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A GENERALIZATION OF LEVINGER'S THEOREM TO POSITIVE KERNEL OPERATORS

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Abstract. We prove some inequalities for the spectral radius of positive operators on Banach function spaces. In particular, we prove the following extension of Levinger's theorem. Let K be a positive compact kernel operator on $L^2(X, \mu)$ with the spectral radius $r(K)$. Then the function ϕ defined by $\phi(t) = r(tK + (1-t)K^*)$ is non-decreasing on $[0, \frac{1}{2}]$. We also prove that $\|A + B^*\| \geq 2 \cdot \sqrt{r(AB)}$ for any positive operators A and B on $L^2(X, \mu)$.

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1. Introduction. In general there is no relation between the spectral radius of a sum of operators on a Banach space and the sum of the corresponding spectral radii, so that, under appropriate assumptions, any inequality between these two numbers might be interesting. In [**5**] we proved some inequalities for the spectral radius of a sum of positive compact kernel operators on a Banach function space. We thus extended the corresponding matrix results proved in [**7**]. In this article we show their further generalizations by removing several assumptions from the results in [**5**]. As an application of our main result we obtain an extension of Levinger's theorem to positive compact kernel operators on L^2-spaces. This beautiful result, stated without proof in [**11**], asserts that for a non-negative (square) matrix A the function

$$\phi(t) = r(tA + (1-t)A^T)$$

is non-decreasing on $[0, \frac{1}{2}]$ and is non-increasing on $[\frac{1}{2}, 1]$. In particular, for all $t \in [0, 1]$, the following inequality holds

$$r(tA + (1-t)A^T) \geq r(A).$$

This theorem was generalized in Bapat [**3**], where an elementary proof is given. Recently, Alpin and Kolotilina [**2**, Theorem 7] further extended Bapat's result. Our Theorem 8 includes their extension as a special case. Finally, Theorem 10 proves an inequality that seems to be new even in the finite-dimensional case. For the theory of Banach function spaces and Banach lattices we refer the reader to the books [**13**], [**12**] and [**1**]. Here we shall recall some relevant facts.

Let μ be a σ-finite positive measure on a σ-algebra $\mathcal{M}$ of subsets of a non-void set X. Let $M(X, \mu)$ be the vector space of all equivalence classes of (almost everywhere equal) complex measurable functions on X. A Banach space $L \subseteq M(X, \mu)$ is called a *Banach function space* if $f \in L$, $g \in M(X, \mu)$, and $|g| \leq |f|$ imply that $g \in L$ and

$\|g\| \le \|f\|$. Throughout the paper, it is assumed that the dimension of L is greater than one and that X is the carrier of L, that is, there is no subset Y of X of strictly positive measure with the property that $f = 0$ a.e. on Y for all $f \in L$ (see [**13**]). The cone of positive elements in L is denoted by L_+. A non-negative function $f \in L_+$ is said to be *strictly positive* if $f(x) > 0$ for almost all $x \in X$. The norm of L is said to be a *weakly Fatou norm* if there exists a finite constant $k \ge 1$ such that $0 \le f_\tau \uparrow f$ in L implies that $\|f\| \le k \cdot \sup_\tau \|f_\tau\|$.

By L' we denote the *associate space* (also called the Köthe dual) of all $g \in M(X, \mu)$ such that

$$\varphi_g(f) = \int_X f\, g\, d\mu$$

defines a bounded linear functional φ_g on L. The space L' is also a Banach function space with respect to the associate norm $\|\cdot\|'$ defined by

$$\|g\|' = \|\varphi_g\| = \sup\left\{\int_X |f\, g|\, d\mu : f \in L,\ \|f\| \le 1\right\},$$

and it may be considered as a closed subspace of the dual Banach lattice L^*. In view of the definition of $\|\cdot\|'$ the following generalized Hölder's inequality holds

$$\int_X |f\, g|\, d\mu \le \|f\|\, \|g\|'$$

for $f \in L$ and $g \in L'$. Note that the set X is also the carrier of the associate space L', and L' separates points of L (see [**13**, Theorem 112.1]). For any non-negative functions f and g on X we introduce the following notation

$$\langle f, g\rangle = \int_X f\, g\, d\mu.$$

For brevity, the integration over the whole set X will be denoted by $\int d\mu(x)$ or even $\int dx$.

By an *operator* on a Banach function space L we always mean a linear operator on L. The spectrum and the spectral radius of a bounded operator T on L are denoted by $\sigma(T)$ and $r(T)$, respectively. An operator T on L is said to be *positive* if $Tf \in L_+$ for all $f \in L_+$. Given operators S and T on L, we write $S \ge T$ if the operator $S - T$ is positive. It should be recalled that a positive operator T on L is automatically bounded and that $r(T)$ belongs to the spectrum of T. An operator K on L is called a *kernel operator* if there exists a $\mu \times \mu$-measurable function $k(x, y)$ on $X \times X$ such that, for all $f \in L$ and for almost all $x \in X$,

$$\int_X |k(x, y) f(y)|\, d\mu(y) < \infty \quad \text{and} \quad (Kf)(x) = \int_X k(x, y) f(y)\, d\mu(y).$$

One can check that a kernel operator K is positive iff its kernel k is non-negative almost everywhere. We say that K is *reducible* if there exists a set $A \in \mathcal{M}$ such that $\mu(A) > 0$, $\mu(A^c) > 0$ and $k = 0$ a.e. on $A \times A^c$. Otherwise, if there is no such set, K is said to be *irreducible*.

Let K be a positive kernel operator on L with kernel k. It is easily seen that L' is invariant under the adjoint operator K^*. We denote by K' the restriction of K^* to L'.

One can show [**13**, Section 97] that K' is also a positive kernel operator with the kernel $k'(x, y) = k(y, x)$ $(x, y \in X)$. The following important observation was already stated in [**6**] for general Banach lattices.

PROPOSITION 1. *Let L be a Banach function space with a weakly Fatou norm. If K is a kernel operator on L, then $r(K') = r(K)$.*

Proof. It follows from [**13**, Theorem 107.7] (see also the equality (2) on p. 393 of [**13**]) that the space L can be (not necessarily isometrically) embedded into $(L')'$ as a Banach space. Then we have $r(K) \geq r(K') \geq r((K')') \geq r(K)$, and so $r(K') = r(K)$. □

The following important result is contained in [**9**, Theorems 4.13 and 3.14].

THEOREM 2. *Let K be an irreducible positive kernel operator on a Banach function space L such that $r(K)$ is a pole of the resolvent $(\lambda - K)^{-1}$. Then $r(K) > 0$, $r(K)$ is an eigenvalue of K of algebraic multiplicity one, and the corresponding eigenspace is spanned by a strictly positive function.*

It is well known that the assumption that $r(K)$ is a pole of the resolvent $(\lambda - K)^{-1}$ is satisfied if some power of K is a compact operator. In this case Theorem 2 is known as the theorem of Jentzsch and Perron (see [**9**, Theorem 5.2]).

We will also need the following simple result.

PROPOSITION 3. *Assume that a positive operator T on a Banach function space L is the norm limit of a sequence $\{T_n\}_{n\in\mathbb{N}}$ of positive operators on L such that $T_1 \geq T_2 \geq \ldots \geq T$. Then*

$$r(T) = \lim_{n\to\infty} r(T_n).$$

Proof. The sequence $\{r(T_n)\}_{n\in\mathbb{N}}$ is non-increasing and bounded below by $r(T)$, so that $r(T) \leq \lim_{n\to\infty} r(T_n)$. Since the spectral radius is upper semicontinuous, the equality holds in this inequality. □

2. General Banach function spaces. Throughout this section, let L be a Banach function space with a weakly Fatou norm. For brevity, we denote by $L^\infty_{++}(X, \mu)$ the set of all strictly positive functions $f \in L^\infty(X, \mu)_+$ satisfying $1/f \in L^\infty(X, \mu)_+$. For $d \in L^\infty(X, \mu)_+$ the *multiplication operator* D is a positive operator on L defined by $Df = df$. Clearly, D is invertible iff $d \in L^\infty_{++}(X, \mu)$.

The following lemma that extends [**5**, Lemma 2.2] is needed in the proof of Theorem 5.

LEMMA 4. *Let K be a positive kernel operator on L with $r(K) = 1$. Let d and e be strictly positive functions in $L^\infty_{++}(X, \mu)$, and let D and E be the corresponding multiplication operators on L. Let $f \in L_+$ and $g \in L'_+$ be strictly positive functions such that Kf is a strictly positive function satisfying*

$$\frac{Kf}{f} = \frac{K'g}{g} \quad \textit{and} \quad \langle Kf, g\rangle = 1.$$

Then

$$\langle DKEu, v\rangle \geq \exp\left(\int_X Kf\, g\, \log(d\, e)\, d\mu\right) \tag{1}$$

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for any $u \in L_+$ and for any nonnegative measurable function v on X satisfying $u\,v = f\,g$. If, in addition, $\langle Ku, v\rangle < \infty$, then

$$\langle Ku, v\rangle \geq \exp\left(\int_X Kf\,g\,\log\left(\frac{K\,u}{u}\,\frac{f}{Kf}\right)d\mu\right) \geq 1. \tag{2}$$

Proof. Since $\langle Kf, g\rangle = 1$, the integral in (1) exists, while it will be seen below that the integral in (2) exists provided $\langle Ku, v\rangle < \infty$. In fact, there is no loss of generality in assuming that $\langle DKEu, v\rangle < \infty$, and consequently, $\langle Ku, v\rangle < \infty$, since it holds

$$\langle Ku, v\rangle \leq \|1/d\|_\infty \cdot \|1/e\|_\infty \cdot \langle DKEu, v\rangle.$$

We will first show the right-hand inequality in (2), that is

$$\int_X Kf\,g\,\log\left(\frac{K\,u}{u}\,\frac{f}{Kf}\right)\,d\mu \geq 0. \tag{3}$$

We consider the special case when $v \in L'_+$. For almost all $x \in X$ we define the probability measure on $\mathcal{M}$ by

$$\nu_x(A) = \frac{1}{(Kf)(x)}\int_A k(x, y)f(y)\,dy,$$

where k is the kernel of K. Using the estimate $|\log(t)| \leq t + \frac{1}{t}$ $(t > 0)$ we obtain that

$$\int Kf\,g\,\left|\log\left(\frac{u}{f}\right)\right|\,d\mu \leq \int Kf\,g\,\left(\frac{u}{f} + \frac{f}{u}\right)\,d\mu = \langle u, K'g\rangle + \langle Kf, v\rangle < \infty. \tag{4}$$

Now, we have

$$\begin{aligned}\int (Kf)(y)\,g(y)\,\log\left(\frac{u(y)}{f(y)}\right)\,dy &= \int f(y)\,(K'g)(y)\,\log\left(\frac{u(y)}{f(y)}\right)\,dy\\ &= \int f(y)\,\log\left(\frac{u(y)}{f(y)}\right)\left(\int k(x, y)\,g(x)\,dx\right)dy.\end{aligned}$$

Because of (4) we can use Fubini's theorem to get

$$\begin{aligned}\int (Kf)(y)\,g(y)\,\log\left(\frac{u(y)}{f(y)}\right)\,dy &= \int g(x)\left(\int k(x, y)f(y)\,\log\left(\frac{u(y)}{f(y)}\right)dy\right)dx\\ &= \int (Kf)(x)\,g(x)\,\left(\int \log\left(\frac{u(y)}{f(y)}\right)d\nu_x(y)\right)dx.\end{aligned}$$

Then, an application of Jensen's inequality gives the inequality

$$\begin{aligned}\int (Kf)(y)\,g(y)\,\log\left(\frac{u(y)}{f(y)}\right)\,dy &\leq \int (Kf)(x)\,g(x)\,\log\left(\int \frac{u(y)}{f(y)}\,d\nu_x(y)\right)dx\\ &= \int (Kf)(x)\,g(x)\,\log\left(\frac{(K\,u)(x)}{(Kf)(x)}\right)dx,\end{aligned}$$

from which (3) follows. To prove the general case, define sequences $\{u_n\}_{n\in\mathbb{N}}$ and $\{v_n\}_{n\in\mathbb{N}}$ of strictly positive functions by $u_n = u + f/n$ and $v_n = fg/u_n$. Since $v_n \leq n\,g$, we have

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$v_n \in L'_+$, and so

$$\int_X Kf\, g \log\left(\frac{K\, u_n}{u_n}\, \frac{f}{Kf}\right) d\mu \geq 0, \tag{5}$$

by the special case of (3). Since

$$\frac{Ku_n}{u_n} - \frac{Kf}{f} = \frac{u}{u_n}\left(\frac{Ku}{u} - \frac{Kf}{f}\right),$$

it holds that

$$\left\{x \in X : \frac{(Ku_n)(x)}{u_n(x)} \geq \frac{(Kf)(x)}{f(x)}\right\} = \left\{x \in X : \frac{(Ku)(x)}{u(x)} \geq \frac{(Kf)(x)}{f(x)}\right\},$$

and the sequence $\{\frac{Ku_n}{u_n}\}_{n\in\mathbb{N}}$ is non-decreasing on this set. Then, by the Monotone Convergence Theorem,

$$\lim_{n\to\infty} \int_X Kf\, g \log^+\left(\frac{Ku_n}{u_n}\, \frac{f}{Kf}\right) d\mu = \int_X Kf\, g \log^+\left(\frac{Ku}{u}\, \frac{f}{Kf}\right) d\mu, \tag{6}$$

where the limit is finite. Namely, using the inequality $\log^+ t \leq t$ $(t > 0)$ we obtain that

$$0 \leq \int_X Kf\, g \log^+\left(\frac{Ku}{u}\, \frac{f}{Kf}\right) d\mu \leq \int_X Kf\, g\, \frac{Ku}{u}\, \frac{f}{Kf}\, d\mu = \langle Ku, v\rangle < \infty.$$

This shows that the integral in (3) is defined (and its value belongs to $[-\infty, \infty)$). Similarly, we obtain that

$$\lim_{n\to\infty} \int_X Kf\, g \log^-\left(\frac{Ku_n}{u_n}\, \frac{f}{Kf}\right) d\mu = \int_X Kf\, g \log^-\left(\frac{Ku}{u}\, \frac{f}{Kf}\right) d\mu,$$

which together with (6) gives that

$$\lim_{n\to\infty} \int_X Kf\, g \log\left(\frac{Ku_n}{u_n}\, \frac{f}{Kf}\right) d\mu = \int_X Kf\, g \log\left(\frac{Ku}{u}\, \frac{f}{Kf}\right) d\mu.$$

In view of (5) this completes the proof of (3).

We now define the probability measure λ on $\mathcal{M}$ by

$$\lambda(A) = \int_A Kf\, g\, d\mu.$$

An application of Jensen's inequality gives that

$$\log(\langle Ku, v\rangle) = \log\left(\int \frac{Ku}{u}\, \frac{f}{Kf}\, d\lambda\right) \geq \int \log\left(\frac{Ku}{u}\, \frac{f}{Kf}\right) d\lambda$$
$$= \int Kf\, g \log\left(\frac{Ku}{u}\, \frac{f}{Kf}\right) d\mu,$$

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so that the left-hand inequality holds in (2). Similarly, we have

$$\log(\langle DKEu, v\rangle) = \log\left(\int d\,e\,\frac{K(Eu)}{Eu}\,\frac{f}{Kf}\,d\lambda\right) \geq \int \log\left(d\,e\,\frac{K(Eu)}{Eu}\,\frac{f}{Kf}\right) d\lambda$$
$$= \int Kf\,g\,\log(d\,e)\,d\mu + \int Kf\,g\,\log\left(\frac{K(Eu)}{Eu}\,\frac{f}{Kf}\right) d\mu.$$

Since the last integral is non-negative by (3), this gives (1). □

The following result extends Theorems 2.4 and 2.6 in [**5**]. Its finite-dimensional version was shown in [**7**, Theorem 2.3].

THEOREM 5. *Let $K_1, K_2, \ldots, K_n$ be positive kernel operators on L. Assume that f_1, $f_2, \ldots, f_n \in L_+$ and $g_1, g_2, \ldots, g_n \in L'_+$ are strictly positive functions satisfying*

$$K_i f_i = r(K_i) f_i, \quad K_i' g_i = r(K_i) g_i$$

and be normalized so that

$$f_i \cdot g_i = h\,(i = 1, 2, \ldots, n) \quad \text{and} \quad \int_X h\,d\mu = 1.$$

Furthermore, let $d_1, \ldots, d_n$ and $e_1, \ldots, e_n$ be in $L^\infty(X, \mu)_+$, and let $D_1, \ldots, D_n$ and $E_1, \ldots, E_n$ be the corresponding multiplication operators on L. Then

$$r\left(\sum_{i=1}^{n} D_i K_i E_i\right) \geq \sum_{i=1}^{n} r(K_i) \exp\left(\int_X h \log(d_i e_i)\,d\mu\right) \tag{7}$$

adopting the convention $\exp(-\infty) = 0$. *In particular, for all positive numbers $t_1, \ldots, t_n$,*

$$r(t_1 K_1 + \ldots + t_n K_n) \geq t_1 r(K_1) + \ldots + t_n r(K_n). \tag{8}$$

Proof. If, for some i, $d_i e_i = 0$ on the set of positive measure, then $\int_X h \log(d_i e_i)\,d\mu = -\infty$, which together with the monotonicity of the spectral radius convinces us that there is no loss of generality in assuming that $\{d_i\}_{i=1}^n$ and $\{e_i\}_{i=1}^n$ are strictly positive functions. Also, we may assume that $r(K_i) > 0$ for all i.

Consider first the case when $\{d_i\}_{i=1}^n$ and $\{e_i\}_{i=1}^n$ are in $L^\infty_{++}(X, \mu)$. Denote $K = D_1 K_1 E_1 + \ldots + D_n K_n E_n$, pick $\lambda > r(K)$, and set

$$u = (\lambda - K)^{-1} f_1 = \sum_{j=0}^{\infty} \lambda^{-j-1} K^j f_1.$$

Then u is a strictly positive function in L satisfying $Ku \leq \lambda u$. Denoting $v = h/u$ we apply (1) of Lemma 4 for the operator $K_i / r(K_i)$, $i = 1, \ldots, n$, to get

$$\langle D_i K_i E_i u, v\rangle \geq r(K_i) \exp\left(\int_X h \log(d_i e_i)\,d\mu\right).$$

Summing over i gives the inequality

$$\sum_{i=1}^{n} r(K_i) \exp\left(\int_X h \log(d_i e_i)\,d\mu\right) \leq \sum_{i=1}^{n} \langle D_i K_i E_i u, v\rangle = \langle Ku, v\rangle \leq \langle \lambda u, v\rangle = \lambda.$$

Since this is true for any $\lambda > r(K)$, the inequality (7) follows.

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To remove the assumptions on $\{d_i\}_{i=1}^n$ and $\{e_i\}_{i=1}^n$, define $d_i^{(m)} = \max\{d_i, \frac{1}{m}\}$ and $e_i^{(m)} = \max\{e_i, \frac{1}{m}\}$ ($m \in \mathbb{N}$, $i = 1, \dots, n$), and let $D_i^{(m)}$ and $E_i^{(m)}$ be the corresponding multiplication operators on L. Then, by the above,

$$r\left(\sum_{i=1}^{n} D_i^{(m)} K_i E_i^{(m)}\right) \geq \sum_{i=1}^{n} r(K_i) \exp\left(\int_X h \log\left(d_i^{(m)} e_i^{(m)}\right) d\mu\right).$$

When m tends to infinity, the left-hand side approaches $r(K)$ by Proposition 3, while

$$\lim_{m\to\infty} \int_X h \log\left(d_i^{(m)} e_i^{(m)}\right) d\mu = \int_X h \log(d_i e_i)\, d\mu$$

by the Monotone Convergence Theorem (for decreasing sequences). This yields the inequality (7), and the proof is finished. □

A glance at the proof above shows that Theorem 5 also holds in the case when some operators of $K_1, K_2, \dots, K_n$ are positive multiples of the identity operator, or in other words, every K_i is a sum of a positive kernel operator and a non-negative multiple of the identity.

Given a positive operator T on L, let $\mathcal{P}_+(T)$ denote the set of all functions $p(z) = \sum_{k=0}^{\infty} a_k z^k$ such that $a_k \geq 0$ for all k and the convergence radius of p is greater than $r(T)$. Using the spectral mapping theorem one can show easily that $r(p(T)) = p(r(T))$ for all $p \in \mathcal{P}_+(T)$.

THEOREM 6. *Under the assumptions of Theorem* 5, *let* $p_i \in \mathcal{P}_+(K_i)$ *for* $i = 1, \dots, n$. *Then*

$$r(p_1(K_1) + \dots + p_n(K_n)) \geq p_1(r(K_1)) + \dots + p_n(r(K_n)).$$

In particular, if $s_i > r(K_i)$ *for* $i = 1, \dots, n$, *then*

$$r((s_1 - K_1)^{-1} + \dots + (s_n - K_n)^{-1}) \geq \frac{1}{s_1 - r(K_1)} + \dots + \frac{1}{s_n - r(K_n)}.$$

Proof. We first claim that every $p_i(K_i)$, $i = 1, \dots, n$, is the sum of a kernel operator and a non-negative multiple of the identity operator I. If $p_i(z) = \sum_{k=0}^{\infty} a_k z^k$ with $a_k \geq 0$, then $p_i(K_i) - a_0 I$ is the limit (in norm and in order) of an increasing sequence of kernel operators. It follows that it is a kernel operator (see e.g. [**13**, Theorem 94.5]). This proves our claim. Now, according to the remark following the proof of Theorem 5 we may apply the inequality (8) of Theorem 5 for operators $p_1(K_1), \dots, p_n(K_n)$ to get

$$r(p_1(K_1) + \dots + p_n(K_n)) \geq r(p_1(K_1)) + \dots + r(p_n(K_n)) = p_1(r(K_1)) + \dots + p_n(r(K_n)).$$

□

As an extension of Theorem 4.2 in [**8**] we now show that the inequality (7) of Theorem 5 for $n = 1$ can be improved if the operator is of the form $(s - K)^{-1}$, where $s > r(K)$.

THEOREM 7. *Let* K *be a positive operator on* L *with* $r(K) > 0$ *that is a sum of a positive kernel operator and a non-negative multiple of the identity. Assume that* $f \in L_+$ *and* $g \in L'_+$ *are strictly positive functions satisfying* $Kf = r(K)f$, $K'g = r(K)g$ *and* $\langle f, g\rangle = 1$.

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Let d be in $L^\infty(X,\mu)_+$, and let D be the corresponding multiplication operator on L. Then

$$r(DK) \geq r(K)\, \exp\left(\int_X f\, g\, \log(d)\, d\mu\right). \tag{9}$$

Furthermore, for $s > r(K)$ it holds

$$r(D(s-K)^{-1}) \geq r((s-K)^{-1}) \left(\int_X f\, g\, d\, d\mu\right). \tag{10}$$

Proof. The inequality (9) is a special case of (7). Denote $T = (s-K)^{-1}$ and pick $\lambda > r(DT)$. Then $w = (\lambda - DT)^{-1} f$ is a strictly positive function in L satisfying $DTw \leq \lambda w$. Set $u = Tw$ and $v = f \cdot g/u$. If we apply (2) of Lemma 4 for the operator $K/r(K)$, we obtain that $\langle Ku, v\rangle \geq r(K)$, and so

$$\langle T^{-1}u, v\rangle = \langle (s-K)u, v\rangle \leq s - r(K) = \frac{1}{r(T)}.$$

On the other hand, since $\lambda T^{-1} u = \lambda w \geq DTw = du$, we have $\lambda \langle T^{-1}u, v\rangle \geq \langle du, v\rangle$. It follows that $\lambda \geq r(T)\langle du, v\rangle$ which implies (10). □

Observe that (10) is really a sharpening of (9) for the special class of positive operators, since

$$\exp\left(\int_X f\, g\, \log(d)\, d\mu\right) \leq \int_X f\, g\, d\, d\mu$$

by Jensen's inequality. Also, simple examples show that in (9) $\exp(\int_X f\, g\, \log(d)\, d\mu)$ can not be replaced by $\int_X f\, g\, d\, d\mu$. (Consider $K = \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$ on $L = \mathbb{C}^2$.)

3. L^2-spaces. In [**5**] we proved an extension of Levinger's inequality to positive kernel operators on L^2-spaces. Unfortunately, we were able to show it only under some assumptions on the kernel of the operator. We now show that these assumptions are redundant, as we expected. In the finite-dimensional case this result was proved in [**2**, Theorem 7].

THEOREM 8. *Let K be a positive kernel operator on $L^2(X,\mu)$ such that $r(K)$ is an isolated point of $\sigma(K)$ and the corresponding Riesz idempotent has finite rank. Let $d \in L^\infty_{++}(X,\mu)$ be a strictly positive function, and let D be the corresponding multiplication operator on $L^2(X,\mu)$. Then, for any $t \in [0,1]$,*

$$r(tDKD^{-1} + (1-t)K^*) \geq r(K). \tag{11}$$

If, in addition, the operator K is compact and if $\phi : [0,1] \to [0,\infty)$ is defined by

$$\phi(t) = r(tDKD^{-1} + (1-t)K^*),$$

then ϕ is non-decreasing on $[0, \frac{1}{2}]$ and is non-increasing on $[\frac{1}{2}, 1]$.

Proof. Consider first the case when $D = I$, the identity on L. If K is irreducible, then by Theorem 2 there exist strictly positive functions $f, g \in L^2(X,\mu)$ satisfying

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$Kf = r(K)f$, $K^*g = r(K)g$ and $\langle f, g\rangle = 1$, and the inequality (11) follows from Theorem 5 with $K_1 = K$, $K_2 = K^*$, $f_1 = g_2 = f$ and $g_1 = f_2 = g$. For general K pick any strictly positive function $u \in L^2(X, \mu)$. (Such functions exist because the measure μ is σ-finite.) Denote by K_0 an irreducible kernel operator with strictly positive kernel $u(x)u(y)$ $(x, y \in X)$. For each $m \in \mathbb{N}$ define an irreducible positive kernel operator on $L^2(X, \mu)$ by $K_m = K + \frac{1}{m}K_0$. Then $r(K_m) \geq r(K)$, and the left (and, similarly, the right) essential spectra of K_m and K coincide. Now, Proposition XI.6.9 and Theorem XI.6.8 of [**4**] imply that $r(K_m)$ is an isolated point of $\sigma(K_m)$ and the corresponding Riesz idempotent has finite rank. By the first part of the proof, we then have

$$r\left(tK + (1-t)K^* + \frac{1}{m}K_0\right) = r\left(t\left(K + \frac{1}{m}K_0\right) + (1-t)\left(K + \frac{1}{m}K_0\right)^*\right) \geq r\left(K + \frac{1}{m}K_0\right).$$

Letting $m \to \infty$ we get $\phi(t) \geq r(K)$ by Proposition 3, which proves (11) in the case $D = I$. Since $\phi(t) = \phi(1-t)$, it remains to show in this special case that ϕ is non-decreasing on $[0, \frac{1}{2}]$ provided K is compact. Let $0 \leq t < s \leq \frac{1}{2}$. Then, by (11),

$$\begin{aligned}\phi(t) &\leq r(u(tK + (1-t)K^*) + (1-u)(tK + (1-t)K^*)^*)\\ &= r((2ut - u - t + 1)K + (t + u - 2ut)K^*)\end{aligned}$$

for all $u \in [0, 1]$. Put $u = \frac{1-s-t}{1-2t}$ to obtain that $\phi(t) \leq r(sK + (1-s)K^*) = \phi(s)$.

The general case follows from the special one. To show this, let E be the multiplication operator on L the multiplier of which is $\sqrt{d}$, so that $E^2 = D$. Introducing the notation

$$\phi_{K,D}(t) = r(tDKD^{-1} + (1-t)K^*)$$

we have, for all $t \in [0, 1]$,

$$\begin{aligned}\phi_{K,D}(t) &= r(E(tEKE^{-1} + (1-t)E^{-1}K^*E)E^{-1})\\ &= r(tEKE^{-1} + (1-t)(EKE^{-1})^*) = \phi_{EKE^{-1},I}(t).\end{aligned}$$

Since $\phi_{EKE^{-1},I}(t) \geq r(EKE^{-1}) = r(K)$ by the special case, (11) follows. If, in addition, K is compact, then $\phi_{EKE^{-1},I}$ is non-decreasing on $[0, \frac{1}{2}]$ and is non-increasing on $[\frac{1}{2}, 1]$ by the special case, and so the same is also true for $\phi_{K,D}$. This completes the proof. □

We do not know whether Theorem 8 is valid for every positive operator K on $L^2(X, \mu)$. However, we shall show below that for $t = 1/2$ the inequality (11) holds for all positive operators on $L^2(X, \mu)$. To do this, we recall that the *numerical radius* $w(A)$ of a bounded operator A on $L^2(X, \mu)$ is defined by

$$w(A) = \sup\{|\langle Af, f\rangle| : f \in L^2(X, \mu), \|f\|_2 = 1\}.$$

If, in addition, A is positive, then we have

$$w(A) = \sup\{\langle Af, f\rangle : f \in L^2(X, \mu)_+, \|f\|_2 = 1\}.$$

Indeed, this follows from the estimate

$$|\langle Af, f\rangle| \leq \int_X |Af|\,|f|\,d\mu \leq \langle A|f|, |f|\rangle$$

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that holds for any $f \in L^2(X, \mu)$. It is well known [**10**] that

$$r(A) \le w(A) \le \|A\|$$

for all bounded operators A on $L^2(X, \mu)$.

THEOREM 9. *Let A be a positive operator on $L^2(X, \mu)$. Then, for any $t \in [0, 1]$,*

$$\|A\| \ge \|tA + (1-t)A^*\| \ge w(tA + (1-t)A^*) = w(A) \ge r(A) \tag{12}$$

and

$$\|(tA + (1-t)A^*)^2\| \ge w((tA + (1-t)A^*)^2) \ge w(A^2) \ge (r(A))^2. \tag{13}$$

Furthermore, if d is in $L^\infty_{++}(X, \mu)$ and D is the corresponding multiplication operator on $L^2(X, \mu)$, then

$$r(DAD^{-1} + A^*) \ge 2\, r(A) \tag{14}$$

Proof. The equality in (12) follows from

$$\langle (tA + (1-t)A^*)f, f\rangle = t\langle Af, f\rangle + (1-t)\langle f, Af\rangle = \langle Af, f\rangle,$$

which holds for all $f \in L^2(X, \mu)_+$. The remaining inequalities in (12) are clear. Similarly, only the second inequality in (13) needs a proof. This relation is a consequence of the following inequality

$$\langle (tA + (1-t)A^*)^2 f, f\rangle \ge \langle A^2 f, f\rangle$$

that holds for every $f \in L^2(X, \mu)_+$, since it is equivalent to $t(1-t)\|Af - A^*f\|_2^2 \ge 0$. Setting $t = 1/2$ in (12) we obtain (14) in the case $D = I$, since $r(A + A^*) = w(A + A^*) = \|A + A^*\|$. The general case can be obtained from the special one as in the proof of Theorem 8. Namely, if E is the multiplication operator on L with the multiplier $\sqrt{d}$, then

$$\begin{aligned} r(DAD^{-1} + A^*) &= r(E(EAE^{-1} + E^{-1}A^*E)E^{-1}) \\ &= r(EAE^{-1} + (EAE^{-1})^*) \ge 2\, r(EAE^{-1}) = 2\, r(A). \end{aligned}$$ □

An application of Berberian's trick concerning 2×2 operator matrices gives the following result which seems to be new even in the finite-dimensional case.

THEOREM 10. *Let A and B be positive operators on $L^2(X, \mu)$. Then*

$$\|A + B^*\| \ge 2 \cdot \sqrt{r(AB)}.$$

If, in addition, A and B are compact kernel operators, then, for each $t \in [0, 1]$,

$$\max\{\|tA + (1-t)B^*\|, \|tB + (1-t)A^*\|\} \ge \sqrt{r(AB)}.$$

Proof. Let T be a positive operator on $L^2(X, \mu) \oplus L^2(X, \mu)$ defined by 2×2 operator matrix

$$T = \begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}.$$

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Then $r(T+T^*) = \|T+T^*\| = \|A+B^*\|$ and $(r(T))^2 = r(T^2) = r(AB)$. By (14), we obtain that

$$\|A+B^*\| = r(T+T^*) \geq 2\,r(T) = 2\sqrt{r(AB)}.$$

If, in addition, A and B are compact kernel operators, then T is a compact kernel operator as well. Then, for each $t \in [0, 1]$,

$$\begin{aligned}\sqrt{r(AB)} = r(T) &\leq r(tT + (1-t)T^*) \leq \|tT + (1-t)T^*\| \\ &= \max\{\|tA + (1-t)B^*\|, \|tB + (1-t)A^*\|\},\end{aligned}$$

where we have used (11). This completes the proof. □

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