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FINITE DINILPOTENT GROUPS OF SMALL DERIVED LENGTH

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Dedicated to Mike (M. F.) Newman on the occasion of his 65th birthday

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Abstract

A finite dinilpotent group G is one that can be written as the product of two finite nilpotent groups, A and B say. A finite dinilpotent group is always soluble. If A is abelian and B is metabelian, with $|A|$ and $|B|$ coprime, we show that a bound on the derived length given by Kazarin can be improved. We show that G has derived length at most 3 unless G contains a section with a well defined structure; in particular if G is of odd order, G has derived length at most 3.

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1. Introduction

If a finite group G can be written as the product AB of two nilpotent subgroups, A and B, we will call G a *dinilpotent* group. If A and B are of coprime order and G is soluble, Hall and Higman proved that the derived length of G is at most the sum of the nilpotency classes of A and B (as a special case of [3, Theorem 1.2.4]). Wielandt proved that a dinilpotent group G must indeed be soluble if the factors are of coprime order([9]) and Kegel then proved that a dinilpotent group is always soluble ([8]). However a bound for the derived length of dinilpotent groups has proved elusive.

When A and B are coprime, the bound of Hall and Higman is best possible for small values of the nilpotency classes of A and B. However it seemed likely that for larger values of the nilpotency classes this bound is too large and should be replaced by a function of the derived lengths of A and B. Such a bound has recently been

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provided by Kazarin [7] in a more general setting. We denote by $d(H)$ the derived length of a soluble group H. For a dinilpotent group G with A and B of coprime order he establishes that $d(G) \leq 2d(A)d(B) + d(A) + d(B)$ and if G is of odd order then $d(G) \leq d(A)d(B) + \max\{d(A), d(B)\}$ ([7, Theorem 3]) and (in the proof of [7, Corollary]) he observes that if A is abelian then $d(G) \leq 2d(B) + 1$ and if further G is of odd order then $d(G) \leq 2d(B)$.

The purpose of this paper is to give more precise information about the derived length of the dinilpotent group G in the case when A and B are of coprime orders and A is abelian, B metabelian. In this case, Kazarin's bounds give G of derived length at most 5 and, if G is of odd order, of derived length at most 4. We will show that the bounds can be improved to 4 and 3, respectively and that these bounds are best possible. Our main result is however rather more technical and shows that in most situations the bound will be 3 and that the groups with derived length 4 have a well defined structure. In particular, we obtain that the derived length is at most 3 if G has odd order.

If A is abelian and B is metabelian then A wr B, the wreath product of A and B has derived length 3 and so the bound of 3 can not be improved. If we take $G = GL(2, 3)$, then $G = AB$, where A is a Sylow 3-subgroup and B is a Sylow 2-subgroup. We then have G of derived length 4, A abelian and B metabelian, so that the bound of 4 for dinilpotent groups of even order can not be improved. This group is typical of the groups of derived length 4. We say that a group G is of type (E) if it has the following structure: $F(G)$ is an extraspecial 2-group, $G/F(G)$ is dihedral of order $2q$ for some odd prime q and $F(G)/\Phi(F(G))$ is either a minimal normal subgroup of $G/\Phi(F(G))$ or the product of 2 minimal normal subgroups of $G/\Phi(F(G))$. We give examples to show that for any odd prime q both these possibilities occur.

Our main result is then the following theorem.

THEOREM 1. *Suppose that G is a finite group and $G = AB$ with A abelian and B metabelian and nilpotent. Suppose further that the order of A and the order of B are coprime. Then G is soluble of derived length at most* 4. *Further, the derived length is at most* 3 *unless G has a section of type (E); in which case it has derived length* 4.

2. Preliminaries

We begin with the observation that groups of type (E) are easy to find and it is probably not difficult to classify them completely. For a given odd prime q we show that we can construct groups of type (E). Let D denote the dihedral group of order $2q$ and let U be a faithful irreducible module for D over the field of 2 elements. Then $|U| = 2^r$, where r is the order of 2 modulo q if this order is even and twice the order of 2 modulo q if this order is odd. It is not difficult to see that U is isomorphic to

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its dual V (see Doerk and Hawkes [2, Definition B.6.6] for the definition of duality). It then follows that the trivial module is a quotient of $U \otimes V$ and we can use the construction of Huppert [5, Hilfssatz 6.7.22] to give an extraspecial group F of order p^{2r+1} on which D acts so that F' is trivial and $F/F' \cong U \oplus V$ as D-modules. Put $G = FD$. Then G is clearly a group of type (E). For another example, we note that there exists a non-singular D-invariant quadratic form on U (Huppert and Blackburn [6, Theorem 7.8.13 and Theorem 7.8.30]). Thus D may be regarded as a subgroup of one of the two orthogonal groups $GO_r^{\epsilon}(2)$ ($\epsilon = +1$ or -1; see [1, page (xii)]). It then follows follows from Huppert [5, Satz 3.13.8 and Bemerkung 3.13.9(b)] that there is an extraspecial group F of order 2^{r+1} whose automorphism group contains a subgroup $D_0 \cong D$ for which the action of D_0 on F/F' is the same as that of D on U. We set $G = FD$ and again G is clearly a group of type (E). We can vary these examples to produce non-splitting examples of a similar structure.

LEMMA 1. *Let p be a prime and K a field of characteristic p. Let G be a p-nilpotent group, P a Sylow p-subgroup of G and $Q = O_{p'}(G)$ a Hall p'-subgroup of G. Suppose that U is a faithful irreducible KG-module. Then if Q is abelian and P is nonabelian, the semidirect product of U and P has derived length at least* 3.

PROOF. Note that for p an odd prime, the result is an immediate corollary of Kazarin [7, Lemma 9]. A direct proof is easy however and we include it here.

We assume that the result is false and G has been chosen to have order as small as possible with UP metabelian. Thus if P_0 is a nonabelian maximal subgroup of P and U_0 is an irreducible submodule of U_{QP_0} then it is an easy consequence of Clifford's Theorem (Huppert [5, Hauptsatz V.17.3]) that U_0 and $QP_0/C_{QP_0}(U_0)$ satisfy the hypotheses of the lemma and hence $U_0P_0 \leq UP$ has derived length at least 3, a contradiction. It follows that P_0 is abelian and so every maximal subgroup of P is abelian. We then have that P is generated by two elements, x and y say, $\Phi(P)$, the Frattini subgroup of P, is central (and so $\Phi(P) = \zeta(P)$, the centre of P), and P' is central of order p. For a maximal subgroup P_0 of P and an irreducible submodule U_0 of U_{QP_0}, it is again an easy consequence of Cliffords Theorem that, if any element of $\zeta(P)$ centralises U_0, it will centralise all irreducible components of U_{QP_0} and hence U, a contradiction. Set $G_1 = (QP_0)/C_{QP_0}(U_0)$ and let Q_1 and P_1 denote the images of Q and P_0 in G_1. We now claim that U_0, regarded as a KG_1-module by deflation contains a submodule isomorphic to KP_1 when restricted to P_1. To see this note first that we may assume that K is algebraically closed. Then $(U_0)_{Q_1}$ can be written as a direct sum of homogeneous components by Clifford's Theorem. Since Q_1 is abelian and K is algebraically closed we have that the homogeneous components are one dimensional and moreover no element of P_1 can fix every homogeneous component. It now follows that P_1 acts faithfully and transitively as permutation group on the

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homogeneous components. But then P_1 acts regularly as permutation group on the homogeneous components (Wielandt [10, Proposition 4.4]). It is then clear that U_0 is isomorphic to KP_1 (as KP_1-module).

Suppose now that $|\zeta(P)| > 2$. We have shown above that $\zeta(P) \cap C_{QP_0}(U_0) = 1$ and hence we have $(U_0)_{\zeta(P)}$ contains a submodule V say isomorphic to $K\zeta(P)$. If now c is an element of order p in P' and $C = \langle c \rangle$ and W is the unique maximal submodule of V then W_C contains a submodule isomorphic to KC unless $C = \zeta(P)$, in which case we must have $p \geq 3$ and W uniserial of length $p - 1$. In either case we have that for some element $w \in W$, $wc \neq w$. Next suppose that $|\zeta(P)| = 2$. Then P contains a cyclic subgroup of order 4; we may assume that P_0 has been chosen to be cyclic (of order 4). In this case we have $C_{P_0}(U_0) = 1$ and so $(U_0)_{P_0}$ contains a submodule V isomorphic to KP_0. If W is the unique maximal submodule of V, then $W_{\zeta(P)}$ contains a submodule isomorphic to $K\zeta(P)$. Again if $1 \neq c \in P'$ there is an element $w \in W$ such that $wc \neq w$.

We now translate the claims above in the semidirect product PU. We have an element $1 \neq c \in P'$ and w in the radical of U such that $wc - w \neq 0$. In the semidirect product, $w \in [U, P]$ and $wc - w$ may be written $[w, c]$. But both w and c are in $(UP)'$ and so $(UP)'' \neq 1$. This completes the proof of the lemma. □

The next lemma generalises a result from modular representation theory in a form we need.

LEMMA 2. *Let p be a prime, G a group with U an abelian normal p-subgroup and G/U a p-nilpotent group. Then $U = U_1 \times \cdots \times U_t$ where each U_i is normal in G, all chief factors of G contained in U_i are isomorphic as G-modules and if $i \neq j$ no chief factor of U_i is isomorphic to a chief factor of U_j.*

PROOF. When U is elementary abelian, we can regard U as a G/U-module and the result is then essentially a restatement of a theorem of Srinivasan (Huppert and Blackburn [6, Theorem 7.16.10]). We proceed by induction on the length of a G chief series from U to 1; the result is clearly true for 1. By our observation we can assume that U is not elementary abelian. If U has exponent p^a then $U^{p^{a-1}}$ is elementary abelian and moreover isomorphic (as an G-module) to a quotient of $U/\Phi(U)$. Since $U^{p^{a-1}} \leq \Phi(U)$, we have that for some minimal normal subgroup V of G contained in U U/V contains a G-chief factor isomorphic to V (as G-modules). Now by our inductive hypothesis U/V can be written as a direct product $U/V = (U_1^*/V) \times (U_1^*/V) \times \cdots \times (U_t^*/V)$, where the U_j^*/V satisfy the requirements of the lemma and U_1^*/V has been chosen so that each chief factor of G contained in U_1^*/V is isomorphic to V. For $i > 1$ we have the length of U_i^* is less than the length of U and so $U_i^* = V \times U_i$, since no chief factor of U_i^*/V is isomorphic to V. Set

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$U_1 = U_1^*$. Then it is easy to see that $U = U_1 \times \cdots \times U_t$ and that the U_i satisfy the requirements of the lemma. □

The next result is a technical one we need in the proof of the main theorem.

LEMMA 3. *Let p, q be distinct primes and suppose $G = AB$, where A is the unique minimal normal subgroup of G and is of q-power order and B is cyclic of order p^i. Let U be a faithful irreducible FG-module, where F is a finite field of characteristic p. Let V be the radical of U_B. Then U_B is a free FB-module (of rank t say) and for any element $1 \neq a \in A$ we have $V + Va = U$. Further, $U/(V \cap Va)$ has dimension at most $2t$.*

PROOF. Recall that the radical of a module is the smallest submodule with completely reducible quotient (Doerk and Hawkes [2, Definition B.3.7] and remarks following). If F is a splitting field for G, then U is induced from a 1-dimensional irreducible for A (by Clifford's Theorem) and so by the Mackey Subgroup Theorem (Huppert [5, Satz V.16.9]) U_B is a free FB-module. It then follows easily that U_B is free for any field F of characteristic p. If the dimension of U over F is t, then $t = p^i r$ and U_B is free of rank r. Note that if $a \in A$, then Va is the radical of U_{B^a}. There are now two cases to consider.

Suppose first that U_A is reducible, so that $U_A = U_0 \oplus \cdots \oplus U_{p^s-1}$, where $s \leq t$ and each U_j is a distinct irreducible FA-module of dimension $p^{t-s}r$. If $B = \langle b \rangle$, then B permutes the U_j, say $U_0 b^j = U_j$, with $0 \leq j \leq p^s - 1$. We then have that $Y = \{u - ub : u \in U_0\}$ is a subspace of U contained in V. Now suppose $1 \neq a \in A$. Then $[b, a] \neq 1$ and so $[b, a]$ does not act trivially on some U_j; we may suppose that U_0 has been chosen so that $[b, a]$ does not act trivially on U_0. Let $W = \{u - ub^a : u \in U_0\}$. Suppose now that $W \cap V \neq 0$. Since $U = U_0 \oplus \cdots \oplus U_{p-1}$, we have $u - ub^a = x + y$, with $x \in U_0$ and $y \in U_1$. Since $U_0 b^a = U_1$ we then have $x = u$ and since $x - xb \in V$ we also have $y + xb \in U_1 \cap V = 0$, giving $y = -xb$. Thus we now have $ub^a = ub$ and so $u[b, a] = u$. But then $[b, a]$ acts trivially on U_0, contradicting the choice of a. We thus have $W \cap V = 0$. Since W has dimension r and V has dimension $(p-1)r$, we have $W + V$ has dimension pr and so $U = W + V$. Then we have $U = V + Va$ since $W \leq Va$.

Next we suppose that U_A is irreducible. It follows that A is cyclic of order q and $p | q - 1$. Let E be the field of order $|F|^{pr}$. Then we can regard U as the additive group of E, A as a subgroup of the multiplicative group of E and B as a subgroup of the Galois group of E over F. Note that q divides $|F|^{pr} - 1$ but not $|F|^s - 1$ for any $s < pr$. If D denotes the subfield of E fixed element-wise by B, then E has dimension p as a vector space over D. We now regard E as a DG-module and we then have E_B is isomorphic to DB as DB-module. The radical W of E_B then has dimension $p - 1$ (over D). Since the radical of E_{B^a} is Wa for any $a \in A$ and $Wa \neq W$

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if $a \neq 1$ (otherwise W would be G-invariant, a contradiction) we have $W + Wa = E$. Since W regarded as an FB-module has dimension $r(p-1)$ and E/W is trivial as FB-module, W is the radical of E as FB-module. But U is isomorphic to E as FG-module and the result follows.

The final statement of Lemma 3 comes immediately from the fact that the free FB-module of rank t modulo its radical has dimension t. □

3. Proof of Theorem 1

We suppose that G satisfies the following hypothesis:

(*) $G = AB$ with A abelian, B metabelian and nilpotent and A and B of coprime orders. Further, G has no section isomorphic to one of the groups $P(p, i)$.

We want to show that if G satisfies (*) then G has derived length at most 3. So we suppose that G has been chosen to have order as small as possible with derived length greater than 3 and satisfying (*). We begin with some standard reductions.

Since any quotient of a group satisfying (*) also satisfies (*), it follows quickly that G has a unique minimal normal subgroup N whose quotient G/N has derived length 3. We also have that $F(G)$ is a p-group for some prime p. Further if $\pi(A)$ is the set of primes dividing $|A|$, then G has $\pi(A)$-length 1. If $p \in \pi(A)$ then A centralises $F(G)$ and so is contained in $F(G)$ (Huppert [5, Satz 3.4.2]). Thus $A = F(G)$, $G/F(G) \cong B$ and G clearly has derived length at most 3, a contradiction. Hence we must have $p \in \pi(B)$. If H is the Hall p'-subgroup of B then centralises $F(G)$ and so $H \leq F(G)$, giving $H = 1$. Thus B is a p-group. If $B = F(G)$ then $G/B \cong A$ and again G has derived length at most 3, a contradiction.

We now have $M = F(G)A$ a normal subgroup of G with G/M a nontrivial p-group. We suppose first that G/M is nonabelian, so that there are elements x and y in B with $[x, y]$ not in M. It then follows from Huppert [5, Satz 3.4.2] that there is a chief factor $F(G)/K$ of G with $[x, y] \notin C_G(F(G)/K)$. Let H/K be a complement for $F(G)/K$ in G/K. Then the semidirect product $(F(G)/K)(H/C_H(F(G)/K)) \cong G/C_H(F(G)/K)$ satisfies the hypotheses of Lemma 1 and so has Sylow p-subgroup of derived length at least 3, a contradiction. Thus we must have G/M abelian. Note that this immediately gives a bound of 4 for the derived length, since G/M and $M/F(G)$ are abelian and $F(G)$ is metabelian. Our aim now is to show that $F(G)$ must be abelian unless G has a section isomorphic to some P_i.

Since we have assumed that G has derived length greater than 3 and is minimal, we have $N = G''' < G'' \leq F(G)$. Let L denote the smallest normal subgroup of G contained in $F(G)$ for which every chief factor X/Y of G with $L \leq X < Y \leq F(G)$ satisfies $G/C_G(X/Y)$ abelian. Note that $L \leq G''$. Also we have that G/L is nilpotent-by-abelian and so since any nilpotent subgroup of G is metabelian we have G/L of

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derived length at most 3. In particular $1 \neq L$ and so $N \leq L$. Since G'' is not abelian and $G''' = N$ we have $N \leq \zeta(G'')$ and so G'' has nilpotency class 2. Suppose that $F(G) \neq L$.

Suppose that L is abelian and let D be a maximal abelian normal subgroup containing L. Then $D \leq F(G)$ and we must have $F(G)/D$ nonabelian. Let $E/D = (F(G)/D)'$ and suppose that L is not contained in $\zeta(E)$. We choose L/K to be a chief factor of G with $\zeta(E) \cap L \leq K$. If x is a p-power element not in $F(G)$ we have L/K as an $\langle x \rangle$-module is nontrivial and so for some element $yK \in L/K$ we have $[x, y] \notin K$. If $c \in F(G)'$, then if P is a Sylow p-subgroup of G containing x, c and $[x, y]$ are both in P'. Thus $[c, [x, y]] = 1$. Since E is generated by $F(G)'$ and D we have $[x, y] \in \zeta(E)$, a contradiction. It follows that L is not abelian.

Now let $F(G)/K$ be a chief factor with $L \leq K$. Then $F(G)/K$ is complemented in G, by H say. We then have that H satisfies (*). Moreover $L \leq H''$, since if not there is a chief factor L/J of G with L not contained in $H''J$. But then L/J is a chief factor of H with $H'' \cap L \leq J$ and $H/C_H(L/J)$ abelian. But then we have $G/C_G(L/J)$ abelian, a contradiction. It follows that $L \leq H''$ and then H has derived length 4, a contradiction. Thus we must have $F(G) = L$.

Since $L \leq G'' \leq F(G)$ we have $G'' = F(G)$. Then we have $N = G''' \leq \zeta(F(G))$ and so $F(G)$ is of nilpotency class 2. Moreover, since $F(G)'$ is elementary abelian, p^{th} powers are central in $F(G)$, giving $\Phi(F(G))$ central in $F(G)$. We have $G/F(G)$ p-nilpotent and so by Lemma 2 we can write $F(G)/N = (U_1/N) \times \cdots \times (U_n/N)$, where all chief factors between U_i and N are isomorphic and if $i \neq j$ no chief factor between U_i and N is isomorphic to a chief factor between U_j and N. Note now that no chief factor $F(G)/K$ can have $G/C_G(F(G)/K)$ nilpotent, for we would then have $G/C_G(F(G)/K)$ abelian. It follows that $F(G)/N$ is the metanilpotent residual of G/N and so is complemented, by H/N say (Huppert [5, Satz 6.7.15]). If $F(G)/K$ is a chief factor of G we put $E = KH$. If K is nonabelian then E satisfies (*) and has derived length 4, a contradiction. Hence K must be abelian. Suppose that $F(G)/\Phi(G) = (V_1/\Phi(G)) \times \cdots \times (V_m/\Phi(G))$ with $V_i/\Phi(G)$ a chief factor of G. If $m > 2$ then the product of any $m - 1$ of the V_i is abelian and so in particular $[V_i, V_j] = 1$ and then since $F(G)$ is generated by the V_i we have $F(G)$ abelian, a contradiction. Thus $F(G)/\Phi(G)$ is either a minimal normal subgroup or the product of two minimal normal subgroups of $G/\Phi(G)$.

We now consider the structure of $G/F(G)$. We have B a Sylow p-subgroup of G and we let K be a maximal subgroup of B containing $F(G)$. Then KA is a normal subgroup of index p in G and also satisfies (*). It follows that KA must have derived length 3 and hence that $(KA)''$ must be properly contained in $F(G)$. Regarded as a $\mathbb{Z}_p(KA)$-module, $F(G)/\Phi(G)$ is completely reducible by Clifford's Theorem and so if $F(G)/L$ is a chief factor of G with $(KA)'' \leq L$ we have that KA acts on each composition factor of $F(G)/L$ as an abelian group. It follows that K

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centralises $F(G)/L$. If $F(G)/\Phi(G)$ is irreducible or the direct sum of two isomorphic irreducibles, then we must have $K \leq F(G)$ and hence $K = F(G)$. Now suppose that $F(G)/\Phi(G) = (U/\Phi(G)) \times (V/\Phi(G))$, with $U/\Phi(G)$, $V/\Phi(G)$ irreducible. If B has two distinct maximal subgroups containing $F(G)$, it has at least $p+1$ maximal subgroups containing $F(G)$. Thus we can find distinct maximal subgroups K_1, K_2, K_3 each containing $F(G)$. We can not have both K_1 and K_2 centralising $U/\Phi(G)$, for then we would have B centralising $U/\Phi(G)$ and G acting on $U/\Phi(G)$ as an abelian group, a contradiction; suppose K_1 centralises $U/\Phi(G)$. On the other hand, K_3 must centralise one of $U/\Phi(G)$, $V/\Phi(G)$, $U/\Phi(G)$ say. We then have B centralises $U/\Phi(G)$, a contradiction. Thus we may assume that B has a unique maximal subgroup containing $F(G)$. It nows follows that $B/F(G)$ is cyclic and moreover that $B/F(G)$ acts faithfully on one of $U/\Phi(G)$ and $V/\Phi(G)$, $U/\Phi(G)$ say, and then $(B/F(G))^p$ centralises $V/\Phi(G)$.

Note that $F(G)A$ is normal in G; we choose $F(G)A_0$ normal in G and so that $(F(G)A)/(F(G)A_0)$ is a chief factor. We then have BA_0 satisfies (*) and so has derived length at most 3. Thus we must have that BA_0 acts as an abelian group on some chief factor $F(G)/W$. Since BA_0 cannot act as an abelian group on $F(G)/W$, we must have B centralises $(F(G)A_0)/F(G)$ but not $(F(G)A)/(F(G)A_0)$. It now follows from Higman's Lemma [3] that $A = A_0 \times A_1$ with $(F(G)A_1)/F(G)$ a chief factor of G. If $F(G)/\Phi(G)$ is irreducible, we have BA_1 of derived length 4 and so $G = BA_1$, giving $A = A_1$. Hence suppose that $F(G)/\Phi(G)$ is reducible, so that $F(G)/\Phi(G) = (U/\Phi(G)) \times (V/\Phi(H))$, with $U/\Phi(G)$ and $V/\Phi(G)$ chief factors of G. If A_1 does not centralise either of $U/\Phi(G)$ and $V/\Phi(G)$ then again BA_1 has derived length 4 and $A_1 = A$. If A_1 centralises $U/\Phi(G)$ then it cannot centralise $V/\Phi(G)$ also. Moreover we must have A_0 centralises $V/\Phi(G)$, since it must centralise one of $U/\Phi(G)$ and $V/\Phi(G)$ and if it centralised $U/\Phi(G)$ A would centralise $U/\Phi(G)$, a contradiction. Now choose $F(G)A_2$ so that $(F(G)A_0)/(F(G)A_2)$ is a chief factor of G. We have then that BA_1A_2 has derived length at most 3 and so we must have BA_1A_2 acts as an abelian group on $U/\Phi(G)$. Since BA_0 does not act as an abelian group on $U/\Phi(G)$, we again see that $A_0 = A_2 \times A_3$. But then BA_1A_3 has derived length 4 and so $A_3 = A_1$, giving $(F(G)A_1)/F(G)$ a chief factor of G and $A = A_0 \times A_1$. Note that if $A_0 \cong A_1$ as B-modules then we may take a diagonal submodule D and get BD of derived length 4, a contradiction. In particular if $A = A_0 \times A_1$, we must have $|B/F(G)| > 2$.

If $F(G)/\Phi(G) = (U/\Phi(G)) \times (V/\Phi(G))$ is the direct product of two minimal normal subgroups, then U and V are abelian and so $\Phi(G) = U \cap V$ is central in $F(G)$. If $F(G)/\Phi(G)$ is a minimal normal subgroup then $F(G)/\Phi(F(G))$ is indecomposable as $G/F(G)$-module. But $F(G)/\Phi(G)$ is faithful and free as $B/F(G)$-module and so by Lemma 3 it is free as $B/F(G)$-module. But then it is projective as $G/F(G)$-module (Huppert and Blackburn [6, Theorem 7.7.14]) and hence $\Phi(G) = \Phi(F(G))$.

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Since $F(G)' = N$ is elementary abelian, we have p^{th} powers of elements of $F(G)$ are central in $F(G)$ and so again $\Phi(G) \le \zeta(G)$. We also have $B' \le F(G)$ and hence $B'\Phi(G)$ is an abelian normal subgroup of $F(G)$. Regarding $F(G)/\Phi(G)$ as a $B/F(G)$-module we have $B'\Phi(G)/\Phi(G)$ generated by the elements $u^{-1}u^b\Phi(G)$, with $b \in B$, $u \in F(G)$, so that $B'\Phi(G)/\Phi(G)$ is just the radical of $F(G)/\Phi(G)$.

At this point it is convenient to break the proof into a number of different cases. We have $F(G)A/F(G)$ can be a chief factor of G or the direct product of two chief factors of G, $B/F(G)$ is cyclic and can have order either 2 or greater than 2. These give rise to the following cases: $F(G)A/F(G)$ the product of two chief factors with $|B/F(G)| > 2$ and $F(G)A/F(G)$ a chief factor with $|B/F(G)| = p > 2$ or $|B/F(G)| = 2$. Using the Frattini argument we can choose $b \in B$ so that $\langle b \rangle$ normalises A and $G = F(G)A\langle b \rangle$.

Suppose first that $A = A_0 \times A_1$ and $F(G)/\Phi(G) = (U_0/\Phi(G)) \times (U_1/\Phi(G))$, with $[U_1, A_0] \le \Phi(G)$ and $[U_0, A_1] \le \Phi(G)$ and let $V_i/\Phi(G)$ denote the radical of $U_i/\Phi(G)$, $i = 0, 1$. Suppose moreover that $|B/F(G)| = p^r$ and $\langle b \rangle / C_{\langle b \rangle}(A_0)$ has order greater than 2. Let $|U_0/\Phi(G)| = p^{p^r t}$ and $|U_1/\Phi(G)| = p^{pk}$. We then have $V/\Phi(G) = (V_0/\Phi(G)) \times (V_1/\Phi(G))$ is the radical of $F(G)/\Phi(G)$. By Lemma 3 we can find elements a_i such that $V_i^{a_i}/\Phi(G)$ is the radical of $U_i/\Phi(G)$ as $\langle b \rangle^{a_i}$-module and $U_i/\Phi(G) = (V_i/\Phi(G))(V_i^{a_i}/\Phi(G)$, $i = 0, 1$. If $a = a_0 a_1$ then $F(G) = V^a V$. Since V and V^a are abelian normal subgroups of $F(G)$, we have $V^a \cap V \le \zeta(F(G))$. But by Lemma 3, we have $|F(G)/(V^a \cap V)| \le p^{2t+2k} < |F(G)/\Phi(G)|$, since $2 < p^r$. Thus $\zeta(F(G) > \Phi(G)$ and hence must contain either U_0 or U_1. But then since both U_0 and U_1 are abelian, we must have $F(G)$ abelian, a contradiction.

We now suppose that $F(G)A/F(G)$ is a chief factor and hence $|B/F(G)| = p$. We consider the case p odd. Then if $F(G)/\Phi(G)$ is an irreducible H-module, we let $|F(G)/\Phi(G)| = p^{pk}$. If $V/\Phi(G)$ is the radical of $F(G)/\Phi(G)$ as a B-module, it follows from Lemma 3 that if $1 \ne a \in A$ we have $F(G) = V^a V$. Moreover, $V^a \cap V \le \zeta(F(G))$ since V, V^a are abelian. From Lemma 3 we have $|F(G)/(V^a \cap V)| \le p^{2k} < |F(G)/\Phi(G)|$. But then $\zeta(F(G)) = F(G)$, a contradiction. Hence we suppose that $F(G)/\Phi(G) = (U_0/\Phi(G)) \times (U_1/\Phi(G))$ with $U_0/\Phi(G)$ and $U_1/\Phi(G)$ chief factors of G. We let $V_i/\Phi(G)$ be the radical of $U_i/\Phi(G)$ (considered as a B-module) and take $1 \ne a \in A$. As above we see from Lemma 3 that $(V_0V_1)^a \cap (V_0V_1)$ is central and properly contains $\Phi(G)$, giving a contradiction.

We are now left with the case $F(G)A/F(G)$ a chief factor and $|B/F(G)| = 2$. We now have $G/F(G)$ dihedral of order $2q$. Suppose first that $F(G)/\Phi(G)$ is a chief factor and let $V/\Phi(G)$ be the radical of $F(G)/\Phi(G)$ as $\langle b \rangle$-module. Again if $1 \ne a \in A$, we have V, V^a both abelian, $V^a V = F(G)$ and since $p = 2$ we have $V^a \cap V = \Phi(G)$. Thus we may choose generators $u_1, \dots u_k, v_1, \dots, v_k$ for $F(G)$ with $[u_i, u_j] = [v_i, v_j] = 1$ for all pairs $1 \le i, j \le k$. Thus $F(G)'$ is generated by the commutators $[u_i, v_j]$, $1 \le i, j \le k$. For a fixed u_i and $x \in F(G)$ it is easy to

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check that the map $x\Phi(G) \to [u_i, x]$ is a $\langle b\rangle$-module homomorphism with $V/\Phi(G)$ in its kernel. Thus the image is a completely reducible $\langle b\rangle$-submodule of $F(G)'$. It follows that $F(G)'$ is a completely reducible $\langle b\rangle$-module. Since $F(G)'$ is irreducible as $(G/F(G))$-module it cannot be faithful by Lemma 3 and hence it must be trivial. Thus we have $F(G)'$ central in G. Now suppose that $\Phi(G) \neq F(G)'$. Since all chief factors of G in $F(G)/F(G)'$ are noncentral by Lemma 2 and all chief factors of G in $\zeta(F(G)) = \Phi(G)$ are central by Lemma 2, we have a contradiction. Thus $F(G)' = \zeta(G) = \Phi(G) = \Phi(F(G))$ and so $F(G)$ is extraspecial and G is of type (E), a contradiction. A similar argument applies if $F(G)/\Phi(G)$ is the direct product of two minimal normal subgroups of $G/\Phi(G)$, again leading to G being of type (E), a final contradiction.

We are now left with proving that if $G = AB$, A abelian, B metabelian and nilpotent and A and B of coprime order and G has a section of type (E), then G has derived length 4. It is enough to show that every group of type (E) satisfies these conditions and is of derived length 4. That a group of type (E) has derived length 4 is clear. If G is of type (E), then we can write $G = AB$ where A is a (cyclic) Sylow q-subgroup and B is a Sylow 2-subgroup. We need to show that B is metabelian to complete the proof. The proof is similar to the argument above. If b is chosen so that $G = F(G)A\langle b\rangle$ we then let $V/\Phi(G)$ be the radical of $F(G)/\Phi(G)$ as $\langle b\rangle$-module. If $v \in V$ is fixed and $x \in F(G)$ then the map $x\Phi(G) \to [v, x]$ is a $\langle b\rangle$-module homomorphism from $F(G)/\Phi(G)$ to $\Phi(G)$. Since the image is completely reducible we have $V/\Phi(G)$ in the kernel, giving $[v, x] = 1$ for all $x \in V$. Since this is true for any $v \in V$, we have V abelian. That B/V is abelian comes immediately from the definition of V and hence B is metabelian as required. □

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