as subhypergroup of K in a natural way (see Section 4 of Voit [18]), and we have the following commutative diagram

$$\begin{array}{ccccccccc}
H_1 & \overset{g_{2,1}}{\longleftarrow} & H_2 & \overset{g_{3,2}}{\longleftarrow} & H_3 & \cdots & \longleftarrow & \cdots & H \\
\downarrow & & \downarrow & & \downarrow & & & & \downarrow \\
K_1 & \overset{g^{2,1}}{\longleftarrow} & K_2 & \overset{g^{3,2}}{\longleftarrow} & K_3 & \cdots & \longleftarrow & \cdots & K
\end{array}$$

Let $(H_i, g_{i,j}, \mathbb{N})$ be a projective system of hypergroups. For abbreviation we denote the new projective system $(K_i, g^{i,j}, \mathbb{N})$ by $\mathrm{PS}(H_i, g_{i,j}, \mathbb{N})$.

REMARKS 2.3. (1) The hypergroups H_i are compact, totally disconnected, commutative, symmetric or unimodular if and only if the projective limit K has the same property. This is a consequence of the fact that these properties are obviously preserved under substitution and surjective projective limits (see Voit [18, 19]).

(2) Let $(H_i, g_{i,j}, \mathbb{N})$ be a surjective projective system of hypergroups and $(K_i, g^{i,j}, \mathbb{N})$ the associated system according to 2.2. It follows inductively that then the homomorphisms from H_i into K_i are hypergroup isomorphisms. Thus, PS transforms surjective projective systems into isomorphic ones. In particular, as $\mathrm{PS}(H_i, g_{i,j}, \mathbb{N})$ is surjective for any projective system $(H_i, g_{i,j}, \mathbb{N})$, it follows that PS is an idempotent operation.

(3) For a finite sequence of hypergroups H_i, the idea of the construction 2.2 can be also found in Section 4.4 of Voit [19].

We are interested in constructing compact, almost discrete hypergroups. The following proposition yields a sufficient condition such that the limit K of a projective system $\mathrm{PS}(H_i, g_{i,j}, \mathbb{N})$ is compact and almost discrete. Remark 2.3(2) above shows that the conditions are far from being necessary.

PROPOSITION 2.4. *We retain the notation of Section 2.2 and assume that the hypergroups H_i are finite. Then the limits H and K are compact, second countable, and totally disconnected. Furthermore, the following conclusions hold:*

(1) Each $x \in K - H$ is isolated in K.

(2) $x = (x_i)_{i\in\mathbb{N}} \in H$ is not isolated in K if and only if $|g_{i+1,i}^{-1}(x_i)| \geq 2$ for infinitely many $i \in \mathbb{N}$.

(3) If $H = \{e\}$, and if the mappings $g_{i+1,i}\colon H_{i+1} \longrightarrow H_i$ are not injective for infinitely many $i \in \mathbb{N}$, then K is a compact, almost discrete hypergroup.

PROOF. Clearly, H and K are compact, second countable, and totally disconnected.

(1) If $x = (x_i)_{i\in\mathbb{N}} \in K - H$, then there exists $j \in \mathbb{N}$ such that $x_i \in K_i - H_i$ for $i > j$. It follows from the construction of the hypergroups K_i that $(\pi^i)^{-1}(x_i) = \{x_{i+1}\}$ for all $i \geq j$. Hence,

$$W := \{x_1\} \times \cdots \times \{x_j\} \times \prod_{i\geq j+1} K_i \subset \prod_{i\in\mathbb{N}} K_i$$

satisfies $W \cap K = \{(x_i)_{i\in\mathbb{N}}\}$. As W is open in $\prod_{i\in\mathbb{N}} K_i$, $(x_i)_{i\in\mathbb{N}}$ is isolated in K.

(2) Let $x = (x_i)_{i\in\mathbb{N}} \in H$. As $(K_i, g^{i,j}, \mathbb{N})$ is a surjective projective system, x is not isolated in K if and only if $|(g^{i+1,i})^{-1}(x_i)| \geq 2$ for infinitely many $i \in \mathbb{N}$. By the construction of $(K_i, g^{i,j}, \mathbb{N})$, this is equivalent to the fact that $|g_{i+1,i}^{-1}(x_i)| \geq 2$ for infinitely many $i \in \mathbb{N}$.

(3) It follows from the parts (1) and (2) that K is an infinite, compact hypergroup such that each $x \in K - \{e\}$ is isolated. In particular, e must be a cluster point. As K is second countable, we see that K is countably infinite. This completes the proof.

2.5. Let $(H_i, g_{i,j}, \mathbb{N})$ be a projective system of hypergroups as introduced in Section 2.2. Suppose that $\ker g_{i,i-1} \subset g_{i+1,i}(H_{i+1})$ for all $i \geq 2$. We consider the open subhypergroups $W_i := g_{i+2,i}(H_{i+2})$ of H_i. Assume that all W_i are finite and that $g_{i+1,i}$ induces a hypergroup isomorphism between W_{i+1} and W_i for each $i \in \mathbb{N}$. These assumptions in particular imply that the H_i are discrete and, for $i \geq 3$, finite. Moreover, the projective limit H is isomorphic to all W_i.

We now restrict our attention to the case that all H_i are finite. Let K be the limit of the projective system $(K_i, g^{i,j}, \mathbb{N}) = \mathrm{PS}(H_i, g_{i,j}, \mathbb{N})$. By Proposition 2.4, $x = (x_i)_{i\in\mathbb{N}} \in K$ is not isolated if and only if $x \in H \subset K$ and $|g_{i+1,i}^{-1}(x_i)| \geq 2$ for infinitely many $i \in \mathbb{N}$. In particular, if for each $x \in H$ there exist infinitely many $i \in \mathbb{N}$ satisfying $|g_{i+1,i}^{-1}(x_i)| \geq 2$, then K is a compact hypergroup whose non-isolated points form a finite subhypergroup isomorphic to H. Theorem 2.6 below shows that, under a mild restriction, the converse statement is also true.

Before stating this result, we consider an important special case which leads to compact, almost discrete hypergroups. In the setting above, $W_i = \{e\}$ means that $g_{i+2,i+1}(H_{i+2}) = \operatorname{kern} g_{i+1,i}$ holds. If this is true for all $i \in \mathbb{N}$, then the projective system $(H_i, g_{i,j}, \mathbb{N})$ is called *exact*. Using the conclusion above, we see that the projective limit of $\mathrm{PS}(H_i, g_{i,j}, \mathbb{N})$ is a compact, almost discrete hypergroup if $(H_i, g_{i,j}, \mathbb{N})$ is an exact projective system of finite hypergroups such that infinitely many H_i are not trivial.

THEOREM 2.6. *Let K be a compact, countably infinite hypergroup such that the set V of all non-isolated points of K is a finite subhypergroup. Assume that K admits a neighbourhood base of e consisting of open and normal subhypergroups. Then there exists a projective system $(H_i, g_{i,j}, \mathbb{N})$ of finite hypergroups with the following properties:*

- *(i)* $\operatorname{kern} g_{i+1,i} \subset g_{i+2,i+1}(H_{i+2})$, *and $W_i := g_{i+2,i}(H_{i+2})$ is a subhypergroup of H_i isomorphic to V. Moreover, $g_{i+1,i}$ establishes an isomorphism between W_{i+1} and W_i for all $i \in \mathbb{N}$.*
- *(ii)* $|g_{i+1,i}^{-1}(x_i)| \geq 2$ *for all $i \in \mathbb{N}$ and $x_i \in W_i$.*
- *(iii) K is isomorphic to the limit of* $\mathrm{PS}(H_i, g_{i,j}, \mathbb{N})$.

In particular, if K is a compact, almost discrete hypergroup which admits a neighbourhood base of e consisting of open and normal subhypergroups, then there exists an exact projective system $(H_i, g_{i,j}, \mathbb{N})$ of finite hypergroups such that K is isomorphic to the limit of $\mathrm{PS}(H_i, g_{i,j}, \mathbb{N})$.

Before we prove this theorem, we discuss its restrictions. Each compact open neighbourhood of the identity element of a hypergroup contains an open subhypergroup by Vrem [22]. However, we do not know whether for each compact hypergroup these open subhypergroups can be taken to be normal as would be the case for compact groups. As this problem does not appear for commutative hypergroups, we obtain the following corollary.

COROLLARY 2.7. *Let K be a commutative, compact and almost discrete hypergroup. Then there exists an exact projective system $(H_i, g_{i,j}, \mathbb{N})$ of finite commutative hypergroups such that K is isomorphic to the limit of* $\mathrm{PS}(H_i, g_{i,j}, \mathbb{N})$.

The proof of Theorem 2.6 is based on the following lemma:

LEMMA 2.8. *Let K be a hypergroup. If $x \in K - \{e\}$ is isolated in K, then*

$$U_x := \{y \in K : \delta_x = \delta_x * \delta_y = \delta_y * \delta_x, \delta_{\bar{x}} = \delta_{\bar{x}} * \delta_y = \delta_y * \delta_{\bar{x}}\}$$

is an open and compact subhypergroup of K such that $x \notin U_x$. Moreover, $(\delta_x * \delta_{\bar{x}})|_{U_x}$ is a (non-trivial) Haar measure of U_x.

PROOF. In order to check that U_x is a subhypergroup, take $y_1, y_2 \in U_x$. Then $\delta_{y_1} * \delta_{y_2} * \delta_x = \delta_{y_1} * \delta_x = \delta_x$ and thus $\delta_z * \delta_x = \delta_x$ for all $z \in \mathrm{supp}(\delta_{y_1} * \delta_{y_2})$. Similar arguments yield $\delta_x * \delta_z = \delta_x$ and $\delta_z * \delta_{\bar{x}} = \delta_{\bar{x}} * \delta_z = \delta_{\bar{x}}$ for $z \in \mathrm{supp}(\delta_{y_1} * \delta_{y_2})$. As U_x is also closed under involution, U_x is a subhypergroup.

We next note that $x \notin U_x$ since otherwise $\delta_x = \delta_x * \delta_{\bar{x}}$ in contradiction to $x \neq e$.

To show that U_x is open in K, we first recapitulate that the space $\mathcal{C}(K)$ of all compact non-void subsets of K carries a natural topology (for details see Jewett [8] and Michael [9]), and that, by the hypergroup axioms, the mapping

$$K \longrightarrow \mathcal{C}(K), \quad y \longmapsto \mathrm{supp}(\delta_x * \delta_y),$$

is continuous with respect to this topology. As $\delta_x * \delta_e = \delta_x$ and as $\{x\}$ is isolated in $\mathcal{C}(K)$, the set $\{u \in K : \mathrm{supp}(\delta_x * \delta_u) = \{x\}\}$ is open. If we apply this argument also to the other equations which characterize U_x, it follows that U_x is open.

We next investigate the measure $\tau := (\delta_x * \delta_{\bar{x}})|_{U_x}$. Since U_x is open, and since $e \in \mathrm{supp}(\delta_x * \delta_{\bar{x}})$, it follows that τ is a nontrivial, bounded and positive measure on U_x. Moreover, for $y \in U_x$, we have

$$\tau * \delta_y = (\tau * \delta_y)|_{U_x} = \left(\delta_x * \delta_{\bar{x}} * \delta_y - (\delta_x * \delta_{\bar{x}})|_{K-U_x} * \delta_y\right)\Big|_{U_x} = (\delta_x * \delta_{\bar{x}})|_{U_x} - 0 = \tau$$

and, similarly, $\delta_y * \tau = \tau$. This proves that τ is a bounded Haar measure on U_x. Therefore, by Theorem 7.2B of Jewett [8], U_x is compact, and the proof is finished.

PROOF OF THEOREM 2.6. (1) Let $(x_i)_{i \in \mathbb{N}}$ be an enumeration of $K - V$. We construct open and normal subhypergroups U_i of K inductively as follows:

Put $U_1 := K$.

If U_i is constructed, then we consider the compact subhypergroup

$$R_i := U_i \cap U_{x_i} \cap \bigcap_{x \in K - V * U_i} U_x$$

of K, the hypergroup U_x being defined as in Lemma 2.8. As $V * U_i$ is an open set containing V, it follows that $K - V * U_i$ is finite. Hence R_i is open. Now choose U_{i+1} as an open and normal subhypergroup of K such that

$$U_{i+1} \subset R_i, \quad U_{i+1} * \{v\} \cap U_{i+1} * \{u\} = \emptyset, \text{ and } U_{i+1} * \{v\} \neq U_i * \{v\} \quad \text{for } u, v \in V,\ u \neq v.$$

In fact, this is possible by our assumptions and by Lemma 3.2D of Jewett [8]. In particular, we have $U_{i+1} \subset U_i$ for all $i \in \mathbb{N}$ and $\bigcap_{i \in \mathbb{N}} U_i = \{e\}$ where the second statement is a consequence of the facts that $U_i \cap V = \{e\}$ for $i \geq 2$ and that $\bigcap_{i \in \mathbb{N}} U_i \subset \bigcap_{i \in \mathbb{N}} U_{x_i} \subset V$ (see Lemma 2.8). This completes the construction of the U_i.

As U_i is normal in K for each $i \in \mathbb{N}$, the subhypergroup $U_i * V$ is open in K ($i \in \mathbb{N}$). Now we define finite hypergroups H_i and L_i by

$$H_1 := L_1 := (U_1 * V)/U_2, \; H_i := (U_{i-1} * V)/U_{i+1}, \; L_i := (U_i * V)/U_{i+1} \subset H_i \quad (i \geq 2).$$

By the isomorphism theorem (Jewett [8], 14.3A), the coset hypergroups

$$H_i/(U_i/U_{i+1}) = \big((U_{i-1} * V_i)/U_{i+1}\big)/(U_i/U_{i+1}) \quad \text{and} \quad (U_{i-1} * V)/U_i = L_{i-1}$$

are isomorphic, and we shall identify $H_i/(U_i/U_{i+1})$ and L_{i-1} from now on. Taking the canonical projections

$$\pi_i : H_{i+1} \longrightarrow H_{i+1}/(U_{i+1}/U_{i+2}) = L_i \subset H_i$$

and defining $g_{i,j}: H_i \longrightarrow H_j$ by $g_{i,j} = \pi_{j+1} \circ \cdots \circ \pi_i$, we obtain a projective system $(H_i, g_{i,j}, \mathbb{N})$. The isomorphism theorem also shows that

$$g_{i+1,i-1}(H_{i+1}) = g_{i,i-1}(L_i) \simeq (U_i * V)/U_i \quad \text{for } i \geq 2.$$

Moreover, $\varphi_i : V \longrightarrow (U_i * V)/U_i$, $v \longmapsto v * U_i$ is a hypergroup isomorphism. Thus, condition (i) holds. Moreover, $g_{i+2,i}(H_{i+2}) = (U_{i+1} * V)/U_{i+1} \simeq V$, $U_{i+1} \subset U_i$, and $U_{i+1} * \{v\} \neq U_i * \{v\}$ for $v \in V$ ensure that that condition (ii) holds.

(2) We still have to check that K is isomorphic to the projective limit $\tilde{K}$ of the system $\mathrm{PS}(H_i, g_{i,j}, \mathbb{N})$. For this, we first fix $i \in \mathbb{N}$. The obvious homomorphism from $(U_i * V)/U_{i+2}$ onto $(U_i * V)/U_{i+1} \subset K/U_{i+1}$ yields that we may apply part (ii) of Section 1.3 to $\tilde{M}_i := K/U_{i+2}$. It follows that

$$\begin{aligned} r_i : K/U_{i+2} &\longrightarrow S\big(K/U_{i+1}, (U_i * V)/U_{i+1} \longrightarrow (U_i * V)/U_{i+2}\big), \\ \{x\} * U_{i+2} &\longmapsto \begin{cases} \{x\} * U_{i+1} & \text{if } x \in K - U_i * V \\ \{x\} * U_{i+2} & \text{if } x \in U_i * V, \end{cases} \end{aligned} \tag{2.1}$$

is a consistent generalized orbital morphism. As $\{x\} * U_{i+1} = \{x\} = \{x\} * U_{i+2}$ for all $x \in K - U_i * V$ by construction, r_i is bijective and hence an isomorphism. Obviously, r_i is equal to the identity on the common subhypergroup $(U_i * V)/U_{i+2}$ of the hypergroups $S\big(K/U_{i+1}, (U_i * V)/U_{i+1} \longrightarrow (U_i * V)/U_{i+2}\big)$ and K/U_{i+2}.

Let $(K_i, g_{i,j}, \mathbb{N})$ be the projective system associated with the hypergroups $H_i = (U_{i-1} * V)/U_{i+1}$ and $L_i = (U_i * V)/U_{i+1}$ according to Section 2.2. Now we construct inductively hypergroup isomorphisms $\tau_i : K_i \longrightarrow K/U_{i+1}$ for $i \in \mathbb{N}$ as follows: Using $K_1 = H_1 = K/U_2$, we take τ_1 to be the identity. If τ_i is constructed, then τ_i induces an obvious isomorphism

$$\begin{aligned} \tilde{\tau}_i : K_{i+1} = S\big(K_i, (U_i * V)/U_{i+1} \longrightarrow (U_i * V)/U_{i+2}\big) \longrightarrow \\ S\big(K/U_{i+1}, (U_i * V)/U_{i+1} \longrightarrow (U_i * V)/U_{i+2}\big). \end{aligned}$$

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We define $\tau_{i+1}: K_{i+1} \longrightarrow K/U_{i+2}$ by $\tau_{i+1} = r_i^{-1} \circ \tilde{\tau}_i$. By this construction,

$$\begin{array}{ccc} K_i & \xleftarrow{g_{i+1,i}} & K_{i+1} \\ \tau_i \downarrow & & \tau_{i+1} \downarrow \\ K/U_{i+1} & \longleftarrow & K/U_{i+2} \end{array}$$

commutes for each $i \in \mathbb{N}$. Therefore, by part (ii) of Section 1.5, there exists an open homomorphism τ from K onto the projective limit $\tilde{K}$ such that, for $i \in \mathbb{N}$ and $z \in K$, $\tau_i^{-1}(\{z\} * U_{i+1})$ is equal to the i-th component of $\tau(z) \in \tilde{K}$. Finally, $\bigcap_{i \in \mathbb{N}} U_i = \{e\}$ implies that τ is injective. This completes the proof of the theorem.

Let $(H_i, g_{i,j}, \mathbb{N})$ be an exact projective system of hypergroups as introduced in Section 2.2. Then, in particular, the hypergroups H_i are discrete and, for $i \geq 3$, also finite (*cf.* Section 2.5). As above we put $L_i := g_{i+1,i}(H_i)$ and assume that L_i is non-trivial for infinitely many $i \in \mathbb{N}$. We now determine the limit K of $\mathrm{PS}(H_i, g_{i,j}, \mathbb{N})$ explicitly. By Theorem 2.6, this includes all compact, almost discrete hypergroups which admit a neighbourhood base of the identity consisting of normal subhypergroups.

THEOREM 2.9. *Retaining the situation above, we realize K as follows:*

If $W_k := H_k - L_k$ for $k \in \mathbb{N}$, then $K := \bigcup_{k=1}^{\infty} W_k \cup \{e\}$ is the disjoint union of the identity element e and of the sets W_k which are assumed to be embedded into K as open subsets. The sets $\bigcup_{k=n}^{\infty} W_k \cup \{e\}$ $(n \in \mathbb{N})$ form a neighbourhood base of the only non-isolated point $e \in K$.

Let e_k be the identity element and $*_k$ the convolution of H_k. If ω_k is the Haar measure on H_k normalized by $\omega_k(H_k) = 1$, then the convolution on K satisfies

$$\delta_x * \delta_y = \begin{cases} \delta_x & \text{if } x \in W_k,\ y \in W_l,\ k \leq l-2 \\ \delta_y & \text{if } x \in W_k,\ y \in W_l,\ l \leq k-2 \\ \delta_x *_k \delta_{g_{k+1,k}(y)} & \text{if } x \in W_k,\ y \in W_{k+1} \\ \delta_{g_{k+1,k}(x)} *_k \delta_y & \text{if } y \in W_k,\ x \in W_{k+1} \\ (\delta_x *_k \delta_y)|_{W_k} + \tilde{g}_{k+1,k}^{-1}\big((\delta_x *_k \delta_y)|_{L_k}\big) & \text{if } x,y \in W_k,\ x \neq \bar{y} \\ (\delta_x *_k \delta_{\bar{x}})|_{W_k} + \tilde{g}_{k+1,k}^{-1}\big((\delta_x *_k \delta_{\bar{x}})|_{L_k - \{e_k\}}\big) & \\ \quad + \sum_{j=k+2}^{\infty} \delta_x *_k \delta_{\bar{x}}(\{e_k\}) \cdot \big(\prod_{s=k+2}^{j-1} \omega_s(L_s)\big) \omega_j|_{W_j} & \text{if } x,y \in W_k,\ x = \bar{y} \end{cases}$$

where $\tilde{g}_{k+1,k}^{-1}$ is the inverse of the isomorphism $\tilde{g}_{k+1,k}: M_b(H_{k+1}|L_{k+1}) \longrightarrow M_b(L_k)$; cf. Section 1.1.

PROOF. Consider the hypergroups K_i $(i \in \mathbb{N})$ defined by $K_1 := H_1$ and $K_{i+1} := S(K_i, L_i \longrightarrow H_{i+1})$. Using the results of Section 1.3 and induction, we see that K_i is given by $K_i = \bigcup_{k=1}^{i-1} W_k \cup H_i$ with the following convolution:

$$\delta_x * \delta_y = \begin{cases} \delta_x & \text{if } x \in H_k \cap K_i, \\ & \quad y \in H_l \cap K_i, \\ & \quad k \le l-2 \le i-2 \\ \delta_y & \text{if } x \in H_k \cap K_i, \\ & \quad y \in H_l \cap K_i, \\ & \quad l \le k-2 \le i-2 \\ \delta_x *_k \delta_{g_{k+1,k}(y)} & \text{if } x \in W_k, y \in W_{k+1}, \\ & \quad k \le i-1 \\ \delta_{g_{k+1,k}(x)} *_k \delta_y & \text{if } y \in W_k, x \in W_{k+1}, \\ & \quad k \le i-1 \\ \delta_x *_i \delta_y & \text{if } x,y \in H_i \\ (\delta_x *_k \delta_y)|_{W_k} + \tilde{g}_{k+1,k}^{-1}\big((\delta_x *_k \delta_y)|_{L_k}\big) & \text{if } x,y \in W_k, \text{ and either} \\ & k = i-1 \\ & \text{or } k < i-1 \\ & \text{and } x \neq \bar{y} \\ (\delta_x *_k \delta_{\bar{x}})|_{W_k} + \tilde{g}_{k+1,k}^{-1}\big((\delta_x *_k \delta_{\bar{x}})|_{L_k - \{e_k\}}\big) & \\ \quad + \sum_{j=k+2}^{i-1} \delta_x *_k \delta_{\bar{x}}(\{e_k\}) \cdot \big(\prod_{s=k+2}^{j-1} \omega_s(L_s)\big)\, \omega_j|_{W_j} & \\ \quad + \delta_x *_k \delta_{\bar{x}}(\{e_k\}) \cdot \big(\prod_{s=k+2}^{i-1} \omega_s(L_s)\big)\omega_i & \text{if } x,y \in W_k, k \le i-2, \\ & \quad x = \bar{y} \end{cases}$$

K is the projective limit of $(K_i, g^{i,j}, \mathbb{N})$ where the surjective homomorphisms $g^{i+1,i} \colon K_{i+1} \longrightarrow K_i$ satisfy

(2.2) $$g^{i+1,i}(x) := \begin{cases} x & \text{if } x \in \bigcup_{k=1}^{i} W_k \\ g_{i+1,i}(x) & \text{if } x \in H_{i+1} \end{cases}$$

The assertion of Theorem 2.9 now follows from the convolution on the hypergroups K_i and the construction of a projective limit of hypergroups; see the proof of Theorem 2.2 of Voit [18].

REMARK 2.10. It is difficult to find attractive criteria when the application of the construction PS to two given projective systems leads to projective systems with isomorphic limits. Clearly, the information about the structure of a given compact, almost discrete hypergroup will be optimal if the subhypergroups U_i in the proof of Theorem 2.6 are as large as possible.

3. **The dual space.** We next compute the dual space $\hat{K}$ of the hypergroup K as studied in Theorem 2.9 above in terms of the dual spaces of the hypergroups H_i. For details on (irreducible) representations of hypergroups see Bloom and Heyer [1], Jewett [8] and Vrem [20]. For annihilators in $\hat{K}$ we refer to Bloom and Heyer [1, 2] and Voit [16, 18, 19].

THEOREM 3.1. *Let $(H_i, g_{i,j}, \mathbb{N})$ be an exact projective system of finite hypergroups and let $K := \bigcup_{k=1}^{\infty} W_k \cup \{e\}$ be defined as in Theorem 2.9.*

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(1) For each irreducible representation T of H_1 on some Hilbert space $\mathcal{H}$,

$$R_1(T): K \longrightarrow B(\mathcal{H}), \quad R_1(T)(x) := \begin{cases} T(x) & \text{if } x \in H_1 - L_1 \\ T\big(g_{2,1}(x)\big) & \text{if } x \in H_2 - L_2 \\ 1 & \text{otherwise} \end{cases}$$

defines an irreducible representation $R_1(T)$ of K on the same Hilbert space $\mathcal{H}$. This construction leads to an injective mapping R_1 from $\hat{H}_1$ to $\hat{K}$.

(2) If for $i \geq 2$ we have some irreducible representation $T \in \hat{H}_i - A(\hat{H}_i, L_i)$ on some Hilbert space $\mathcal{H}$, then

$$R_i(T): K \longrightarrow B(\mathcal{H}), \quad R_i(T)(x) := \begin{cases} 0 & \text{if } x \in H_k - L_k,\ k < i \\ T(x) & \text{if } x \in H_i - L_i \\ T\big(g_{i+1,i}(x)\big) & \text{if } x \in H_{i+1} - L_{i+1} \\ 1 & \text{otherwise} \end{cases}$$

also defines an irreducible representation $R_i(T)$ of K. This again leads to an injective mapping R_i from $\hat{H}_i - A(\hat{H}_i, L_i)$ to $\hat{K}$.

(3) The sets $R_1(\hat{H}_1)$, $R_i\big(\hat{H}_i - A(\hat{H}_i, L_i)\big)$ and $R_j\big(\hat{H}_j - A(\hat{H}_j, L_j)\big)$ are disjoint subsets of $\hat{K}$ for $1 < i < j$, and we have

$$\hat{K} = R_1(\hat{H}_1) \cup \bigcup_{i \geq 2} R_i\big(\hat{H}_i - A(\hat{H}_i, L_i)\big).$$

PROOF. First fix $l \in \mathbb{N}$. The definition of the hypergroups K_l together with Theorem 3.3 of Voit [19] and an obvious induction lead to the following description of the dual space $\hat{K}_l$ of K_l: If we have some irreducible representation $T \in \hat{H}_1$ on some Hilbert space $\mathcal{H}$, then

$$R_{1,l}(T): K_l \longrightarrow B(\mathcal{H}), \quad R_{1,l}(T)(x) := \begin{cases} T(x) & \text{if } x \in W_1 \\ T\big(g_{2,1}(x)\big) & \text{if } x \in W_2 \\ 1 & \text{otherwise} \end{cases}$$

establishes an irreducible representation $R_{1,l}(T) \in \hat{K}_l$ on $\mathcal{H}$. Moreover, if we have some $T \in \hat{H}_i - A(\hat{H}_i, L_i)$ for $i \geq 2$, then

$$R_{i,l}(T): K_l \longrightarrow B(\mathcal{H}), \quad R_{i,l}(T)(x) := \begin{cases} 0 & \text{if } x \in W_k \text{ and } k < i \\ T(x) & \text{if } x \in W_i\ (i < l) \text{ or } x \in H_l \\ T\big(g_{i+1,i}(x)\big) & \text{if } x \in W_{i+1} \text{ and } i + 1 \leq l \\ 1 & \text{otherwise} \end{cases}$$

again defines an irreducible representation $R_{i,l}(T)$ of K_l on the same Hilbert space $\mathcal{H}$. Moreover, it is clear that the mappings $R_{i,l}$ $(1 \leq i \leq l)$ are obviously injective, and that $\hat{K}_l = R_{1,l}(\hat{H}_1) \cup \bigcup_{i=2}^{l} R_{i,l}\big(\hat{H}_i - A(\hat{H}_i, L_i)\big)$ holds where the union is disjoint.

We now have to extend these results to the projective limit. The natural projections from K onto K_l $(l \in \mathbb{N})$ are given by

$$g_l(x) := \begin{cases} x & \text{if } x \in \bigcup_{i \leq l} W_i \\ g_{l+1,l}(x) & \text{if } x \in W_{l+1} \\ e_l & \text{otherwise} \end{cases} \tag{3.1}$$

It follows from Theorem 5.1 of Voit [18] that the mappings R_i as defined in Theorem 3.1 establish injective mappings from $\hat{H}_1$ (for $i = 1$) and $\hat{H}_i - A(\hat{H}_i, L_i)$ (for $i \geq 2$) respectively to $\hat{K}$, and that each irreducible representation of K appears in this way. As the other assertions are evident, the proof is complete.

REMARKS 3.2. In the setting of Theorem 2.9, the hypergroup K is commutative if and only if all hypergroups H_i are commutative. In this case, Theorem 3.1 may be stated as follows: For each character $\alpha \in \hat{K}$ there exists either a unique $\alpha_1 \in \hat{H}_1$ such that

$$\alpha(x) = \begin{cases} \alpha_1(x) & \text{if } x \in W_1 \\ \alpha_1\big(g_{2,1}(x)\big) & \text{if } x \in W_2 \\ 1 & \text{otherwise} \end{cases}$$

or there exist a unique $i \geq 2$ and a unique $\alpha_i \in \hat{H}_i - A(\hat{H}_i, L_i)$ such that

$$\alpha(x) = \begin{cases} 0 & \text{if } x \in W_k,\ k < i \\ \alpha_i(x) & \text{if } x \in W_i \\ \alpha_i\big(g_{i+1,i}(x)\big) & \text{if } x \in W_{i+1} \\ 1 & \text{otherwise} \end{cases} \tag{3.2}$$

It is easy to decide whether $\hat{K}$ carries a dual hypergroup structure. In fact, by Theorem 5.7 of Voit [18], this is equivalent to the statement that all dual spaces $\hat{K}_l$ carry hypergroup structures. Moreover, the latter statement is true whenever all H_i have dual hypergroups (this follows from Theorem 3.6 of Voit [19] and induction). We also note that K, as a totally disconnected hypergroup, has a dual hypergroup if and only if K is a Pontryagin hypergroup, *i.e.* if Pontryagin's duality theorem holds for K (*cf.* Theorem 6.5(2) of Voit [18]). It is not difficult to investigate the structure of $\hat{K}$ whenever the dual $\hat{K}$ carries a dual hypergroup structure. This will be done in a forthcoming paper by using substitution of open subhypergroups together with inductive limits of suitable finite hypergroups.

We finish our discussion of the dual $\hat{K}$ with the following consequence of Theorem 2.6 above and Theorem 3.6 of Voit [19]:

COROLLARY 3.3. *The following statements are equivalent:*

(1) K is a compact, almost discrete Pontryagin hypergroup.

(2) K is compact, almost discrete and commutative, and has a dual hypergroup $\hat{K}$.

(3) There exists an exact projective system $(H_i, g_{i,j}, \mathbb{N})$ of finite Pontryagin hypergroups such that K is isomorphic to the limit of $\mathrm{PS}(H_i, g_{i,j}, \mathbb{N})$.

4. Examples.

EXAMPLES 4.1. Let $(L_i)_{i\in\mathbb{N}}$ be a sequence of nontrivial finite hypergroups. Set $H_1 := L_1$ and $H_i := S(L_{i-1}, \{e\} \longrightarrow L_i)$ for $i \geq 2$. Using the canonical projections $\pi_i : H_{i+1} \longrightarrow H_{i+1}/L_{i+1} = L_i$ and $g_{i,j} := \pi_j \circ \cdots \circ \pi_{i+1} : H_i \longrightarrow H_j$ for $i > j$, we obtain an exact projective system $(H_i, g_{i,j}, \mathbb{N})$ as assumed in Section 2.5.

For a constant $b \in]0, 1]$, we define the hypergroup structure L^b on the set $\{e, z\}$ with identity element e by $\delta_z * \delta_z = (1 - b)\delta_z + b\delta_e$. Clearly, each hypergroup K with $|K| = 2$ is isomorphic to some L^b.

Now we fix a sequence $(b_i)_{i\in\mathbb{N}} \subset]0,1]$. If we apply the construction above to the hypergroups $L_i := L^{b_i}$, then the convolution on the resulting compact, almost discrete hypergroup $K = \mathbb{N}\cup\{e\}$ with identity element e is given by $\delta_n * \delta_m = \delta_n$ for $n < m$ and by

$$\delta_n * \delta_n = (1-b_n)\delta_n + \sum_{k=n+1}^{\infty} \frac{b_n}{b_k} \prod_{l=n+1}^{k} \frac{b_l}{1+b_l}\delta_k.$$

Clearly, K is a Pontryagin hypergroup by Corollary 3.3. Moreover, our construction just leads to the hypergroups (in particular for constant coefficients b_i) introduced in Dunkl and Ramirez [5] and Spector [13].

EXAMPLES 4.2. Let $(L_i)_{i\in\mathbb{N}}$ be a sequence of nontrivial finite groups. Assume that for each $i \in \mathbb{N}$ the group L_i acts on L_{i+1} as a group of automorphisms. Set $H_1 := L_1$ and, for $i \geq 2$, $H_i := L_i \times L_{i-1}$ as the semidirect product of L_i and L_{i-1}. Then L_i is a normal subgroup of H_i in a natural way. Using the associated natural homomorphisms $\pi_i\colon H_{i+1} \longrightarrow H_{i+1}/L_{i+1} = L_i \subset H_i$ and $g_{i,j} := \pi_j \circ \cdots \circ \pi_{i-1}\colon H_i \longrightarrow H_j$ for $i > j$, we get an exact projective system $(H_i, g_{i,j}, \mathbb{N})$. It is now possible to apply construction 2.2 to this example which leads to a modified projective system of finite hypergroups whose limit is almost discrete; see also Section 2.5. We finally mention that this almost discrete limit can be described explicitly in terms of the groups L_i with the aid of Theorem 2.9.

For a concrete example, we propose to take a finite non-abelian group G which acts on itself by conjugation and then to put $L_i := G$ for all $i \in \mathbb{N}$.

4.3. Hypergroups associated with compact, totally disconnected groups. Let G be an infinite compact, totally disconnected group. Assume that $(G_i)_{i\in\mathbb{N}}$ is a family of open subgroups of G with

$$G_1 = G,\ G_i \supset G_{i+1} \quad \text{for } i \in \mathbb{N} \text{ and } \bigcap_{i\in\mathbb{N}} G_i = \{e\}.$$

Let $(H_i)_{i\in\mathbb{N}}$ be a another sequence of open subgroups of G such that, for all $i \in \mathbb{N}$,

$$H_{i+1} \subset H_i \subset G_{i+1} \subset G_i \quad \text{and } H_i \text{ is normal in } G_i.$$

We define a partition K of G consisting of cosets with respect to the different subgroups H_i as follows:

$$K := \{xH_i : i \in \mathbb{N}, x \in G_i - G_{i+1}\} \cup \big\{\{e\}\big\}.$$

K equipped with the quotient topology is a compact space in which every element except for $\{e\}$ is isolated. The associated canonical projection $\varphi\colon G \longrightarrow K$ is continuous, open, and surjective and hence an orbital mapping. If ω_i is the normalized Haar measure of the compact group H_i, then we consider the probability measures $q_{\{e\}} := \delta_e$ and $q_{xH_i} := \delta_x * \omega_i$ on K for $i \in \mathbb{N}$ and $x \in G_i - G_{i+1}$. We next apply Theorem 13.5A and Lemma 13.6A of Jewett [8] to conclude that there exists a unique hypergroup structure on K such that φ

becomes a unary consistent orbital morphism from G onto the compact, almost discrete hypergroup K. The convolution on K is given explicitly by

$$(4.1)\qquad \delta_{xH_i} * \delta_{yH_j} := \begin{cases} \delta_{xyH_i} & \text{if } i < j \\ \delta_{xyH_j} & \text{if } i > j \\ \sum_{l=i}^{\infty} \sum_{\{zH_l : z \in G_l - G_{l+1}; zH_l \subset xyH_i\}} \frac{1}{|H_i : H_l|} \cdot \delta_{zH_l} & \text{if } i = j \end{cases}$$

where $|H_i : H_l|$ stands for the index of H_l in H_i.

By Theorem 2.6, K can be regarded as the limit of a projective system of finite hypergroups which is constructed by the operation PS of Section 2.2. To construct an appropriate projective system, we consider the projective system $(G_i/H_i, g_{i,j}, \mathbb{N})$ of finite groups G_i/H_i with the connecting homomorphisms

$$g_{i,j}\colon G_i/H_i \longrightarrow G_j/H_j,\ xH_i \longmapsto xH_j \quad \text{for } i > j.$$

Then $(K_i, g^{i,j}, \mathbb{N}) := \mathrm{PS}(G_i/H_i, g_{i,j}, \mathbb{N})$ is a surjective projective system. To show that its limit is isomorphic with K, we use the fact that the hypergroups K_j are constructed by successive substitutions. It follows inductively that K_j is given by

$$K_j := \{xH_i : 1 \leq i \leq j, x \in G_i - G_{i+1}\} \cup \{xH_{j+1} : x \in G_{j+1}\}.$$

with the convolution

$$(4.2)\qquad \delta_{xH_i} * \delta_{yH_k} := \begin{cases} \delta_{xyH_i} & \text{if } i < k \\ \delta_{xyH_k} & \text{if } i > k \\ \sum_{l=i}^{j+1} \sum_{\{zH_l : z \in G_l - G_{l+1}; zH_l \subset xyH_i\}} \frac{1}{|H_i : H_l|} \cdot \delta_{zH_l} & \text{if } i = k. \end{cases}$$

It is now obvious that K is isomorphic to the projective limit of $(K_i, g^{i,j}, \mathbb{N})$.

REMARK 4.4. It is clear by the construction of K in Section 4.3 that K is commutative if and only if G_i/H_i is commutative for each $i \in \mathbb{N}$. Furthermore, if K is commutative, then K is a Pontryagin hypergroup as a consequence of Corollary 3.3. The dual hypergroup $\hat{K}$ can be constructed explicitly from the abelian groups $(G_i/H_i)^\wedge$ by using substitution and inductive spectra (see Voit [18]) in a similar way as in Section 2.2.

EXAMPLES 4.5. Let $(G_i)_{i\in\mathbb{N}}$ be a sequence of open subgroups of a compact totally disconnected group G such that G_{i+1} is normal in G_i for each $i \in \mathbb{N}$. If we set $H_i := G_{i+1}$ and apply the construction of Section 4.3, then we obtain a compact almost discrete hypergroup K which may be regarded as the limit of a projective system of finite hypergroups which are constructed by successive hypergroup joins. Hypergroups coming from groups in this specific way were studied earlier by Spector [12, 13].

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