J. Austral. Math. Soc. (Series A) **49** (1990), 399–417

RANK 1 PRESERVERS ON THE UNITARY LIE RING

W. J. WONG

(Received 6 June 1989)

Communicated by R. Lidl

Dedicated to G. E. (Tim) Wall, in recognition of his distinguished contribution to mathematics in Australia, on the occasion of his retirement

Abstract

The surjective additive maps on the Lie ring of skew-Hermitian linear transformations on a finite-dimensional vector space over a division ring which preserve the set of rank 1 elements are determined. As an application, maps preserving commuting pairs of transformations are determined.

1980 *Mathematics subject classification* (*Amer. Math. Soc.*) (1985 *Revision*): 15 A 57, 15 A 27.

Introduction

Many authors have studied the problem of determining the maps on spaces of matrices which transform rank 1 matrices into rank 1 matrices. For example, Marcus and Moyls [4] found the linear maps on the space of all $n \times n$ matrices over a field having this property, and their result was extended to matrices over any commutative ring, by Waterhouse [7] and McDonald [5]. The present author has considered cases in which the base ring is noncommutative [11, 12]. In another direction, Waterhouse has studied maps on the set of self-adjoint matrices with respect to a nondegenerate quadratic form over a field [8].

In this paper, we determine the additive surjective maps on the unitary Lie ring $\mathrm{U}(V)$ of skew-Hermitian transformations relative to a nondegenerate skew-Hermitian form on a finite-dimensional vector space V over a division ring D, which preserve the set of rank 1 elements (Theorem 3.1). A variation is given, determining maps which preserve rank-one-plus-scalar transformations (Theorem 4.1), and this is applied to determine maps which preserve pairs of commuting transformations, in the case that D is commutative (Theorem 6.1).

Among the tools used in the paper is a version of the fundamental theorem of projective geometry (Proposition 2.1) which is slightly sharper than the usual form, as stated, for example, in [2].

It is a great pleasure to dedicate this paper to my friend Tim Wall, to whom I shall always be grateful for the support he gave me as a young mathematician, beginning by encouraging me to participate in the Summer Research Institute in Canberra in 1963. I am particularly happy to be writing on a subject which seems appropriate in view of Tim's interest in the classical groups, and especially in view of his important paper on the unitary groups [6].

1. Rank 1 elements of the unitary Lie ring

Throughout the paper, V will denote an n-dimensional vector space over a division ring D. The additive group of all linear transformations on V will be written $\mathbf{L}(V)$. We shall also need the notion of a semilinear map. If $\sigma: D_1 \to D_2$ is a homomorphism between two division rings, and V_1, V_2 are vector spaces over D_1 and D_2, respectively, a map $A: V_1 \to V_2$ is called *σ-semilinear* if it is additive and

$$A(ax) = a^\sigma(Ax),$$

for all x in V, a in D_1. If c is a nonzero element of D, the scalar map $cI: V \to V$ mapping x to cx is semilinear relative to the inner automorphism σ of D given by $a^\sigma = cac^{-1}$. We sometimes write cI simply as c.

We assume that D is provided with an involutory anti-automorphism J, so that $(ab)^J = b^J a^J$, for all a, b in D, and $J^2 = 1$. An element a of D is said to be symmetric if $a^J = a$, skew if $a^J = -a$, and we have two additive groups

$$D_0 = \{a \in D | a^J = a\}, \qquad D_1 = \{a \in D | a^J = -a\}.$$

We shall use the notation $a^{-J} = (a^{-1})^J$. We also assume that V is provided with a *skew-Hermitian* form $(\ ,\ )$, that is, for each x, y in V, there is defined an element (x, y) of D, which is linear in the first variable x, and which satisfies the identity

$$(y, x) = -(x, y)^J.$$

(In particular, (x, x) is skew.) The vectors x, y are said to be *orthogonal* if $(x, y) = 0$. The form is taken to be *nondegenerate*, that is, the only vector which is orthogonal to the whole space V is 0. A vector x is called *isotropic* if $(x, x) = 0$; otherwise it is called *anisotropic*. We shall also assume that the form is *trace-valued*, that is, (x, x) can be expressed in the form $a - a^J$, for every x in V. This condition is automatically satisfied if D does not have characteristic 2, or if J is not the identity on the centre Z of D [2, page 19]. On the other hand, if J is the identity (and so D is commutative), then the condition implies that the form is alternating (symplectic case).

If σ is an automorphism of D, and $A: V \to V$ is a σ-semilinear map, then there exists a unique map $A^*: V \to V$, such that

$$(Ax, y) = (x, A^*y)^\sigma,$$

for all x, y in V. The map A^* is $J\sigma^{-1}J$-semilinear, and is called the *adjoint* of A.

The *unitary Lie ring* on V is the set

$$\mathbf{U}(V) = \{A \in \mathbf{L}(V) | A^* = -A\}.$$

This is a Lie ring, with the Lie product $[A, B] = AB - BA$, and will be the main object of our study.

LEMMA 1.1. *If T is an element of rank 1 in $\mathbf{U}(V)$, then there exist a nonzero vector u of V and a nonzero element a of D_0 such that*

$$Tx = (x, u)au,$$

for all x in V.

PROOF. By nondegeneracy, every linear functional on V has the form $x \to (x, u)$ for some vecctor u in V. Thus T must have the form

$$Tx = (x, u)v,$$

for some u, v in V. A calculation shows that the adjoint has the form

$$T^*x = -(x, v)u.$$

Since $T^* = -T$, it follows easily that $v = au$, where $a \in D_0$. This proves the lemma.

We shall write $T_{u,a}$ for the rank 1 element corresponding to u and a, as in the lemma, that is,

$$T_{u,a}x = (x, u)au.$$

PROPOSITION 1.2. *Every element of* $\mathbf{U}(V)$ *is a sum of elements of rank* 1 *in* $\mathbf{U}(V)$.

PROOF. We use induction on the dimension n.

First suppose that $J = 1$, the symplectic case. We remark that if x and y are vectors of V such that $(x, y) \neq 0$, then, for $a = (x, y)^{-1}$, $T_{y,a}$ maps x to y, and z to 0, for all z orthogonal to y. On the other hand, if $(x, y) = 0$, but $x \neq 0$, choose a vector w which is not orthogonal to x. By the remark, there exist rank 1 elements T_1, T_2, such that $T_1 x = y + w$, $T_2 x = -w$. Then $(T_1 + T_2)x = y$. Let $\mathbf{T}$ denote the additive subgroup of $\mathbf{U}(V)$ generated by its rank 1 elements.

Let $A \in \mathbf{U}(V)$, and let x, y be vectors which are not orthogonal to each other. We wish to show that $A \in \mathbf{T}$. From the last paragraph, we may assume that $Ax = 0$. Then, $(x, Ay) = -(Ax, y) = 0$. If $(Ay, y) \neq 0$, the remark above shows that there exists a rank 1 element T in $\mathbf{U}(V)$ such that $Tx = 0$, $Ty = Ay$. Then $A - T$ maps x and y to 0. If $(Ay, y) = 0$, then we get a rank 1 element T_1 such that $T_1 x = 0$, $T_1 y = Ay - x$. Then $(A - T_1)x = 0$, $(A - T_1)y = x$. Since $(x, y) \neq 0$, there exists a rank 1 element T_2 such that $T_2 x = 0$, $T_2 y = x$. Then $A - T_1 - T_2$ maps x and y to 0. In any case, we have shown that there exists an element T of $\mathbf{T}$ such that $A - T$ is 0 on the nondegenerate plane P spanned by x and y, so that $A - T$ is essentially an element of $\mathbf{U}(W)$, where W is the $(n-2)$-dimensional orthogonal complement of P in V. By induction, $A - T \in \mathbf{T}$, and so $A \in \mathbf{T}$.

Next, suppose that $J \neq 1$, the "proper" unitary case. Let $A \in \mathbf{U}(V)$. If $Ax = 0$, for some anisotropic vector x, then A is essentially an element of $\mathbf{U}(W)$, where W is the $(n-1)$-dimensional orthogonal complement of the nondegenerate subspace spanned by x, and we may apply induction.

Suppose $A \neq 0$. As a function of x and y, (x, Ay) is a nonzero sesquilinear form on V, with $J \neq 1$. Thus the form is not alternating, so that there exists a vector x such that $(x, Ax) \neq 0$. If $a = (x, Ax)^{-1}$, then $a \in D_0$, and $A - T_{Ax,a}$ maps x to 0. If $n = 2$, then $A - T_{Ax,a}$ is of rank 1 or 0. If $n \geq 3$, then we can take x to be anisotropic, by [6, Lemma 2]. We can then apply induction, as in the last paragraph. This proves the proposition.

In the case $J \neq 1$, it can be proved that in fact every element of $\mathbf{U}(V)$ is a sum of elements of the form $T_{u,a}$, where u is anisotropic, except in the case that $n = 2$ and D is the field $\mathbf{F}_4$ of 4 elements. This may be compared with the result of [2, page 41] on the generation of unitary groups by quasi-symmetries.

We shall now characterize lines and planes (one- and two-dimensional subspaces) in V by means of rank 1 elements of $\mathrm{U}(V)$. If $x, y, \ldots$ are vectors, we denote by $\langle x, y, \ldots\rangle$ the subspace of V spanned by $x, y, \ldots$.

LEMMA 1.3. *Let u, v be nonzero vectors in V and let a, b be nonzero elements of D_0.*

Then the image

$$\operatorname{im}(T_{u,a} + T_{v,b}) = \langle u, v\rangle,$$

except when $T_{u,a} + T_{v,b} = 0$. In particular, $T_{u,a} + T_{v,b}$ is 0 or is of rank 1 if and only if $\langle u\rangle = \langle v\rangle$.

PROOF. If $\langle u\rangle = \langle v\rangle$, the result is clear. Assume that $\langle u\rangle \neq \langle v\rangle$. Then there exists a vector x such that $(x, u) \neq 0$, $(x, v) = 0$. Then $(T_{u,a} + T_{v,b})x = (x, u)au$, so that $u \in \operatorname{im}(T_{u,a} + T_{v,b})$. Similarly, $v \in \operatorname{im}(T_{u,a} + T_{v,b})$. Thus, $\operatorname{im}(T_{u,a} + T_{v,b}) = \langle u, v\rangle$, and $T_{u,a} + T_{v,b}$ has rank 2. This proves the lemma.

LEMMA 1.4. (i) *Let u, v be linearly independent vectors in V, $w = ru + sv$, where $r \neq 0$, and let a, b, c be nonzero elements of D_0. Then, $T_{u,a} + T_{v,b} + T_{w,c}$ is of rank 1 if and only if $ra^{-1}r^J + sb^{-1}s^J + c^{-1} = 0$, in which case*

$$T_{u,a} + T_{v,b} + T_{w,c} = T_{z,d}, \quad \textit{where } z = -b^{-1}s^J r^{-J}au + v,\ d = b + s^J cs.$$

(ii) *Suppose $|D_0| > 2$, and let u, v, w be nonzero vectors in V. Then, u, v, w are coplanar if and only if there exist nonzero elements a, b, c of D_0 such that $T_{u,a} + T_{v,b} + T_{w,c}$ is of rank 1.*

PROOF. (i) Let $T = T_{u,a} + T_{v,b} + T_{w,c}$. Then,

$$Tx = (x, u)z_1 + (x, v)z_2,$$

where $z_1 = au + r^J cw$, $z_2 = bv + s^J cw$. Since v, w are linearly independent, $z_2 \neq 0$. From the linear independence of u and v, it follows as in the proof of Lemma 1.3 that T has rank 1 if and only if z_1 is a scalar multiple of z_2. Since $u = -r^{-1}sv + r^{-1}w$,

$$z_1 = -ar^{-1}sv + (ar^{-1} + r^J c)w.$$

From the linear independence of v and w, z_1 is a scalar multiple of z_2 if and only if

$$z_1 = -ar^{-1}sb^{-1}z_2, \quad ar^{-1} + r^J c = -ar^{-1}sb^{-1}s^J c.$$

https://doi.org/10.1017/S1446788700032419 Published online by Cambridge University Press

Multiplying the last equation on the left by ra^{-1} and on the right by c^{-1}, we obtain

$$c^{-1} + ra^{-1}r^J = -sb^{-1}s^J,$$

as asserted.

If this equation is now multiplied on the left by $s^J c$ and on the right by $r^{-J}a$, we find

$$s^J cr = -(b + s^J cs)b^{-1}s^J r^{-J} a,$$

so $z_2 = (b+s^J cs)z$, where $z = -b^{-1}s^J r^{-J} au+v$. It follows that $T = T_{z,d}$, for some d. If x is a vector chosen so that $(x, u) = 0$, $(x, v) = 1$, then $dz = Tx = z_2$, so $d = b + s^J cs$.

(ii) If two of the lines $\langle u\rangle$, $\langle v\rangle$, $\langle w\rangle$ coincide, say $\langle u\rangle = \langle v\rangle$, we can choose a, b so that $T_{u,a}+T_{v,b} = 0$. Thus we may assume that $\langle u\rangle$, $\langle v\rangle$, $\langle w\rangle$ are distinct.

If u, v, w are coplanar, let $w = ru + sv$, and let a be any nonzero element of D_0. Since $|D_0| > 2$, we can choose a nonzero element b of D_0, such that $ra^{-1}r^J + sb^{-1}s^J \neq 0$. Take

$$c = -(ra^{-1}r^J + sb^{-1}s^J)^{-1}.$$

Then $T_{u,a} + T_{v,b} + T_{w,c}$ is of rank 1, by part (i).

Conversely, if $T_{u,a} + T_{v,b} + T_{w,c}$ is of rank 1, say

$$T_{u,a} + T_{v,b} + T_{w,c} = T_{z,d},$$

then $T_{u,a}+T_{v,b} = T_{z,d}-T_{w,c}$. By Lemma 1.3, $\langle u, v\rangle = \langle z, w\rangle$, so u, v, w are coplanar. This proves the lemma.

We note that the condition $|D_0| > 2$ is satisfied in all cases except when $J = 1$ and $|D| = 2$, or $J \neq 1$ and $|D| = 4$, by the following result of Dieudonné [1].

LEMMA 1.5 [1, LEMMA 1]. *If D is not commutative, it is generated by D_0, except when D_0 is the centre Z of D, and D is a quaternion division algebra over Z, of characteristic different from 2.*

2. Fundamental theorem of projective geometry

We shall use a form of the fundamental theorem of projective geometry similar to that in [3].

Proposition 2.1. *Let V_1, V_2 be n-dimensional vector spaces over division rings D_1, D_2, respectively, where $n \geq 3$. Suppose that there is a mapping $L \to L'$ from the set of all lines in V_1 into the set of all lines in V_2, with the properties*

(i) *the lines L' span the vector space V_2,*

(ii) *if $L_1 \subseteq L_2 + L_3$, then $L_1' \subseteq L_2' + L_3'$.*

Then there exist a homomorphism $\sigma: D_1 \to D_2$, and a σ-semilinear monomorphism $P: V_1 \to V_2$, such that $\langle PV_1 \rangle = V_2$, and $L' = \langle PL \rangle$, for all lines L in V_1. In particular, the mapping $L \to L'$ is injective.

Proof. By (i), there exist lines $L_1, \dots, L_n$ in V_1, such that $V_2 = L_1' \oplus \cdots \oplus L_n'$. We assert that, for $1 \leq m \leq n$, $L_1 + \cdots + L_m$ is a direct sum, and, if L is any line in $L_1 + \cdots + L_m$, then L' is a line in $L_1' \oplus \cdots \oplus L_m'$. We prove this by induction, the assertion being trivial for $m = 1$. Assume it is true for a value of m less than n. Since L'_{m+1} is not in $L_1' \oplus \cdots \oplus L_m'$, L_{m+1} is not in $L_1 + \cdots + L_m$, and so the sum $L_1 + \cdots + L_m + L_{m+1}$ is direct. If L is a line in $L_1 + \cdots + L_m + L_{m+1}$, then there is a line M in $L_1 + \cdots + L_m$, such that $L \subseteq M + L_{m+1}$. Applying the induction hypothesis and (ii), we see that

$$L' \subseteq M' + L'_{m+1} \subseteq L_1' \oplus \cdots \oplus L_m' \oplus L'_{m+1}.$$

This proves the assertion.

In particular, the case $m = n$ shows that $V_1 = L_1 \oplus \cdots \oplus L_n$. We can now apply [3, 1.11] to obtain σ and P as required.

If M_1, M_2 are distinct lines in V_1, express V_1 as a direct sum of lines $M_1, M_2, \dots, M_n$. Then

$$V_2 = \langle PV_1 \rangle = \langle PM_1 \rangle + \langle PM_2 \rangle + \cdots + \langle PM_n \rangle = M_1' + M_2' + \cdots + M_n'.$$

Since V_2 has dimension n, $M_1' \neq M_2'$. Thus the mapping $L \to L'$ is injective. This proves this proposition.

3. Rank 1 preservers

We now state the main theorem of the paper.

Theorem 3.1. *Let $F: \mathrm{U}(V) \to \mathrm{U}(V)$ be a surjective, additive map, such that, whenever A is an element of $\mathrm{U}(V)$ of rank 1, $F(A)$ also has rank 1. Suppose that $n \geq 3$, and $|D_0| > 2$. Then, there exist an automorphism σ of D, a σ-semilinear automorphism P of V, and a nonzero element c of D_0, such that*

$$F(A) = cPAP^*,$$

for all A in $\mathrm{U}(V)$.

We remark that, in order for $F(A)$ to be linear, the inner automorphism of D induced by c must be equal to $\sigma^{-1}J\sigma J$.

The rest of the section is devoted to a proof of the theorem. We assume its hypotheses throughout.

Lemma 3.2. *There is a mapping $L \to L'$ of the set of all lines of V into itself, such that, if $u \in L$, $a \in D_0$, then $F(T_{u,a}) = T_{u',a'}$, where $u' \in L'$.*

Proof. Suppose that u, v belong to the same line L, and let $F(T_{u,a}) = T_{u',a'}$, $F(T_{v,b}) = T_{v',b'}$. Since $T_{u,a}+T_{v,b}$ is either 0 or of rank 1, $T_{u',a'}+T_{v',b'} = F(T_{u,a}+T_{v,b})$ is either 0 or of rank 1. By Lemma 1.3, u' and v' belong to the same line L'. Thus the mapping $L \to L'$ exists as required. This proves the lemma.

Lemma 3.3. *There exist an endomorphism σ of D, a σ-semilinear monomorphism $P: V \to V$, and a mapping $h: V \times D_0 \to D_0$, such that*

$$F(T_{u,a}) = T_{Pu,h(u,a)},$$

for all $u \in V$, $a \in D_0$. If u, v are linearly independent, then Pu, Pv are linearly independent.

Proof. We check that the mapping $L \to L'$ satisfies the conditions of Proposition 2.1. It follows from Proposition 1.2 and the surjectivity of F that every element of $\mathrm{U}(V)$ is a sum of elements of the form $F(T_{u,a})$. If u belongs to a line L, then the image of V under $F(T_{u,a})$ is in L'. Thus every element of $\mathrm{U}(V)$ has image in the span of the lines L'. Since any vector v is in the image of the element $T_{v,1}$ of $\mathrm{U}(V)$, it follows that the lines L' span V.

Next, suppose that L_1, L_2, L_3 are lines such that $L_1 \subseteq L_2 + L_3$, where we assume that $L_2 \neq L_3$. Choose nonzero vectors u, v, w in L_2, L_3, L_1, respectively, and, by Lemma 1.4, choose nonzero elements a, b, c of D_0, such that $T_{u,a} + T_{v,b} + T_{w,c}$ has rank 1. If $F(T_{u,a}) = T_{u',a'}$, $F(T_{v,b}) = T_{v',b'}$, $F(T_{w,c}) = T_{w',c'}$, then $T_{u',a'} + T_{v',b'} + T_{w',c'} = T_{z,d}$, for some z, d. If $T_{u',a'} + T_{v',b'} \neq 0$, it follows from Lemma 1.3 that $\langle u', v'\rangle = \langle z, w'\rangle$, so $w' \in \langle u', v'\rangle$, that is, $L_1' \subseteq L_2' + L_3'$.

If $T_{u',a'} + T_{v',b'} = 0$, then F is not injective. Since F is assumed to be surjective, it follows that D must be infinite. By Lemma 1.5, D_0 must have more than 3 elements. As in Lemma 1.4, we can now choose nonzero elements e, f of D_0, such that $T_{u,e} + T_{v,b} + T_{w,f}$ has rank 1, where $e \neq a$. Then $F(T_{u,e}) = T_{u',e'}$, where $e' \neq a'$, since $F(T_{u,e} - T_{u,a})$ is of rank 1, not 0. Now the argument above, with e, f replacing a, c, shows that $L_1' \subseteq L_2' + L_3'$.

By Proposition 2.1, we obtain an endomorphism σ of D, and a σ-semilinear monomorphism P of V into itself, such that $L' = \langle PL \rangle$, for all lines L in V. This means that $F(T_{u,a})$ can be expressed in the form asserted. The last statement follows from the injectivity of the mapping $L \to L'$, given by Proposition 2.1. This proves the lemma.

We may assume that $h(0, a) = 0$, for all a. Also, $h(u, 0) = 0$, for all u.

Lemma 3.4. *There exists a nonzero element c of D_0 such that*

$$h(u, a) = ca^\sigma,$$

for all u in V, a in D_0.

Proof. If u is a nonzero vector, application of F to the equation $T_{u,a+b} = T_{u,a} + T_{u,b}$ shows that

$$h(u, a+b) = h(u, a) + h(u, b).$$

Suppose that u, v are linearly independent vectors in V. let $w = u + v$, and choose a in D_0, distinct from 0 and -1. By Lemma 1.4,

$$T_{u,a} + T_{v,1} + T_{w,c} = T_{z,d},$$

where $c = -(a^{-1} + 1)^{-1}$, $z = -au + v$, $d = c + 1 = (a+1)^{-1}$. Applying F, we find

$$T_{Pu,h(u,a)} + T_{Pv,h(v,1)} + T_{Pw,h(w,c)} = T_{Pz,h(z,d)}.$$

Since Pu and Pv are linearly independent, and $Pw = Pu + Pv$, we see by Lemma 1.4 that $T_{Pz,h(z,d)} = T_{z',d'}$, where

$$z' = -h(v, 1)^{-1} h(u, a) Pu + Pv.$$

Since $Pz = -a^\sigma Pu + Pv$, and Pu, Pv are linearly independent, we have

$$-h(v, 1)^{-1} h(u, a) = -a^\sigma.$$

This holds also for $a = 0$. Since $|D_0| > 2$, the set $\{a \in D_0 | a \neq -1\}$ generates D_0 as an additive group. Thus,

$$h(u, a) = h(v, 1) a^\sigma,$$

for all a in D_0. This holds for every u linearly independent of v; since $h(v, a) = h(u, 1)a^\sigma$, by symmetry, we see that, in fact, $h(u, a) = ca^\sigma$, for all u, where $c = h(v, 1) \in D_0$. Since F does not map $T_{v,1}$ on 0, c must be nonzero. This proves the lemma.

LEMMA 3.5. *For all* a *in* D, $a^{J\sigma J} = ca^{\sigma}c^{-1}$.

PROOF. Let a be a nonzero element of D, $b = a^J a$. Then $T_{au,1} = T_{u,b}$. Apply the mapping F, using Lemmas 3.3 and 3.4. We obtain $T_{P(au),c} = T_{Pu,d}$, where $d = cb^{\sigma}$. Since $P(au) = a^{\sigma}(Pu)$, we find that

$$a^{\sigma J}ca^{\sigma} = d = ca^{J\sigma}a^{\sigma}.$$

Cancelling a^{σ} and applying J, we obtain the result.

LEMMA 3.6. *The endomorphism* σ *is an automorphism of* D, *and* P *is a* σ*-semilinear automorphism of* V.

PROOF. Let $u_1, \dots, u_n$ be a basis of V. Since $\langle PV\rangle = V$, $Pu_1, \dots, Pu_n$ is also a basis of V. Let $v_1, \dots, v_n$ be the dual basis, that is, $(v_i, Pu_j) = \delta_{ij}$. If $u = \sum_j b_j u_j$, then $(v_i, Pu) = b_i^{\sigma J}$, and a calculation using Lemmas 3.3 and 3.4 shows that

$$(v_i, F(T_{u,a})v_j) = (ab_i)^{\sigma J}cb_j^{\sigma}.$$

By Lemma 3.5, $(ab_i)^{\sigma J}c = c(ab_i)^{J\sigma}$, so

$$(v_i, F(T_{u,a})v_j) \in cD^{\sigma}.$$

Since F is surjective and all elements of $\mathbf{U}(V)$ are sums of elements of rank 1, we find that

$$(v_i, Av_j) \in cD^{\sigma},$$

for all A in $\mathbf{U}(V)$.

If $d \in D$, let $v = d^J Pu_1 + Pu_2$, $A = T_{v,1}$. Then

$$(v_1, Av_2) = (v_1, d^J Pu_1 + Pu_2) = d.$$

Thus, $cD^{\sigma} = D$, so $D^{\sigma} = D$. Hence σ is an automorphism of D, and so P is a semilinear automorphism of V, by [3]. This proves the lemma.

PROOF OF THEOREM 3.1. Since σ is an automorphism of D, the adjoint of P is defined, as a $J\sigma^{-1}J$-semilinear map P^* satisfying the identity $(Px, y) = (x, P^*y)^{\sigma}$. From Lemmas 3.3, 3.4 and 3.5, if $A = T_{u,a}$, then

$$\begin{aligned} F(A)x &= (x, Pu)ca^{\sigma}Pu = (P^*x, u)^{J\sigma J}ca^{\sigma}Pu = c(P^*, xu)^{\sigma}a^{\sigma}Pu \\ &= cP((P^*x, u)au) = cPAP^*x. \end{aligned}$$

https://doi.org/10.1017/S1446788700032419 Published online by Cambridge University Press

By Proposition 1.2, it follows that $F(A) = cPAP^*$, for all A in $\mathbf{U}(V)$. This proves the theorem.

4. Rank-one-plus-scalar preservers

The following result is a variation of Theorem 3.1 which will be of use later. We now assume that D is a finite-dimensional division algebra over a field K, and that the involutory anti-automorphism J of D is linear over K. Then K may be identified as a subset of $D_0 \cap Z$, and $\mathbf{U}(V)$ is a finite-dimensional vector space over K.

THEOREM 4.1. *Let τ be an automorphism of K, and let $F: \mathbf{U}(V) \to \mathbf{U}(V)$ be a bijective, τ-semilinear map, such that, whenever A is an element of $\mathbf{U}(V)$ of rank 1, $F(A)$ is the sum of a rank 1 element of $\mathbf{L}(V)$ and a scalar map. Suppose that $n \geq 5$, and $|D_0| > 2$. Then, there exist an extension of τ to an automorphism σ of D, a σ-semilinear automorphism P of V, a nonzero element c of D_0, and a τ-semilinear map $g: \mathbf{U}(V) \to Z_1 = D_1 \cap Z$, such that*

$$F(A) = cPAP^* + g(A)I,$$

for all A in $\mathbf{U}(V)$.

We shall sketch the modifications to the proof of Theorem 3.1 which are needed to prove this result. First we note that if an element A of $\mathbf{L}(V)$ has an expression in the form $A = B + C$, where B has rank less than 3 and C is scalar, then, since $n \geq 5$, B and C are uniquely determined. In particular, if $A \in \mathbf{U}(V)$, so that $B^* + C^* = A^* = -A = -B - C$, then $B^* = -B$, $C^* = -C$, and so B and C both belong to $\mathbf{U}(V)$. In particular, $C = dI$, where $d \in Z_1$.

It now follows that Lemmas 1.3, 1.4, still hold if "rank 1" is replaced by "rank 1 plus scalar" (and 0 by "scalar"). A modified Lemma 3.2 holds, in which a mapping $L \to L'$ of lines is found, such that, if $u \in L$, then $F(T_{u,a}) = T_{u',a'} + \text{scalar}$, where $u' \in L'$. A modified Lemma 3.3 gives a σ-semilinear monomorphism P of V into itself, and a mapping $h: V \times D_0 \to D_0$, such that $F(T_{u,a}) = T_{Pu,h(u,a)} + \text{scalar}$. Lemmas 3.4 and 3.5 hold. If $a \in K$, and u is any nonzero vector in V then

$$F(T_{u,a}) = F(aT_{u,1}) = a^\tau F(T_{u,1}).$$

This leads to the equation $a^\sigma = a^\tau$. Thus, σ is an extension of τ. Since D is finite-dimensional over K, it follows that σ is an automorphism of D, and so P is a σ-semilinear automorphism of V. The argument of the proof of Theorem 3.1 now shows that, if A is an element of rank 1 in $\mathbf{U}(V)$, then $F(A) - cPAP^*$ is a scalar map. By Proposition 1.2, the same holds for every element of $\mathbf{U}(V)$, and so F has the form asserted in the theorem.

5. Centralizers

If $A \in \mathbf{L}(V)$, then A defines an additive map $\theta_A: \mathbf{U}(V) \to \mathbf{U}(V)$, given by

$$\theta_A(B) = AB + BA^*.$$

If $A \in \mathbf{U}(V)$, then the kernel of θ_A is the centralizer

$$C_{\mathbf{U}(V)}(A) = \{B \in \mathbf{U}(V) | AB = BA\}.$$

In this section we shall study this centralizer. From now on we shall assume that, either the characteristic of D is not 2, or J is not the identity on the centre Z.

LEMMA 5.1. (i) *There exists an element e of Z such that $e + e^J = 1$.*
(ii) $D = eD_0 \oplus D_1 = D_0 \oplus e^J D_1$.
(iii) *Every element A of $\mathbf{U}(V)$ has the form $A = B - B^*$, where $B \in \mathbf{L}(V)$.*

PROOF. The assumption on D implies that there exists an element a of Z such that $a + a^J \neq 0$. Let $e = a(a + a^J)^{-1}$, so that $e + e^J = 1$.

If $a \in D$, then

$$a = e(a + a^J) + (e^J a - ea^J).$$

This shows that $D = eD_0 + D_1$. If $a = eb$, where $b \in D_0$, and $a \in D_1$, then $b = a + a^J = 0$. Thus $D = eD_0 \oplus D_1$. Since $e^J e \in D_0$, $D = D_0 \oplus D^J D_1$. The decomposition of an element a of D according to this direct sum is given by

$$a = (ea + e^J a^J) + e^J(a - a^J).$$

If $A \in \mathbf{U}(V)$, then $A = eA - (eA)^*$. This proves the lemma.

Every rank 1 element of $\mathbf{L}(V)$ has the form

$$x \to (x, v)u,$$

where $u, v \in V$, and the adjoint of this map is the map

$$x \to -(x, v)u.$$

Thus the mapping

$$u \bullet v: x \to (x, v)u + (x, u)v$$

is an element of $\mathbf{U}(V)$, and all elements of $\mathbf{U}(V)$ are sums of such elements, by Lemma 5.1. Note that, if $a \in D$, then $(au)\bullet v = u\bullet(a^J v)$. Also, $u\bullet(au) = 0$, if $a \in D_1$. If $V = V_1 \oplus \cdots \oplus V_m$, then $\mathbf{U}(V)$ is a direct sum of all $V_i \bullet V_j$, $i \leq j$, where $V_i \bullet V_j$ is the subgroup generated by $\{u \bullet v | u \in V_i, v \in V_j\}$. In particular, if $v_1, \ldots, v_n$ is a basis of V, then every element of $\mathbf{U}(V)$ is uniquely expressible in the form

$$\sum_{i \leq j} v_i \bullet a_{ij} v_j, \qquad a_{ij} \in D, \ a_{ii} \in eD_0.$$

If $A \in \mathbf{L}(V)$, and V is written as the direct sum of subspaces V_i invariant under A, then, since $\theta_A(u\bullet v) = (Au)\bullet v + u\bullet(Av)$, each $V_i \bullet V_j$ is invariant under θ_A, and so the kernel of θ_A is the direct sum of the kernels of the restrictions of θ_A to the various $V_i \bullet V_j$, $i \leq j$.

From now on, we shall assume that D is commutative, so that we can use the usual elementary divisor theory for a linear transformation A. If $f(t)$ is an element of the polynomial ring $D[t]$, and $v \in V$, define $f(t)v = f(A)v$. This makes V into a $D[t]$-module. We decompose V into a direct sum of indecomposable submodules

$$V = V_1 \oplus \cdots \oplus V_m.$$

Each V_i is a cyclic $D[t]$-module. The order of a generator v of V_i is the monic polynomial $q_i(t)$ of least degree in $D[t]$ such that $q_i(t)v = 0$, and is equal to the characteristic polynomial of the restriction of A to V_i.

If $f(t) = \sum a_j t^j$, we write $f^J(t) = \sum a_j^J t^j$.

Lemma 5.2. *If $i \neq j$, then the kernel of the restriction of θ_A to $V_i \bullet V_j$ is isomorphic as a vector space over D_0 to the space of all polynomials $h(t)$ in $D[t]$, such that $\deg h(t) < \deg q_j(t)$, and*

$$h(t) q_i^J(-t) \equiv 0 \pmod{q_j(t)}.$$

Proof. Set $k = \deg q_i(t)$, so that

$$q_i(t) = a_0 + a_1 t + a_2 t^2 + \cdots + a_{k-1} t^{k-1} + t^k.$$

Let v, w be generators of V_i, V_j as cyclic $D[t]$-modules. Since v, Av, $A^2 v$, $\ldots$, $A^{k-1} v$ form a basis of V_i, every element of $V_i \bullet V_j$ has a unique expression in the form $\sum_{r=0}^{k-1} A^r v \bullet w_r$, where the w_r belong to V_j. Since V_j has a basis consisting of the elements $A^s w$, $0 \leq s < \deg q_j(t)$, we see that every element of $V_i \bullet V_j$ has a unique expression in the form

$$B = \sum_{r=0}^{k-1} A^r v \bullet h_r(A) w,$$

https://doi.org/10.1017/S1446788700032419 Published online by Cambridge University Press

where the $h_r(t)$ are polynomials of degree less than $\deg q_j(t)$. We calculate that

$$\begin{aligned}\theta_A(B) &= \sum_{r=0}^{k-1}(A^{r+1}v \bullet h_r(A)w + A^r v \bullet h_r(A)Aw)\\ &= \sum_{r=0}^{k-1} A^r v \bullet (h_{r-1}(A) + h_r(A)A - a_r^J h_{k-1}(A))w,\end{aligned}$$

where $h_{-1}(t) = 0$. It follows that $\theta_A(B) = 0$, if and only if

$$h_{r-1}(t) + h_r(t)t - a_r^J h_{k-1}(t) \equiv 0 \pmod{q_j(t)},$$

for $r = 0, \dots, k-1$. If these congruences hold, then

$$h_{k-1}(t)q_i^J(-t) = -\sum_{r=0}^{k-1}(-t)^r(h_{r-1}(t) + h_r(t)t - a_r^J h_{k-1}(t)) \equiv 0 \pmod{q_j(t)}.$$

Conversely, if $h_{k-1}(t)$ is a polynomial of degree less than $\deg q_j(t)$, satisfying

$$h_{k-1}(t)q_i^J(-t) \equiv 0 \pmod{q_j(t)},$$

then the congruences determine the $h_r(t)$ completely, since $\deg h_r(t) < \deg q_j(t)$. The correspondence $B \to h_{k-1}(t)$ gives the asserted isomorphism. This proves the lemma.

LEMMA 5.3. *The kernel of the restriction of θ_A to $V_i \bullet V_i$ is isomorphic, as a vector space over D_0, to the space of all polynomials $h(t)$ in $D[t]$ of degree less than $k = \deg q_i(t)$, for which the coefficient of t^{k-1} lies in D_0, such that $h(t)q_i(t) + h^J(-t)q_i^J(-t) = 0$.*

PROOF. Set $k = \deg q_i(t)$, so that

$$q_i(t) = a_0 + a_1 t + a_2 t^2 + \cdots + a_{k-1}t^{k-1} + t^k,$$

and let v be a generator of V_i as a cyclic $D[t]$-module. Every element of $V_i \bullet V_i$ is uniquely expressible as a sum of elements of the form $A^r v \bullet bA^s v$, where $0 \le r \le s \le k-1$, $b \in D$, and $b \in eD_0$ if $r = s$. Thus,

$$V_i \bullet V_i = \mathbf{W} \oplus \mathbf{X},$$

where $\mathbf{W}$ is the set of all sums of elements of the form $A^r v \bullet bA^s v$, where $0 \le r \le s < k-1$, and $\mathbf{X}$ is the set of all sums of all elements of the form $A^r v \bullet bA^{k-1}v$, where $0 \le r \le k-1$. Let $\mathbf{Y}$ be the image of $\mathbf{W}$ under θ_A.

Since

$$\theta_A(A^r v \bullet bA^s v) = A^{r+1} v \bullet bA^s v + A^r v \bullet bA^{s+1}, \quad \text{if } r+1 < s < k-1,$$
$$\theta_A(A^r v \bullet bA^{r+1} v) = A^{r+1} v \bullet e(b+b^J)A^{r+1} v + A^r v \bullet bA^{r+2}, \quad \text{if } r < k-2,$$
$$\theta_A(A^r v \bullet ebA^r v) = A^r v \bullet bA^{r+1}, \quad \text{if } r < k-1,\ b \in D_0,$$

it follows from Lemma 5.1(ii) that

$$V_i \bullet V_i = \mathbf{Y} + \mathbf{Z},$$

where $\mathbf{Z}$ consists of the sums of elements of the form $A^r v \bullet ebA^r v$, where $0 \le r \le k-1$, $b \in D_0$, or of the form $A^r v \bullet e^J bA^{r+1} v$, where $0 \le r < k-1$, $b \in D_1$. Computing dimensions as vector spaces over D_0, we have

$$\dim \mathbf{X} = \dim \mathbf{Z} = k, \quad \text{if } J = 1,$$
$$\dim \mathbf{X} = \dim \mathbf{Z} = 2k-1, \quad \text{if } J \neq 1,$$

and so $\dim \mathbf{Y} \ge \dim \mathbf{W}$. Since $\mathbf{Y}$ is an image of $\mathbf{W}$, $\dim \mathbf{Y} = \dim \mathbf{W}$, and so

$$V_i \bullet V_i = \mathbf{Y} \oplus \mathbf{Z}.$$

Let $\phi\colon \mathbf{X} \to \mathbf{Z}$ be the map given by θ_A followed by projection into $\mathbf{Z}$. Then the image of θ_A is the direct sum of $\mathbf{Y}$ with the image of ϕ, so that the cokernels of θ_A and ϕ are isomorphic. It follows that the kernel of θ_A is isomorphic with the kernel of ϕ.

We now associate polynomials with the elements of $\mathbf{X}$ and $\mathbf{Z}$, in the following way. If $B \in \mathbf{X}$,

$$B = \sum_{r=0}^{k-2} A^r v \bullet b_r A^{k-1} v + A^{k-1} v \bullet eb_{k-1} A^{k-1} v, \qquad b_r \in D,\ b_{k-1} \in D_0,$$

we define a polynomial

$$h_B(t) = \sum_{r=0}^{k-1} b_r (-t)^r.$$

If $C \in \mathbf{Z}$,

$$C = \sum_{r=0}^{k-1} A^r v \bullet eb_r A^r v + \sum_{r=0}^{k-2} A^r v \bullet e^J c_r A^{r+1} v, \qquad b_r \in D_0,\ c_r \in D_1,$$

we define a polynomial

$$g_C(t) = \sum_{r=0}^{k-1} (-1)^{r+1} b_r t^{2r} + \sum_{r=0}^{k-2} (-1)^{r+1} c_r t^{2r+1}.$$

We now assert that

$$g_{\phi(B)}(t) = h_B(t) q_i(t) + h_B^J(-t) q_i^J(-t),$$

for all B in $\mathbf{X}$.

To show this, it is enough to consider the case where B is of the form $A^r v \bullet bA^{k-1}v$. If $r \leq k-2$, then

$$\begin{aligned}&\theta_A(A^r v \bullet bA^{k-1}v)\\&\qquad = A^{r+1}v \bullet bA^{k-1}v + A^r v \bullet b(-a_0 v - a_1 Av - \cdots - a_{k-1}A^{k-1}v).\end{aligned}$$

Also, if $b \in D_0$,

$$\theta_A(A^{k-1}v \bullet ebA^{k-1}v) = A^{k-1}v \bullet b(-a_0 v - a_1 Av - \cdots - a_{k-1}A^{k-1}v).$$

Now it is straightforward, though tedious, to compute $g_{\phi(B)}(t)$ in all cases, and to verify that the asserted relation holds. Since $g_{\phi(B)}(t) = 0$ if and only if $\phi(B) = 0$, we see that the kernel of ϕ consists of the vectors B in $\mathbf{X}$ for which $h_B(t)$ satisfies the condition given in the statement of the lemma. This proves the lemma.

By the elementary divisor theory, each polynomial $q_i(t)$ is a power of an irreducible polynomial. It follows that either $q_i(t)$ and $q_j^J(-t)$ are relatively prime or else one divides the other.

LEMMA 5.4. (i) *If $q_i(t)$ and $q_j^J(-t)$ are relatively prime, then the kernel of the restriction of θ_A to $V_i \bullet V_j$ is* 0.

(ii) *If $i \neq j$, and $q_i(t)$ and $q_j^J(-t)$ are not relatively prime, then the kernel of the restriction of θ_A to $V_i \bullet V_j$ is isomorphic which the space of all polynomials $h(t)$ in $D[t]$ for which*

$$\deg h(t) < \min\{\deg q_i(t), \deg q_j(t)\}.$$

(iii) *If $q_i(t)$ and $q_i^J(-t)$ are not relatively prime, and $k = \deg q_i(t)$, then the kernel of the restriction of θ_A to $V_i \bullet V_i$ is isomorphic with the space of all polynomials $h(t)$ in $D[t]$ of the form*

$$h(t) = \sum_{r=0}^{k-1} b_r t^r,$$

where $b_{k-1}, b_{k-3}, \ldots \in D_0$, $b_{k-2}, b_{k-4}, \ldots \in D_1$.

PROOF. Suppose that $q_i(t)$ and $q_j^J(-t)$ are relatively prime. If $i \neq j$, the condition

$$h(t)q_i^J(-t) \equiv 0 \pmod{q_j(t)}$$

of Lemma 5.2 implies that $h(t)$ is divisible by $q_j(t)$. Since $\deg h(t) < \deg q_j(t)$, $h(t) = 0$. A similar argument applies in the case $i = j$, by use of Lemma 5.3.

Suppose $q_i(t)$ and $q_j^J(-t)$ are not relatively prime, where $i \neq j$. By symmetry, we may suppose that $\deg q_j(t) \leq \deg q_i(t)$. Then $q_j(t)$ divides $q_i^J(-t)$, so that the congruence in Lemma 5.2 is automatically satisfied, and the condition on $h(t)$ is just that $\deg h(t) < \deg q_j(t)$.

Finally, suppose that $q_i(t)$ and $q_i^J(-t)$ are not relatively prime. Then $q_i^J(-t) = (-1)^k q_i(t)$, where $k = \deg q_i(t)$. The condition of Lemma 5.3 then becomes

$$h(t) + (-1)^k h^J(-t) = 0,$$

which is equivalent to $h(t)$ having the form asserted. This proves the lemma.

LEMMA 5.5. (i) *If A is an element of $\mathbf{L}(V)$ with*

$$\dim \ker \theta_A > n^2 - 2n, \qquad J \neq 1,$$

or

$$\dim \ker \theta_A \geq \tfrac{1}{2}(n^2 - n), \qquad J = 1,$$

then A is a scalar map, or the sum of a rank 1 transformation with a scalar map.

(ii) *If A is a rank 1 element of $\mathbf{U}(V)$, then $C_{\mathbf{U}(V)}(A) = \ker \theta_A$ has dimension*

$$\begin{aligned} \dim \ker \theta_A &= n^2 - 2n + 2, \qquad \textit{if } J \neq 1, \\ \dim \ker \theta_A &= \tfrac{1}{2}(n^2 - n), \qquad \textit{if } J = 1. \end{aligned}$$

PROOF. Suppose first that $J \neq 1$. From Lemma 5.4, $\dim \ker \theta_A$ is equal to the sum of all $\min\{\deg q_i(t), \deg q_j(t)\}$, where i, j range over all pairs such that $q_i(t)$, $q_j^J(-t)$ are not relatively prime. If n_i is the number of $q_j^J(-t)$ which are not relatively prime to $q_i(t)$, it follows that

$$\dim \ker \theta_A \leq \sum_i n_i \deg q_i(t).$$

If N is the largest of the n_i, then since $\sum_i \deg q_i(t) = n$, we see that $\dim \ker \theta_A \leq Nn$. If $\dim \ker \theta_A > n^2 - 2n$, then $N = n$ or $N = n - 1$. If $N = n$, then there are n elementary divisors, all equal to $t - a$, for some a. In this case, A is a scalar map. If $N = n - 1$, then either there are $n - 1$ elementary divisors, all equal to some $t - a$, and one elementary divisor equal to some $t - b$, or else there are $n - 2$ elementary divisors, all equal to some $t - a$, and one elementary divisor equal to $(t - a)^2$. In this case, A is the sum of a rank 1 transformation and a scalar map.

https://doi.org/10.1017/S1446788700032419 Published online by Cambridge University Press

If $J = 1$, a similar argument shows that $\dim\ker\theta_A \leq \frac{1}{2}(\mathrm{N}+1)n$, with equality only if $A = 0$. If $\dim\ker\theta_A \geq \frac{1}{2}(n^2 - n)$, then it follows that $N = n$ or $N = n - 1$, as before.

If u is a nonzero isotropic vector and a is a nonzero element of D_0, then the elementary divisors of $T_{u,a}$ are t^2 and t ($n-2$ times). If u is anisotropic, the elementary divisors are $t - b$, where $b \in D_1$, and t ($n-1$ times). Calculation using Lemma 5.4 gives the value of $\dim\ker\theta_A$ as asserted. This proves the lemma.

6. Preservers of commuting pairs

We assume that D is a finite-dimensional extension field over a field K, and that the involutory automorphism J fixes the elements of K. We can now characterize maps preserving zero products in the Lie algebra $\mathbf{U}(V)$ (cf. [9], [10] for the case of the Lie algebra $\mathbf{L}(V)$).

THEOREM 6.1. *Let τ be an automorphism of K, and let $F: \mathbf{U}(V) \to \mathbf{U}(V)$ be a bijective, τ-semilinear map, such that, whenever A and B are of elements of $\mathbf{U}(V)$ which commute, $F(A)$ and $F(B)$ commute. Suppose that $n \geq 5$ and $|D_0| > 2$. Assume that the characteristic of K is not 2 in the case that $J = 1$. Then, there exist an extension of τ to an automorphism σ of D, a σ-semilinear automorphism P of V, a nonzero element of c of D_0, and a τ-semilinear map $g: \mathbf{U}(V) \to D_1$, such that σ commutes with J, P^*P is a scalar map, and*

$$F(A) = cPAP^* + g(A)I,$$

for all A in $\mathbf{U}(V)$.

PROOF. The hypothesis implies that

$$F(C_{\mathbf{U}(V)}(A)) \subseteq C_{\mathbf{U}(V)}(F(A)),$$

and so

$$\dim_K C_{\mathbf{U}(V)}(A) \leq \dim_K C_{\mathbf{U}(V)}(F(A)),$$

From Lemma 5.5, we see first that F maps the space of scalar maps in $\mathbf{U}(V)$ onto itself, and then that if A has rank 1, then $F(A)$ must be a sum of a rank 1 element and a scalar map. Theorem 4.1 now shows that F has the form asserted.

The fact that σ commutes with J follows from Lemma 3.5. If A commutes with B, then the fact that $F(A)$ commutes with $F(B)$ shows that

https://doi.org/10.1017/S1446788700032419 Published online by Cambridge University Press

$AP^*PB = BP^*PA$. Write $Q = P^*P$. If u, v are orthogonal, then $A = T_{u,1}$ commutes with $B = T_{v,1}$. We compute that

$$AQBx = (x, v)(Qv, u)u, \quad BQAx = (x, u)(Qu, v)v,$$

for all x in V. If u, v are linearly independent, it follows that $(Qv, u) = (Qu, v) = 0$. Since the vectors which are orthogonal to u and linearly independent of u generate the hyperplane orthogonal to u, it follows that Q maps this hyperplane on itself. This is true for all hyperplanes, so that Q must be a scalar map. This proves the theorem.

Theorems analogous to our Theorems 4.1 and 6.1 were proved for the space of self-adjoint matrices by Waterhouse [8].

References

[1] J. Dieudonné, 'On the structure of unitary groups', *Trans. Amer. Math. Soc.* **72** (1952), 367–385.

[2] J. Dieudonné, *La géométrie des groupes classiques*, (Springer-Verlag, Berlin, 1955).

[3] A. J. Hahn, 'Cayley algebras and the isomorphisms of the orthogonal groups over arithmetic and local domains', *J. Algebra* **45** (1977), 210–246.

[4] M. Marcus and B. N. Moyls, 'Transformations on tensor product spaces', *Pacific J. Math.* **9** (1959), 1215–1221.

[5] B. McDonald, *R-linear endomorphisms of* $(R)_n$ *preserving invariants*, (Mem. Amer. Math. Soc., no 287, Providence, R.I., 1983).

[6] G. E. Wall, 'The structure of a unitary factor group', *Inst. Hautes Études Sci. Publ. Math.* **1** (1959), 7–23.

[7] W. C. Waterhouse, 'Automorphisms of $\det(X_{ij})$: The group scheme approach', *Adv. in Math.* **65** (1987), 171–203.

[8] W. C. Waterhouse, 'Linear transformations on self-adjoint matrices: The preservation of rank-one-plus-scalar', *Linear Algebra Appl.* **74** (1986), 73–85.

[9] W. Watkins, 'Linear maps that preserve commuting pairs of matrices', *Linear Algebra Appl.* **14** (1976), 29–35.

[10] W. J. Wong, 'Maps on simple algebras preserving zero products II: Lie algebras of linear type', *Pacific J. Math.* **92** (1981), 469–488.

[11] W. J. Wong, 'Rank 1 preserving maps on linear transformations over noncommutative local rings', *J. Algebra* **113** (1988), 263–293.

[12] W. J. Wong, 'Maps on spaces of linear transformations over semisimple algebras', *J. Algebra* **115** (1988), 386–400.

University of Notre Dame
Notre Dame, Indiana 46556
U.S.A