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Continuous Adjacency Preserving Maps on Real Matrices

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Abstract. It is proved that every adjacency preserving continuous map on the vector space of real matrices of fixed size, is either a bijective affine tranformation of the form $A \mapsto PAQ + R$, possibly followed by the transposition if the matrices are of square size, or its range is contained in a linear subspace consisting of matrices of rank at most one translated by some matrix R. The result extends previously known theorems where the map was assumed to be also injective.

1 Introduction and Background

We denote by $M_{m,n}$ the space of all $m \times n$ real matrices. Two matrices $A, B \in M_{m,n}$ are said to be *adjacent* (or coherent) if $\operatorname{rank}(A - B) = 1$. A map $\phi\colon M_{m,n} \to M_{m,n}$ is called *adjacency preserver*, or is said to *preserve adjacency* if $\phi(A)$ and $\phi(B)$ are adjacent whenever A and B are adjacent, $A, B \in M_{m,n}$. Note that it is not required here *a priori* that adjacency of $\phi(A)$ and $\phi(B)$ implies adjacency of A and B. In [12] the following theorem was proved.

Proposition 1.1 *Let $m, n \geq 2$ and let $\phi\colon M_{m,n} \to M_{m,n}$ be a continuous injective (*i.e., *one-to-one) adjacency preserving map. Then,* either *ϕ is of the form*

$$\phi(A) = PAQ + R, \quad A \in M_{m,n}, \tag{1}$$

where P and Q are invertible matrices of dimension $m \times m$ and $n \times n$, respectively, and R is any $m \times n$ matrix, or *$m = n$ and ϕ is of the form*

$$\phi(A) = PA^{\mathrm{tr}}Q + R, \quad A \in M_n, \tag{2}$$

where A^{tr} stands for the transpose of A, and the matrices P, Q, R are as above.

The remarkable implication is that the affine character of ϕ is not an assumption but a conclusion. The above result was motivated by the so called fundamental theorem of the geometry of matrices which characterizes bijective maps on matrices (over more general fields or even skew fields) that preserve the adjacency in both directions. The study of this kind of problems was initiated by Hua [3–10]. An interested reader

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can find the precise formulation of the fundamental theorem of the geometry of matrices and analogous results for symmetric matrices, skew-symmetric matrices, and hermitian matrices in [15]. Some recent improvements and applications of this kind of results can be found in [13, 14].

The fundamental theorem of the geometry of matrices is a purely algebraic result and the proof combines purely algebraic methods with the use of the fundamental theorem of the affine geometry (see [15]). In [12] a completely different approach based on some topological results was used to prove Proposition 1.1. This result gives a nice form of ϕ under rather weak assumptions. Beside the main assumption of preserving the adjacency of matrices, we have two additional assumptions of continuity and injectivity. Note also that after composing ϕ by a translation, we may always assume that $\phi(0) = 0$. We will usually assume that this harmless normalization has been already done.

It is not surprising that the assumption of continuity is indispensable. To see this, one can consider any injective map from $M_{m,n}$ into a linear span of any rank one matrix from $M_{m,n}$. This example given in [12] is degenerate in the sense that the range of ϕ is contained in a linear space of matrices of rank at most one. In [12] a more sophisticated example of a nondegenerate injective adjacency preserving map that is far from being affine was given.

What about the injectivity assumption? One of the main tools in the proof of Proposition 1.1 was the invariance of domain theorem, a corollary of which may be stated as follows (see, *e.g.*, [2, p. 344]): *There is no continuous injective map* $\phi\colon U \to \mathbb{R}^q$, *where* U *is a non-empty open set in* $\mathbb{R}^p$, $p > q$. It is an obvious consequence of the adjacency preserving property that $\operatorname{rank}(A - B) = 1$ implies $\phi(A) \neq \phi(B)$. This is clearly a weaker condition than injectivity but still strong enough to make the following question natural: can we prove Proposition 1.1 without the injectivity assumption? The answer is negative because the map

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \mapsto \begin{bmatrix} a-d & b+c \\ 0 & 0 \end{bmatrix}$$

is continuous (even linear) and it is easy to check that it preserves the adjacency. However, this is a degenerate map. As a side remark, note that degenerate *linear* adjacency preserving maps on $M_{m,n}$ exist if and only if $m \leq R(n)$ (assuming $2 \leq m \leq n$), where $R(n)$ is the Hurwitz-Radon number (write $n = (2a+1)2^{b+4c}$, where a, b, c are nonnegative integers and $0 \leq b \leq 3$; then $R(n) = 2^b + 8c$); see [11], where additional information and references are found. We are not aware of results concerning existence and description of degenerate continuous (but nonlinear) adjacency preserving maps.

In this paper we prove that every continuous adjacency preserving map ϕ satisfying $\phi(0) = 0$ defined on $M_{m,n}$ with $m, n \geq 2$ is either degenerate (and then its range is contained in a linear space consisting of matrices of rank at most one), or has one of the forms (1) or (2) with $R = 0$.

2 Main Result

As usual, we identify $m \times n$ matrices with linear operators mapping $\mathbb{R}^n$ into $\mathbb{R}^m$. Then, of course, the elements of $\mathbb{R}^n$ are identified with $n \times 1$ column matrices. Note that for nonzero vectors $x \in \mathbb{R}^m$ and $y \in \mathbb{R}^n$ the matrix xy^{tr} has rank one, and every matrix of rank one can be written in this form. The elements of the standard bases of $\mathbb{R}^m$, $\mathbb{R}^n$, and $M_{m,n}$ will be denoted by $e_1, \dots, e_m$, $f_1, \dots, f_n$, and $E_{ij} = e_i f_j^{\mathrm{tr}}$, $1 \le i \le m$, $1 \le j \le n$, respectively.

Let x be any vector from $\mathbb{R}^m$. Then we denote by L_x the linear space of all matrices of the form xy^{tr} where y is any vector from $\mathbb{R}^n$. Similarly, for every $y \in \mathbb{R}^n$ we define $R_y = \{xy^{\mathrm{tr}} : x \in \mathbb{R}^m\}$. We also use the notations $L_i := L_{e_i}$ and $R_j := R_{f_j}$. Note that two different rank one matrices xy^{tr} and uv^{tr} are adjacent if and only if x and u are linearly dependent or y and v are linearly dependent. Thus, if $\mathcal{S} \subseteq M_{m,n}$ is a subset of matrices all of whose elements have rank at most one and are pairwise adjacent, then $\mathcal{S} \subseteq L_x$ for some $x \in \mathbb{R}^m$ or $\mathcal{S} \subseteq R_y$ for some $y \in \mathbb{R}^n$.

We state our main result:

Theorem 2.1 *Let $2 \le m \le n$ and let $\phi\colon M_{m,n} \to M_{m,n}$ be a continuous map. Assume that $\phi(A)$ and $\phi(B)$ are adjacent whenever A and B are adjacent, $A, B \in M_{m,n}$. Then, when $m \ne n$, either $\phi(M_{m,n}) \subseteq A + L_x$ for some $A \in M_{m,n}$ and some $x \in \mathbb{R}^m$, or ϕ is of the form* (1). *When $m = n$, in addition to these two forms, we can also have one of the following two possibilities: $\phi(M_n) \subseteq A + R_y$ for some $A \in M_n$ and some $y \in \mathbb{R}^n$ or ϕ is of the form* (2).

The case when $2 \le n < m$ can be treated in the same way. The case $m = 1$ is not interesting: a map $\phi\colon M_{1,n} \to M_{1,n}$ is adjacency preserving if and only if ϕ is injective.

The result of Theorem 2.1 is stated for real matrices only, not for complex ones. In fact, our proof of the theorem relies on [13, Proposition 3.3], the main tool of whose proof is an extension of the fundamental theorem of affine geometry for real planes due to Carter and Vogt [1]. An analogue of their result does not hold true in the complex case.

3 Proofs

The rest of the paper is devoted to the proof of Theorem 2.1. We assume the hypotheses of the theorem, and assume in addition and without loss of generality that $\phi(0) = 0$.

We start with preliminary observations. First notice that for every nonzero $x \in \mathbb{R}^m$ the set $\phi(L_x)$ consists of matrices of rank at most one that are mutually adjacent. Hence, either $\phi(L_x) \subseteq L_z$ for some $z \in \mathbb{R}^m$, or $\phi(L_x) \subseteq R_y$ for some $y \in \mathbb{R}^n$. Note that because of the adjacency preserving property the restriction of ϕ to L_x is an injective continuous map. Clearly, L_x is isomorphic to $\mathbb{R}^n$ and R_y has dimension m. So, by the invariance of domain theorem the possibility $\phi(L_x) \subseteq R_y$ cannot occur unless $m = n$.

In particular, after composing ϕ by an appropriate equivalence transformation $A \mapsto PAQ$ for some invertible fixed P and Q, we may assume that either $\phi(L_1) \subseteq L_1$,

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or $\phi(L_1) \subseteq R_1$. Here, $L_1 = L_{e_1}$ (the set of all matrices having nonzero entries only in the first row) and $R_1 = R_{f_1}$. Of course, the second case is possible only when $m = n$. In this special case we can compose ϕ by the transposition map, and therefore, we may assume with no loss of generality that $\phi(L_1) \subseteq L_1$.

In our next step we will prove that for every nonzero $x \in \mathbb{R}^m$ we have $\phi(L_x) \subseteq L_z$ for some $z \in \mathbb{R}^m$. Assume, to the contrary, that there exists $x \in \mathbb{R}^m$, $x \neq 0$, such that $\phi(L_x) \subseteq R_w$ for some $w \in \mathbb{R}^n$ (then, of course, $m = n$ and x is linearly independent of e_1). We may and we do assume that $w = f_1$. So, $\phi(L_x) \subseteq R_1$. By the invariance of domain theorem we see that the set $\mathcal{O}_1$ of all vectors $y \in \mathbb{R}^n$ with the property that $\phi(e_1 y^{\mathrm{tr}})$ has a nonzero entry outside the first column is an open dense subset of $\mathbb{R}^n$. Similarly, the set $\mathcal{O}_2$ of all vectors $y \in \mathbb{R}^n$ with the property that $\phi(xy^{\mathrm{tr}})$ has a nonzero entry outside the first row is open and dense in $\mathbb{R}^n$. Therefore, we can find a nonzero y such that $\phi(e_1 y^{\mathrm{tr}})$ is a rank one matrix whose nonzero entries are all in the first row and at least one of the second, third, $\dots$, n-th entry in the first row is nonzero. At the same time, $\phi(xy^{\mathrm{tr}})$ is a rank one matrix whose nonzero entries are all in the first column and at least one of the second, third, $\dots$, n-th entry in this column is nonzero. This contradicts the fact that $e_1 y^{\mathrm{tr}}$ and xy^{tr} are adjacent.

Therefore, for every nonzero $x \in \mathbb{R}^m$ we have $\phi(L_x) \subseteq L_z$ for some $z \in \mathbb{R}^m$. Now, there are two possibilities. The first one is that $\phi(L_x) \subseteq L_1$ for every nonzero $x \in \mathbb{R}^m$. In other words, every rank one matrix is mapped into L_1.

The second possibility is that there exists a nonzero $x \in \mathbb{R}^m$ such that $\phi(L_x) \subseteq L_z$ with z and e_1 being linearly independent. Then, clearly, also x and e_1 are linearly independent. After composing ϕ by appropriate equivalence transformations we may assume that $\phi(L_2) \subseteq L_2$. Now, we know that for every nonzero $y \in \mathbb{R}^n$ we have either $\phi(R_y) \subseteq L_z$ for some $z \in \mathbb{R}^m$, or $\phi(R_y) \subseteq R_w$ for some $w \in \mathbb{R}^n$. We already know that $\phi(e_1 y^{\mathrm{tr}}) \in L_1$ and $\phi(e_2 y^{\mathrm{tr}}) \in L_2$. Thus, the first possibility cannot occur. So, for every nonzero $y \in \mathbb{R}^n$ we have $\phi(R_y) \subseteq R_w$ for some $w \in \mathbb{R}^n$. Because $\phi(L_1)$ is not contained in any one-dimensional subspace of L_1, we can find nonzero vectors y, u, w, v such that $\phi(e_1 y^{\mathrm{tr}}) = e_1 w^{\mathrm{tr}}$ and $\phi(e_1 u^{\mathrm{tr}}) = e_1 v^{\mathrm{tr}}$ with v and w linearly independent. Then $\phi(R_y) \subseteq R_w$, $\phi(R_u) \subseteq R_v$, and both pairs of vectors y, u and w, v are linearly independent. Hence, after composing ϕ by appropriate equivalence transformations we may assume that $\phi(L_i) \subseteq L_i$ and $\phi(R_i) \subseteq R_i$, $i = 1, 2$.

We denote by $\mathcal{R} \subseteq M_{m,n}$ the set of all rank one $m \times n$ matrices. We summarize the above obtained conclusions in the following lemma.

Lemma 3.1 *Let $2 \leq m \leq n$ and let $\phi\colon M_{m,n} \to M_{m,n}$ be a continuous adjacency preserving map satisfying $\phi(0) = 0$. Then one of the following four possibilities holds true:*

(1) *$\phi(\mathcal{R}) \subseteq L_x$ for some $x \in \mathbb{R}^m$;*
(2) *$m = n$ and $\phi(\mathcal{R}) \subseteq R_y$ for some $y \in \mathbb{R}^n$;*
(3) *for every $x \in \mathbb{R}^m$ there exists $z \in \mathbb{R}^m$ such that $\phi(L_x) \subseteq L_z$ and for every $y \in \mathbb{R}^n$ there exists $w \in \mathbb{R}^n$ such that $\phi(R_y) \subseteq R_w$;*
(4) *$m = n$ and for every $x \in \mathbb{R}^n$ there exists $y \in \mathbb{R}^n$ such that $\phi(L_x) \subseteq R_y$ and for every $u \in \mathbb{R}^n$ there exists $z \in \mathbb{R}^n$ such that $\phi(R_u) \subseteq L_z$.*

Moreover, in cases (3) *and* (4), *we may assume after composing* ϕ *by equivalence transformations, and by the transposition in case* (4), *that* $\phi(L_i) \subseteq L_i$ *and* $\phi(R_i) \subseteq R_i$, $i = 1, 2$.

Lemma 3.2 *Let* $2 \le m \le n$ *and let* $\phi\colon M_{m,n} \to M_{m,n}$ *be a continuous adjacency preserving map satisfying* $\phi(0) = 0$. *If* $\phi(\mathcal{R}) \subseteq L_x$ *for some* $x \in \mathbb{R}^m$, *then* $\phi(M_{m,n}) \subseteq L_x$. *If* $m = n$ *and* $\phi(\mathcal{R}) \subseteq R_y$ *for some* $y \in \mathbb{R}^n$, *then* $\phi(M_{m,n}) \subseteq R_y$.

Proof The second case ($m = n$ and $\phi(\mathcal{R}) \subseteq R_y$ for some $y \in \mathbb{R}^n$) is reduced to the first one upon transposition. So we need to consider the first case only.

Let $\phi\colon M_{m,n} \to M_{m,n}$ be a continuous map preserving the adjacency, satisfying $\phi(0) = 0$, and $\phi(\mathcal{R}) \subseteq L_x$ for some $x \in \mathbb{R}^m$. We have to prove that $\phi(M_{m,n}) \subseteq L_x$. With no loss of generality we may assume that $L_x = L_1$. We will prove that then all rank two matrices are mapped into L_1. Clearly, every rank two matrix is adjacent to some rank one matrix and is therefore mapped into a matrix of rank at most two. Assume first that there is a rank two matrix A that is mapped into a rank two matrix. We may, and we do assume that $A = E_{11} + E_{22}$. Since $\phi(E_{11} + E_{22})$ is adjacent to $\phi(E_{11}) \in L_1$, we have $\phi(E_{11} + E_{22}) = e_1 x^{\mathrm{tr}} + z y^{\mathrm{tr}}$ with both pairs e_1, z and x, y linearly independent. Thus we can find invertible $m \times m$ matrix P and invertible $n \times n$ matrix Q such that $Pe_1 = e_1$, $Pz = e_2$, $x^{\mathrm{tr}} Q = e_1^{\mathrm{tr}}$, and $y^{\mathrm{tr}} Q = e_2^{\mathrm{tr}}$. After replacing ϕ by $T \mapsto P\phi(T)Q$, we may assume that $\phi(\mathcal{R}) \subseteq L_1$ and $\phi(E_{11} + E_{22}) = E_{11} + E_{22}$. Any matrix of the form

$$E_{11} + \lambda E_{12} \tag{3}$$

is adjacent to $E_{11} + E_{22}$ and is mapped into L_1, and hence, it is mapped into $E_{11} + \mu E_{12}$ for some real μ. The map $\lambda \mapsto \mu$ is injective because the restriction of ϕ to L_1 is injective. Because $\phi(E_{11} + E_{22}) = E_{11} + E_{22}$, the continuity of ϕ yields that the $(2, 2)$-entry of $\phi(E_{11} + aE_{12} + bE_{22})$ is nonzero for every $\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right] \in \mathbb{R}^2$ belonging to some open neighbourhood $\mathcal{U}$ of the point $\left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right] \in \mathbb{R}^2$. Replacing $\mathcal{U}$ by an open smaller neighbourhood, if necessary, we may assume that the second coordinate of every point from $\mathcal{U}$ is nonzero. For every $\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right] \in \mathcal{U}$ the matrix

$$E_{11} + aE_{12} + bE_{22} \tag{4}$$

is adjacent to any matrix of the form (3). So, its image has a nonzero $(2, 2)$-entry and is adjacent to infinitely many matrices of the form (3). Thus,

$$\phi\left(\begin{bmatrix} 1 & a & 0 & \dots & 0 \\ 0 & b & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix}\right) = \begin{bmatrix} 1 & * & 0 & \dots & 0 \\ 0 & f(a,b) & 0 & \dots & 0 \\ 0 & * & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & * & 0 & \dots & 0 \end{bmatrix} \tag{5}$$

with $f(a, b) \neq 0$. Because the ϕ-image of

$$(a + 1)E_{11} + bE_{21} \tag{6}$$

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is contained in L_1 and is adjacent to (5) it has to be of the form $E_{11} + h(a,b)E_{12}$. The map $h\colon \mathcal{U} \to \mathbb{R}$ is continuous and injective. This contradicts the invariance of domain theorem.

So, we have proved that every matrix of rank two is mapped into a matrix of rank at most one. In the next step we will show that every rank two matrix is mapped into L_1. Assume that this is not true. Then we may assume, after composing ϕ by equivalence transformations, that $\phi(E_{11} + E_{22}) = E_{21}$. It follows that matrices of the form (3) are mapped into scalar multiples of E_{11}. For vectors $\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right]$ close to $\left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right]$ the ϕ-image of matrix (4) has rank one and nonzero $(2,1)$-entry and is adjacent to some nonzero scalar multiple of E_{11}, and is therefore mapped into a matrix having nonzero entries only in the first colum. Therefore the matrix (6) is mapped into $f(a,b)E_{11}$ for all $\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right]$ sufficiently close to $\left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right]$. Once again we get a contradiction applying the invariance of domain theorem.

Hence, every rank two matrix is mapped into L_1 and in the next step we will show that every rank three matrix A is mapped into L_1. With no loss of generality we assume that $A = E_{11} + E_{22} + E_{33}$. Because all rank two matrices are mapped into L_1, the matrix $\phi(A)$ has rank at most two. Assume that $\phi(A) \notin L_1$. Then with no loss of generality either $\phi(A) = E_{11} + E_{22}$, or $\phi(A) = E_{21}$. The set of all matrices from L_1 that are adjacent to $\phi(A)$ is in the first case equal to $\{E_{11} + \lambda E_{12} : \lambda \in \mathbb{R}\}$ while in the second case this set is the linear span of E_{11}. We know that ϕ maps injectively the set of all matrices of the form

$$\begin{bmatrix} 1 & 0 & a & \dots & 0 \\ 0 & 1 & b & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix}$$

into L_1. The ϕ-images of these matrices must be adjacent to $\phi(A)$, again a contradiction with the invariance of domain theorem.

Thus, all rank three matrices are mapped into L_1. Repeating the same argument we see that all matrices of rank four, five, ... are mapped into L_1, as desired.

Next we quote [13, Proposition 3.3]:

Proposition 3.3 *Let $\phi\colon M_{m,n} \longrightarrow M_{m,n}$ be an adjacency preserving map such that $\phi(0) = 0$ and the following two properties hold:*

(a) *For every pair $A, B \in M_{m,n}$ of matrices of rank at most one, and for every $\lambda \in \mathbb{R}$, we have*

$$\phi(A + \lambda B) - \phi(A) \in \operatorname{span}\{\phi(A + B) - \phi(A)\};$$

(b) $\operatorname{rank}(\phi(E_{11}) - \phi(E_{22})) = 2$.

Then ϕ is either of the form (1) *with $R = 0$, or $m = n$ and ϕ is of the form* (2) *with $R = 0$.*

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Proof of Theorem 2.1 After composing ϕ by a translation we may, and do, assume that $\phi(0) = 0$. So, we are in a position to apply Lemma 3.1. If ϕ has one of the first two forms described in the lemma then our result follows directly from Lemma 3.2. So, we have to consider the cases when ϕ satisfies the third or the fourth condition described in Lemma 3.1. Then, by the lemma, we may assume that ϕ satisfies the third condition and that $\phi(L_i) \subseteq L_i$ and $\phi(R_i) \subseteq R_i$, $i = 1, 2$. In particular, $\phi(E_{ii}) \in \operatorname{span}\{E_{ii}\}$, $i = 1, 2$. It follows that $\operatorname{rank}(\phi(E_{11}) - \phi(E_{22})) = 2$.

Our result will follow from Proposition 3.3. To apply this proposition we have to show that for every pair of matrices $A, B \in M_{m,n}$ with $\operatorname{rank} B = 1$ and every real number λ we have $\phi(A + \lambda B) - \phi(A) \in \operatorname{span}\{\phi(A + B) - \phi(A)\}$.

We will start with the special case that $A = 0$. Write $B = xy^{\mathrm{tr}}$ and let $z \in \mathbb{R}^m$ and $w \in \mathbb{R}^n$ be vectors such that $\phi(L_x) \subseteq L_z$ and $\phi(R_y) \subseteq R_w$. For any $\lambda \neq 0$ both $\phi(B)$ and $\phi(\lambda B)$ are rank one matrices belonging to $L_z \cap R_w = \operatorname{span}\{zw^{\mathrm{tr}}\}$. Thus, $\phi(\lambda B) \in \operatorname{span}\{\phi(B)\}$, as desired.

Now, let $A \in M_{m,n}$ be any matrix. We define a new map $\psi \colon M_{m,n} \to M_{m,n}$ by

$$\psi(Y) = \phi(A + Y) - \phi(A).$$

This new map is also continuous adjacency preserving map with $\psi(0) = 0$. So, we can apply Lemma 3.1 once again. If we have the first possibility, then, by Lemma 3.2, the range of ϕ is contained in $\phi(A) + L_x$ and we are done. Similarly we treat the second possibility. In the remaining two cases we already know that $\psi(\lambda B) \in \operatorname{span}\{\psi(B)\}$, or equivalently, $\phi(A + \lambda B) - \phi(A) \in \operatorname{span}\{\phi(A + B) - \phi(A)\}$. ■

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