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PASTING INFINITE LATTICES

E. FRIED and G. GRÄTZER

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Abstract

In an earlier paper, we investigated for finite lattices a concept introduced by A. Slavík: Let A, B, and S be sublattices of the lattice L, $A \cap B = S$, $A \cup B = L$. Then L *pastes* A and B together over S, if every amalgamation of A and B over S contains L as a sublattice. In this paper we extend this investigation to infinite lattices. We give several characterizations of pasting; one of them directly generalizes to the infinite case the characterization theorem of A. Day and J. Ježek. Our main result is that the variety of all modular lattices and the variety of all distributive lattices are closed under pasting.

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1. Introduction

In [4], we investigate for finite lattices a concept introduced by A. Slavík [12] (see also G. Grätzer [6], Exercise 12 of Section V.4). Let L be a lattice. Let A, B, and S be sublattices of L, $A \cap B = S$, $A \cup B = L$. Then L *pastes* A and B together over S, if every amalagamation of A and B over S contains L as a sublattice (A. Slavík used the term "A-decomposable".) Pasting generalizes the classical concept of *gluing*: see R. P. Dilworth and M. Hall [2].

A. Day and J. Ježek [1] proved the following theorem: let $\mathbf{V}$ be a nonmodular variety of lattices; if $\mathbf{V}$ is closed under the pasting of finite lattices, then $\mathbf{L} = \mathbf{V}$.

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In [4], we answered a question raised in [1]: N_5 (the five-element nonmodular lattice) cannot be obtained from M_3 (the five-element modular nondistributive lattice) by pasting. In fact, the variety **M** of all modular lattices is closed under the pasting of finite lattices.

In this paper we extend the investigations of [4] to infinite lattices. In Section 2, we give several characterizations of pasting. One of them directly generalizes the characterization theorem of A. Day and J. Ježek [1] to the infinite case.

In Section 3, we investigate convex sublattices and ideal lattices of pasted lattices.

Section 4 contains the generalization of the main result from [4]: the variety **M** of all modular lattices is closed under pasting.

In Section 5 we prove that two distributive lattices pasted together yield a distributive lattice.

Section 6 contains some concluding comments.

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2. Characterizations

We start with a precise definition of pasting:

Definition 1. Let L be a lattice. Let A, B, S be sublattices of L, $A \cap B = S$, $A \cup B = L$. Let f_A and f_B be the embeddings of A and B, respectively, into L. Then L *pastes* A and B together over S, in notation, $L = \mathrm{Paste}(A, B, S)$, if whenever g_A and g_B are embeddings of A and B into a lattice K satisfying $xg_A = xg_B$ for all $x \in S$, then there is a homomorphism h of L into K satisfying $f_A h = g_A$ and $f_B h = g_B$ (see Figure 1).

Note that the homomorphism h is always an embedding; this follows from the proofs of Theorems 5 and 6: we prove that h is one-to-one.

To characterize pasting, we start with a simple lemma from B. Jónsson [11]:

Lemma 2. *Let A, B, and S be lattices, $A \cap B = S$. On $P = A \cup B$, we define a binary relation $\leq$ as follows:*

(i) *for $x, y \in A$ (and for $x, y \in B$), $x \leq y$ in P if and only if $x \leq y$ in A (respectively, $x \leq y$ in B);*

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$$\begin{tikzcd} & A \arrow[d, "f_A"'] \arrow[ddr, "g_A"] & \\ B \arrow[r, "f_B"] \arrow[drr, "g_B"'] & L \arrow[dr, dashed, "h"] & \\ & & K \end{tikzcd}$$

Figure 1

(ii) *for* $x \in A$ *and for* $y \in B$, $x \leq y$ *in* P *if and only if there exists an* $s \in S$ *with* $x \leq s$ *in* A *and* $s \leq y$ *in* B; *and dually, for* $y \leq x$.
Then P *is a poset* $P(A, B, S)$. *We shall use the notations* P *and* $P(A, B, S)$ *interchangeably. The poset* P *contains* A *and* B *as subposets.*

If L pastes A and B together over S, then L as a poset is isomorphic to $P(A, B, S)$; the converse, however, is false in general. The poset $P(A, B, S)$ may be a lattice, but it may not be a pasting. There are many lattice constructions in the literature that put together lattices to obtain a new lattice. Two examples should suffice: the *S-verklebte Summen* of Ch. Hermann [9] and the *hinged-product* of E. Fried and G. Grätzer [3]. Most of these constructions put lattices together to form a poset; it is then proved that this poset is a lattice. It is important to remember that pasting is a lot more than $P(A, B, S)$ being a lattice.

Based on Lemma 2, we define $\wedge$ and $\vee$ as partial operations on P:

DEFINITION 3. Let A, B, and S be lattices, $A \cap B = S$. On the set $P = A \cup B$, the partial algebra $\text{Part}(A, B, S)$ with the partial binary operations $\wedge$ and $\vee$ is defined as follows:

(i) if $x \leq y$ in P, then $x \wedge y = x$ and $x \vee y = y$ in $\text{Part}(A, B, S)$;

(ii) if $x, y, z \in A$ and $x \wedge y = z$ in A, then $x \wedge y = z$ in $\text{Part}(A, B, S)$; and similarly for $x \vee y = z$ in A;

(iii) same as (ii), for $x, y, z \in B$.

Observe that, in general, $\text{Part}(A, B, S)$ is not a partial lattice (or even a weak partial lattice) as defined in [6].

We now have everything we need to characterize pasting. However, for easier applicability in the proof in Section 4, we introduce an additional concept:

DEFINITION 4. Let A, B, and S be lattices, $A \cap B = S$. Let $a \in A$ and $b \in B$. An $\{a, b\}$*-sequence* in $\operatorname{Part}(A, B, S)$ is a sequence of elements of S: $x_1, \ldots, x_n$ satisfying

(i) $x_1 \leq \cdots \leq x_n$;

(ii$_a$) $x_1 \leq a$ (*starts below* a), $x_2 \leq b \vee x_1$, $x_3 \leq a \vee x_2, \ldots, x_{2k} \leq b \vee x_{2k-1}$, $x_{2k+1} \leq a \vee x_{2k}, \ldots$; the *target* t of this sequence is $a \vee x_n$, if n is even (*target above* a), $b \vee x_n$, if n is odd (*target above* b); or

(ii$_b$) $x_1 \leq b$ (*starts below* b), $x_2 \leq a \vee x_1$, $x_3 \leq b \vee x_2, \ldots, x_{2k} \leq a \vee x_{2k-1}$, $x_{2k+1} \leq b \vee x_{2k}, \ldots$; the *target* t of this sequence is $a \vee x_n$, if n is odd (*target above* a), $b \vee x_n$, if n is even (*target above* b).

For example, if $a \in S$, the singelton a is an $\{a, b\}$-sequence (starts below a) with target element $a \vee b$ (above a). The sequence x_1, x_2 ($x_1, x_2 \in S$ and $x_1 \leq x_2$) with $x_1 \leq a$ and $x_2 \leq b \vee x_1$ is an $\{a, b\}$-sequence (starts below a) with target $a \vee x_2$ (above a).

The elements in (ii$_a$) form an increasing sequence in A: $a \leq a \vee x_2 \leq \cdots \leq a \vee x_{2k} \leq \cdots$ and in B: $b \leq b \vee x_1 \leq \cdots \leq b \vee x_{2k+1} \leq \ldots$. The target is the last element of the first or the second sequence.

We describe pasting in terms of an order condition (Ord) and a sequence condition (Seq).

THEOREM 5. *Let L be a lattice. Let A, B, and S be sublattices of L, $A \cap B = S$, $A \cup B = L$. Then $L = \operatorname{Paste}(A, B, S)$ if and only if the following two conditions hold.*

(Ord) *For $a \in A$ and $b \in B$, if $a \leq b$, then there exists an $s \in S$ with $a \leq s$ in A and $s \leq b$ in B, and dually for $b \leq a$.*

(Seq) *Let $a \in A - B$ and $b \in B - A$, and let $c = a \vee b$ in L. Then there exists an $\{a, b\}$-sequence with target c, and dually.*

Another characterization of pasting is in terms of ideals:

THEOREM 6. *Let L be a lattice. Let A, B, and S be sublattices of L, $A \cap B = S$, $A \cup B = L$. Then $L = \operatorname{Paste}(A, B, S)$ if and only if the following two conditions hold.*

(Ord) *For $a \in A$ and $b \in B$, if $a \leq b$, then there exists an $s \in S$ with $a \leq s$ in A and $s \leq b$ in B, and dually for $b \leq a$.*

(Id) *If X and Y are ideals of L satisfying $X - Y \subseteq A - B$ and $Y - X \subseteq B - A$, then $X \subseteq Y$ or $Y \subseteq X$; and dually, for dual ideals.*

For finite lattices, in the presence of (Ord), (Id) takes on a simpler form:

(Id$_{\text{fin}}$) if $x, y \in L$ satisfy $(x \wedge y, x] \subseteq A - B$ and $(x \wedge y, y] \subseteq B - A$, then $x \leq y$ or $y \leq x$; and dually.

Indeed, since $(x \wedge y, x] \subseteq (x] - (y]$, $(\mathrm{Id}_{\mathrm{fin}})$ obviously implies (Id) for finite lattices. Conversely, if L is finite and (Id) holds, then take $x, y \in L$ satisfying $(x \wedge y, x] \subseteq A - B$ and $(x \wedge y, y] \subseteq B - A$. We claim that $(x] - (y] \subseteq A - B$. Indeed, if $x \wedge y \in A - B$, then by (Ord), there exists a $s \in S$ with $x \wedge y < s$ and $s < y$, contradicting that $(x \wedge y, y] \subseteq B - A$. We get a similar contradiction if $x \wedge y \in B - A$. Finally, let $x \wedge y \in S$, and let $u \in (x] - (y]$; if $u \in B$, then $u \vee s \in B$, contradicting that $u \vee s \in (x \wedge y, x] \subseteq A - B$. Similarly, $(y] - (x] \subseteq B - A$. Thus (Id) applies, and we obtain $(x] \subseteq (y]$ or $(y] \subseteq (x]$, that is, $x \leq y$ or $y \leq x$, completing the proof of $(\mathrm{Id}_{\mathrm{fin}})$.

Some of the steps in the proofs of Theorems 5 and 6 are implicit in A. Day and J. Ježek [1]. We give here complete proofs.

Crucial to the proofs is the definition of ideals in Part (A, B, S):

Definition 7. Let A, B, and S be lattices, $A \cap B = S$, and $P = A \cup B$. An *ideal* I of the partial algebra $\mathrm{Part}(A, B, S)$ is a subset of P with the following two conditions:

(i) if $x \in I$, $y \in P$, and $y \leq x$, then $y \in I$;

(ii) if x, y, and $z \in I$ and $x \vee y = z$ in $\mathrm{Part}(A, B, S)$, then $z \in I$.

For a subset X of P, $(X]$ will denote the smallest ideal containing X; if $X = \{x\}$, we write $(x]$ for $(X]$. An *A-ideal* (*B-ideal*) I is of the form $I = (X]$, for some X in A (respectively, in B). The lattice of ideals will be denoted by $\mathrm{Id}(\mathrm{Part}(A, B, S))$. *Dual ideals* can be defined dually.

First, we need to know how to manipulate sequences and how sequences relate to ideals of $\mathrm{Part}(A, B, S)$.

Lemma 8. *Let A, B and S be lattices, $A \cap B = S$, and $P = A \cup B$. Let $a \in A$ and $b \in B$. Then the following statements hold in* $\mathrm{Part}(A, B, S)$.

(I) *Let $x_1, \ldots, x_n$ be an $\{a, b\}$-sequence with target t. Then $x_1, \ldots, x_n, x_n, x_n, \ldots$ is again an $\{a, b\}$-sequence; if we add an even number of x_n-s, the target of the new sequence is t.*

(II) *If there exists an $\{a, b\}$-sequence that starts below a, then for every $\{a, b\}$-sequence with target t there exists another $\{a, b\}$-sequence that starts below a and has target t.*

(III) *Let $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$ be $\{a, b\}$-sequences that start below a with targets t and u, respectively. Then $x_1 \vee y_1, \ldots, x_n \vee y_n$ is an $\{a, b\}$-sequence that starts below a with target $t \vee u$.*

(IV) *Let $a \leq a'$ and $b \leq b'$, and let $x_1, \ldots, x_n$ be an $\{a, b\}$-sequence with target t above a. Then $x_1, \ldots, x_n$ is an $\{a', b'\}$-sequence with target $t \vee a'$.*

(V) *We are given an $\{a, b\}$-sequence with target t above b. Let $a' \leq t$, $a' \in A$. Then there exists an $\{a, b\}$-sequence with target t' above a such that $a' \leq t'$.*

(VI) *Let U be a subset of P. We define the set U^{Seq} as follows: $z \in U^{\mathrm{Seq}}$ if and only if there exist, $a, b \in P$ such that a is a join of elements in $U \cap A$ and b is a join of elements in $U \cap B$, and there exists an $\{a, b\}$-sequence with target t satisfying $z \leq t$. Then U^{Seq} is an ideal of* $\mathrm{Part}(A, B, S)$.

(VII) *If in* (VI), *U is contained in U^{Seq}, then U^{Seq} is the ideal generated by U in* $\mathrm{Part}(A, B, S)$.

(VIII) *Let us assume that any two elements of P have a lower bound in P. Let U be a subset of L of the form $U_A \cup U_B$, where U_A is a nonempty subset of $A - B$ and U_B is a nonempty subset of $B - A$. Then U^{Seq} is the ideal of* $\mathrm{Part}(A, B, S)$ *generated by U.*

PROOF. *Ad* (I). This is trivial

Ad (II). Let the $\{a, b\}$-sequence $x_1, \ldots, x_n$ with target t be given; if it starts below a we have nothing to prove. So let it start under b (condition $4(\mathrm{ii_b})$ applies): $x_1 \leq b$, $x_2 \leq a \vee x_1, \ldots$. Since there is an $\{a, b\}$-sequence starting below a, there is an element $s \in S$ with $s \leq a$. Define $x_0 = x_1 \wedge s$. For the sequence $x_0, x_1, \ldots, x_n$ 4(i) is obvious: $x_0 \leq x_1 \leq \cdots$. Moreover, $x_0 \leq a$, $x_1 \leq b \vee x_0$ $(= b)$, $x_2 \leq a \vee x_1$ (by assumption), and so on, verifying $4(\mathrm{ii_a})$. Obviously, the target of the new sequence is unchanged.

Ad (III) and (IV). These are obvious.

Ad (V). Let $x_1, \ldots, x_n$ be an $\{a, b\}$-sequence with target t above b. Since $a' \in A$ and $t \in B$, the assumption $a' \leq t$ implies that there exists an $s \in S$ with $a' \leq s \leq t = b \vee x_n$. Define $x_{n+1} = x_n \vee s$. Obviously, $x_n \leq x_{n+1}$ and $x_{n+1} \leq b \vee x_n$. Thus $x_1, \ldots, x_n, x_{n+1}$ is an $\{a, b\}$-sequence with target $a \vee x_{n+1}$ above a, and $a' \leq a \vee x_{n+1}$, as claimed.

Ad (VI). Obviously, if $z \in U^{\mathrm{Seq}}$, $w \in P$, and $w \leq z$, then $w \in U^{\mathrm{Seq}}$. Now let $x, y \in U^{\mathrm{Seq}}$, $x \vee y = z$ in $\mathrm{Part}(A, B, S)$; we want to show that $z \in U^{\mathrm{Seq}}$. If $x \vee y = z$ by virtue of 3(i), $z \in U^{\mathrm{Seq}}$ is trivial.

Let $x \vee y = z$ by virtue of 3(ii), that is, $x, y, z \in A$ and $x \vee y = z$ in A. Since $x, y \in U^{\mathrm{Seq}}$, we can choose a_1, a_2 that are joins of elements in $U \cap A$, and b_1, b_2 that are joins of elements in $U \cap B$, an $\{a_1, b_1\}$-sequence $x_1, \ldots, x_n$ with target t_1, an $\{a_2, b_2\}$-sequence $y_1, \ldots, y_m$ with target t_2, satisfying $x \leq t_1$ and $y \leq t_2$. With $a = a_1 \vee a_2$ and $b = b_1 \vee b_2$, a is a join of elements in $U \cap A$ and b is a join of elements in $U \cap B$; by (IV), both sequences are $\{a, b\}$-sequences with targets $t_1' \geq t_1$ and $t_2' \geq t_2$, establishing $x, y \in U^{\mathrm{Seq}}$. So we can assume that $a_1 = a_2 = a$ and $b_1 = b_2 = b$.

If t_1 is the target of $x_1, \ldots, x_n$ above b, then by (V) (with $a' = x$) we can change $x_1, \ldots, x_n$ to a sequence with target t_1' over a, $x \leq t_1'$. Applying (V) again, if necessary, to the sequence $y_1, \ldots, y_m$, we conclude that we can assume that both sequences have targets over a.

By (II), we can further assume that both sequences start below a or both sequences start below b. By (I), therefore, we can assume that the two sequences have the same number of elements. Finally, by (III), we obtain an $\{a,b\}$-sequence with target $t_1 \vee t_2$, establishing that $z = x \vee y \leq t_1 \vee t_2$ is in U^{Seq}.

If $x \vee y = z$ by virtue of 3(iii), we proceed by symmetry. Thus U^{Seq} is an ideal.

Ad (VII). Let W be an ideal containing U. An easy induction on n shows that if we choose $a, b \in P$ such that a is a join of elements in $U \cap A$ and b is a join of elements in $U \cap B$, and an $\{a,b\}$-sequence $x_1, \ldots, x_n$ with target t, then $t \in W$, proving that $U^{\mathrm{Seq}} = (U]$.

Ad (VIII). By (VII), it is enough to prove that for $a \in U_A$, we have $a \in U^{\mathrm{Seq}}$ (and symmetricaly for $b \in U_B$). Choose an element $b \in U_B$. Let c be a common lower bound of a and b. There are two cases to consider.

FIRST CASE: $c \in A$. By Definition 3(i), there exists an $s \in S$ with $c \leq s \leq b$. The sequence s can be regarded as an $\{a,b\}$-sequence that starts below b; its target is $a \vee s = a$ which majorizes a; thus $a \in U^{\mathrm{Seq}}$.

SECOND CASE: $c \in B$. By Definition 3(i), there exists an $s \in S$ with $c \leq s \leq a$. The sequence s, s can be regarded as an $\{a,b\}$-sequence that starts below a; its target is $a \vee s = a$ which majorizes a; thus $a \in U^{\mathrm{Seq}}$. This concludes the proof of (VIII), and therefore that of Lemma 8.

Now we prove Theorems 5 and 6. Let $L = \mathrm{Paste}(A,B,S)$. We shall prove conditions (Ord), (Id), and (Seq) for L.

The MacNeille completion P^c of $P = P(A,B,S)$ is a lattice containing P as a subposet with the property that all joins and meets that exist in P are preserved in P^c (see, for example, [6]). By Definition 1, P is a sublattice of P^c, in fact, the sublattice $P = L$. Thus the partial ordering on L is as described in Lemma 2, verifying condition (Ord) of Theorems 5 and 6.

Now let us take the lattice, $\mathrm{Id}(\mathrm{Part}(A,B,S))$, of all ideals of $\mathrm{Part}(A,B,S)$. The maps $ag_A = (a]$ for $a \in A$, and $bg_B = (b]$ for $b \in B$ embed A and B, respectively, into $\mathrm{Id}(\mathrm{Part}(A,B,S))$, and they agree on S. By Definition 1, there exists a homomorphism h of L into $\mathrm{Id}(\mathrm{Part}(A,B,S))$ extending g_A and g_B.

Next we verify condition (Seq) of Theorem 5. Let $a \in A - B$ and $b \in B - A$, and let $c = a \vee b$ in L. Without loss of generality, we can assume that $c \in A$. Then $ch = ah \vee bh$ in $\mathrm{Id}(\mathrm{Part}(A,B,S))$, that is, $(c] = (a] \vee (b]$. By (VIII) of Lemma 8 ($U_A = \{a\}$ and $U_B = \{b\}$), there exists an $\{a,b\}$-sequence $x_1, \ldots, x_n$ with target t satisfying $c \leq t$. Since $t \leq c$ obviously holds, we conclude that $t = c$, as claimed.

In the next step, we assume conditions (Ord) and (Seq) and we prove condition (Id) of Theorem 6. Let X and Y be ideals of L satisfying $X - Y \subseteq$

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$A - B$ and $Y - X \subseteq B - A$, and let $X \subseteq Y$ and $Y \subseteq X$ both fail. Let $U = (X - Y) \cup (Y - X)$; then (VIII) of Lemma 8 applies with $U_A = X - Y$ and $U_B = Y - X$; note that both U_A and U_B are closed under joins. Now take the ideal U^{Seq} as defined in Lemma 8. Choose any $a \in X - Y$, $b \in Y - X$, and an $\{a, b\}$-sequence $x_1, \ldots, x_n$ with target t. An easy induction on i shows that $a \vee x_i \in X$ or $b \vee x_i \in Y$ holds for all i, hence $t \in X \cup Y$. Thus by (VI) of Lemma 8, $U^{\text{seq}} = X \cup Y$ is an ideal of $\text{Part}(A, B, S)$.

By (Seq), for any $a \in X - Y$ and $b \in Y - X$, there exists an $\{a, b\}$-sequence with target $a \vee b$. Thus $a \vee b \in X \cup Y$, for example $a \vee b \in X$; but then $b \leq a \vee b \in X$ which contradicts that $b \in Y - X$. This proves (Id).

Now let L be a lattice, let A, B, and S be sublattices of L, $A \cap B = S$, $A \cup B = L$. We have just proved that in the presence of condition (Ord), condition (Seq) implies condition (Id). We are now going to prove the converse.

So let us assume that conditions (Ord) and (Id) hold and let $a \in A - B$ and $b \in B - A$. Form the following ideals in L:

$$X_0 = (a], \qquad\qquad Y_0 = (b],$$
$$X_{i+1} = X_i \vee (Y_i \cap A], \quad i = 1, \ldots, \qquad Y_{i+1} = Y_i \vee (X_i \cap B], \qquad i = 1, \ldots.$$

Note that the ideal joins can be taken in $\text{Id}(L)$ or $\text{Id}(\text{Part}(A, B, S))$; indeed, X_i and $(Y_i \cap A]$ are both A-ideals, hence their join in $\text{Id}(L)$ and in $\text{Id}(\text{Part}(A, B, S))$ is the same; similarly for Y_i and $(X_i \cap B]$.

Let X be the union of the X_i, $i = 1, \ldots$, and let Y be the union of the Y_i, $i = 1, \ldots$. It is clear that $X \cup Y$ is the $\text{Part}(A, B, S)$ ideal generated by a and b.

Observe that $X - Y \subseteq A - B$. Indeed, if $x \in X - Y$ and $x \in B$, then $x \in X_i$ for some i, and hence $x \in Y_{i+1}$, contradicting that $x \in Y$ fails. Similarly, $Y - X \subseteq B - A$. Thus by (Id), $X \subseteq Y$ or $Y \subseteq X$, for example $X \subseteq Y$. Then $a \in Y$, and so $a \vee b \in Y$, that is, $a \vee b$ belongs to the $\text{Part}(A, B, S)$ ideal generated by a and b. It follows form Lemma 8(VIII) that $a \vee b$ is the target of some $\{a, b\}$-sequence, verifying (Seq).

To complete the proofs of Theorems 5 and 6, it remains to show that (Ord) and (Seq) imply that $L = \text{Paste}(A, B, S)$. To prove this, by Definition 1, we have to take a lattice K, and embeddings g_A and g_B of A and B into K satisfying $xg_A = xg_B$ for all $x \in S$.

We define a map h of L into K by $xh = xg_A$ for $x \in A$ and $xh = xg_B$ for $x \in B$. We have to show that h is an embedding.

h is isotone. Indeed, if $x \leq y$ and $x, y \in A$ or B, then $xh \leq yh$ since h is isotone on A and B. If, say, $x \in A - B$ and $y \in B - A$, then by (Ord), there is an $s \in S$ with $x \leq s \leq y$, and then $xh \leq sh$ and $sh \leq yh$ by the previous case, hence $h \leq yh$.

h is a homomorphism. Again, we can assume that $x \in A - B$, $y \in B - A$, and $x \vee y \in A$. By (Seq), there exists an $\{x, y\}$-sequence $x_1, \ldots, x_n$ with target $x \vee y$; without loss of generality we can assume that the sequence starts below y.

We proceed by induction on n. If $n = 1$, then $x \vee y = x \vee x_1$, where $x_1 \leq y$ and $x_1 \in S$. Then

$$\begin{aligned}(x \vee y)h &= (x \vee x_1)h \\ &= xh \vee x_1 h && \text{(since } h \text{ is an embedding of } A) \\ &\leq xh \vee yh && \text{(since } x_1 h \leq yh).\end{aligned}$$

Since h is isotone, the reverse inequality is trivial, $(x \vee y)h = xh \vee yh$.

Let $n > 1$, and set $x' = x \vee x_1$. If $x' \in B$, then

$$\begin{aligned}(x \vee y)h &= (x' \vee y)h \\ &= x'h \vee yh && \text{(since } h \text{ is an embedding of } B) \\ &= xh \vee x_1 h \vee yh && \text{(since } h \text{ is an embedding of } A) \\ &= xh \vee yh,\end{aligned}$$

as required. If $x' \in A - B$, then $x_2, \ldots, x_n$ is an $\{x', y\}$-sequence which starts below x' with target $x' \vee y$. Therefore,

$$\begin{aligned}(x \vee y)h &= (x' \vee y)h \\ &= x'h \vee yh && \text{(by induction)} \\ &= xh \vee x_1 h \vee yh && \text{(since } h \text{ is an embedding of } A) \\ &= xh \vee yh,\end{aligned}$$

completing the induction.

Using the dual argument, we establish that $(x \wedge y)h = xh \wedge yh$.

Finally, h is one-to-one. In case $x < y$, we can use (Ord) to verify this. So we can assume that $x \in A - B$, $y \in B - A$, and, say, $x \vee y \in A$. If $xh = yh$, then using that h is a homomorphism we obtain that

$$xh = xh \vee yh = (x \vee y)h.$$

Since $x < x \vee y$, $xh = (x \vee y)h$ contradicts that h is one-to-one. This completes the proofs of Theorems 5 and 6.

In the finite case, in the presence of (Ord), condition (Id) is equivalent to the following condition:

(Cov) For $s \in S$, all the covers of s in L are in A or all are in B; and dually.

To verify this, we shall use the form $(\mathrm{Id}_{\mathrm{fin}})$ of (Id). Let (Cov) fail; then there is an $s \in S$ covered by a in $A - B$ and b in $B - A$. The elements a and b violate $(\mathrm{Id}_{\mathrm{fin}})$ since

$$(a \wedge b, a] = (s, a] = \{a\} \subseteq A - B$$

and similarly for $(a \wedge b, b]$. Conversely, let $(\mathrm{Id}_{\mathrm{fin}})$ fail with a and b; then $a \wedge b \in S$ (if not, then say, $a \wedge b \in A - B$; by (Ord), there is an $s \in S$ with $a \wedge b < s < a$, contradicting that $(a \wedge b, b] \subseteq B - A$) and the covers of s are neither all in A nor are all in B.

Thus we obtain a result of A. Day and J. Ježek [1]:

Corollary 9. *Let L be a finite lattice. Let A, B, and S be sublattices of L, $A \cap B = S$, $A \cup B = L$. Then L pastes A and B together over S if and only if the conditions* (Ord) *and* (Cov) *hold.*

3. Some applications

From the result of Day and Ježek (Corollary 9 above), in [4], we derived the result that if a finite lattice is pasted together, then the same holds for its intervals. Now that we have characterizations of pasting in the infinite case, it is natural to ask whether this result can be extended.

Corollary 10. *Let the lattice L paste the lattices A and B over S. Let C be a convex sublattice of L for which $C \cap S$ is not empty. Define $A_1 = A \cap C$, $B_1 = B \cap C$, $S_1 = S \cap C$. Then C pastes A_1 and B_1 over S_1.*

Rather than proving this directly, we now introduce the concept of the rank; with it, we can quantify how the pasting of A_1 and B_1 is related to the pasting of A and B.

Definition 11. Let A, B and S be lattices, $A \cap B = S$, $P = A \cup B$. For $x, y \in P$, we define the *rank*, $\mathrm{rank}(x, y)$, *of x and y* in $\mathrm{Part}(A, B, S)$ as follows:

(i) if $x, y \in A$ or $x, y \in B$, then $\mathrm{rank}(x, y)$ is 0;

(ii) if there exists an $\{x, y\}$-sequence $x_1, \dots, x_n$ with target t satisfying $x, y \leq t$, then the smallest such n is $\mathrm{rank}(x, y)$;

(iii) otherwise, $\mathrm{rank}(x, y)$ is undefined.

The following observation is useful:

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COROLLARY 12. *Under the conditions of Definition* 11, *for* $x, y \in P$ *and* $s \in S$, *if* $\operatorname{rank}(x, y)$ *is defined, then so is* $\operatorname{rank}(x \vee s, y \vee s)$ *and*

$$\operatorname{rank}(x \vee s, y \vee s) \leq \operatorname{rank}(x, y).$$

PROOF. If the $\{x, y\}$-sequence $x_1, \dots, x_n$ establishes that $\operatorname{rank}(x, y) = n$, then the $\{x \vee s, y \vee s\}$-sequence $x_1 \vee s, \dots, x_n \vee s$ establishes that

$$\operatorname{rank}(x \vee s, y \vee s) \leq n.$$

COROLLARY 13. *In Theorem* 5, *condition* (Seq) *can be replaced by the following condition*:

(R) *for every* $x, y \in L$, $\operatorname{rank}(x, y)$ *is defined, and dually.*

Now we can state Corollary 10 in a stronger form:

COROLLARY 14. *Let the lattice L paste the lattices A and B over S. Let C be a convex sublattice of L for which $C \cap S$ is not empty. Define* $A_1 = A \cap C$, $B_1 = B \cap C$, $S_1 = S \cap C$. *Then for* $x, y \in C$,

$$\operatorname{rank}_C(x, y) \leq \operatorname{rank}_L(x, y) + 1$$

where $\operatorname{rank}_C(x, y)$ *and* $\operatorname{rank}_L(x, y)$ *is the rank in C and L, respectively. In particular,* $\operatorname{rank}_C(x, y)$ *is always defined.*

PROOF. If $x, y \in A$ or B, then $\operatorname{rank}_C(x, y) = \operatorname{rank}_L(x, y) = 0$. Now let $\operatorname{rank}_L(x, y) = n > 0$ be established by the $\{x, y\}$-sequence $x_1, \dots, x_n$ in L. If $z = x \wedge y \in S$, then Corollary 12 (or its proof) shows that

$$\operatorname{rank}_C(x, y) \leq \operatorname{rank}_L(x, y).$$

Now if $z \notin S$, then, say, $z \in A - B$. By (Ord), there is an $s \in S$ with $z < s \leq y$. If $x_1, \dots, x_n$ is an $\{x, y\}$-sequence starting below y, then $x_1 \vee s, \dots, x_n \vee s$ is an $\{x, y\}$-sequence in C establishing that

$$\operatorname{rank}_C(x, y) \leq \operatorname{rank}_L(x, y).$$

Finally, if $x_1, \dots, x_n$ is a $\{x, y\}$-sequence starting below x, then $s, x_1 \vee s, \dots$ is an $\{x, y\}$-sequence in C establishing that

$$\operatorname{rank}_C(x, y) \leq \operatorname{rank}_L(x, y) + 1.$$

This completes the proof of Corollary 14.

The lattice of Figure 2 (S is represented by the black-filled, $A - S$ by the unfilled, and $B - S$ by the dot-filled elements) shows an example where

$$\operatorname{rank}_C(x, y) = \operatorname{rank}_L(x, y) + 1.$$

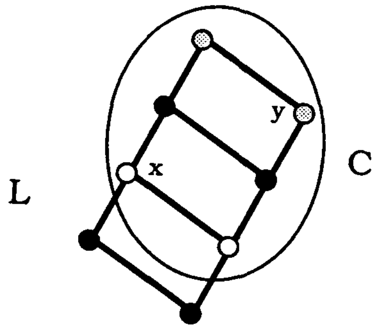

Figure 2

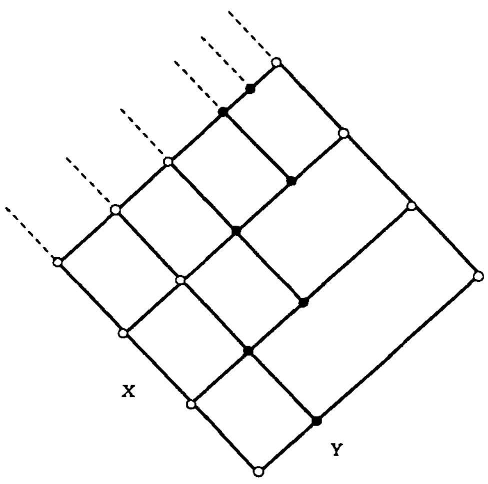

Figure 3

Thus the result of Corollary 14 is best possible.

In the infinite case we can pass from lattices to ideal lattices and ask whether from pasted lattices we get again pasted lattices.

Let $L = \mathrm{Paste}(A, B, S)$. Then $\mathrm{Id}(A)$, the ideal lattice of A, has a natural embedding into L: $\varphi_A \colon X \to (X]_L$.

Viewing $\mathrm{Id}(A)$ as a sublattice of $\mathrm{Id}(L)$, and similarly, $\mathrm{Id}(B)$ as a sublattice of $\mathrm{Id}(L)$, observe that $\mathrm{Id}(A) \cap \mathrm{Id}(B) = \mathrm{Id}(S)$. Indeed, if I is an ideal of L and $I \in \mathrm{Id}(A) \cap \mathrm{Id}(B)$, then for every $i \in I$ there is an $a \in A$ with $a \in I$ and $i \leq a$; so for this $a \in I$, there is an element $b \in b$ with $b \in I$ and $a \leq b$. By (Ord), there is an element $s \in S$ with $a \leq s \leq b$. Thus $i \leq s$ for some $s \in S$, proving that $I \in \mathrm{Id}(S)$. The converse is obvious. Finally, every ideal I of L

is in $\mathrm{Id}(A)$ or $\mathrm{Id}(B)$. Indeed, if $I \notin \mathrm{Id}(A)$, then there is an $i \in I$ such that for all $j \in I$ with $i \leq j$, we have $j \in B$; similarly, if $I \notin \mathrm{Id}(B)$, then there is an $k \in I$ such that for all $j \in I$ with $k \leq j$ we have $j \in A$. But then $i \vee k \in I$ is in neither A nor B, a contradiction.

Thus $\mathrm{Id}(L) = \mathrm{Id}(A) \cup \mathrm{Id}(B)$ and $\mathrm{Id}(A) \cap \mathrm{Id}(B) = \mathrm{Id}(S)$. It is logical to ask whether $\mathrm{Id}(L)$ pastes $\mathrm{Id}(A)$ and $\mathrm{Id}(B)$ together over $\mathrm{Id}(S)$.

Lemma 15. *Let L be the lattice of Figure 3. Let S be the sublattice of elements marked by solid dots; let A be the sublattice of the elements of S and the elements to the left; let B be the sublattice of the elements of S and the elements to the right. Then L pastes A and B together over S but* $\mathrm{Id}(L)$ *does not paste* $\mathrm{Id}(A)$ *and* $\mathrm{Id}(B)$ *together over* $\mathrm{Id}(S)$.

Proof. Choose X to be the left-axis and Y the right-axis. We claim that $\mathrm{rank}(X, Y)$ in $\mathrm{Id}(L)$ is not defined. Indeed, let us form the ideals:

$$X_0 = X; \qquad\qquad Y_0 = Y;$$
$$X_{i+1} = X_i \vee (Y_i \cap A), \quad i = 1, \dots; \qquad Y_{i+1} = Y_i \vee (X_i \cap B), \quad i = 1, \dots.$$

It is easy to see that X_i is the ideal generated by X and the $(2i-1)$st element of S, while Y_i is generated by Y and the $(i+1)$st elements of X. If $\mathrm{rank}(X, Y)$ were defined, $\mathrm{rank}(X, Y) = n$, then either X_n or Y_n would equal L, a contradiction.

4. Pasting modular lattices

In this section we prove the following result, answering Problem 1 of [4]:

Theorem 16. *The variety* **M** *of all modular lattices is closed under pasting.*

In the proof, we shall use the following notation:

Definition 17. For elements x, y, z of the lattice L, $N_5(x,\ y \leq z)$ denotes that x, y, and z satisfy the relations:

$$y \leq z, \qquad x \vee y = x \vee z, \qquad x \wedge y = x \wedge z.$$

If, in addition, $y < z$, then we write $N_5(x,\ y < z)$. If $N_5(x,\ y < 3)$, then x, y, z generate a sublattice isomorphic to N_5.

In the finite case, in [4], we proved Theorem 16 by way of contradiction: let L be a finite nonmodular lattice, and let L be the pasting of the modular sublattices A and B over S. We can choose L as the smallest such lattice.

Since L is nonmodular, it contains elements x, y, z satisfying $N_5(x,\ y < z)$. Clearly, $x \wedge z = 0$ and $x \vee y = 1$; indeed if $x \wedge z = u$ and $x \vee y = v$, and $0 < u$ or $v < 1$, then applying the Corollary 10 to $C = [u, v]$, we get a smaller lattice, contradicting the minimality of L. Thus L is *almost modular*: no proper interval of L can contain an N_5, hence the interval is modular. The proof heavily uses almost modularity.

In the general case there is no chance of finding a minimal or maximal N_5. Instead, we use the concept of the rank introduced in Section 3.

The proof of Theorem 16 relies heavily on the following two lemmas.

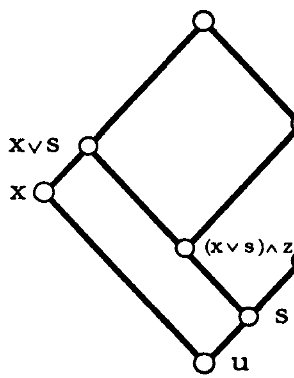

Figure 4

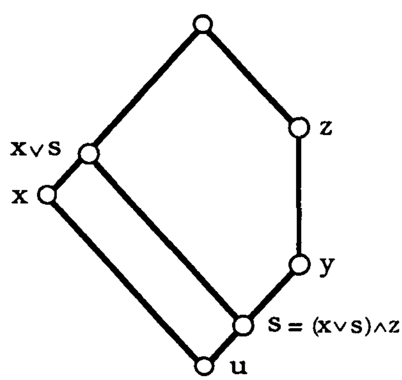

Figure 5

Lemma 18. *Let L be a lattice, and let $x, y, z, s \in L$. If $N_5(x,\ y < z)$ and $x \wedge y \leq s < y$, then*

(i) $N_5(x, s < (x \vee s) \wedge z)$ (*see Figure* 4)

or

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(ii) $s = (x \vee s) \wedge z$ *and* $N_5(x \vee s, y < z)$ (*see Figure* 5).

Proof. Since $s \leq (x \vee s) \wedge z \leq x \vee s$, it follows that $x \vee s = x \vee ((x \vee s) \wedge z)$; obviously, $x \wedge ((x \vee s) \wedge z) = x \wedge s$ $(= x \wedge z)$. Hence, $N_5(x,\ s \leq (x \vee s) \wedge z)$. Now (i) follows; (ii) is even easier.

Lemma 19. *Let L be a lattice, and let $x, y, z, s \in L$. If $N_5(x,\ y < z)$ and $x \wedge y \leq s < x$, then*

(i) $N_5(z,\ s < x \wedge (s \vee z))$ (*see Figure* 6)

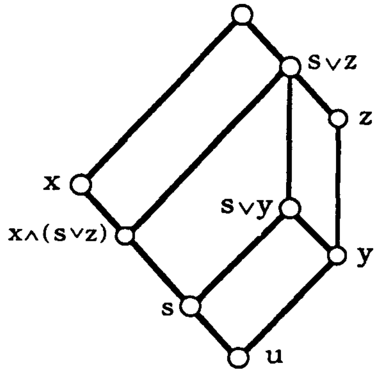

Figure 6

or

(ii) $s = x \wedge (s \vee z)$ *and* $N_5(x,\ s \vee y < s \vee z)$ (*see Figure* 7)

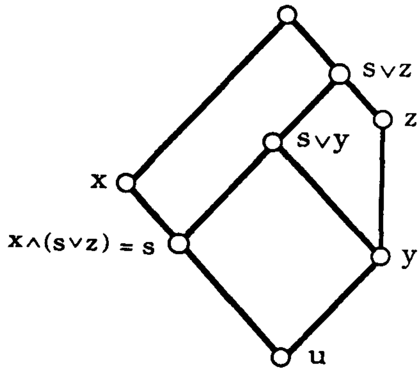

Figure 7

or

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(iii) $s = x \wedge (s \vee z)$, $s \vee y = s \vee z$, *and* $N_5(s,\ y < z)$ (*see Figure* 8).

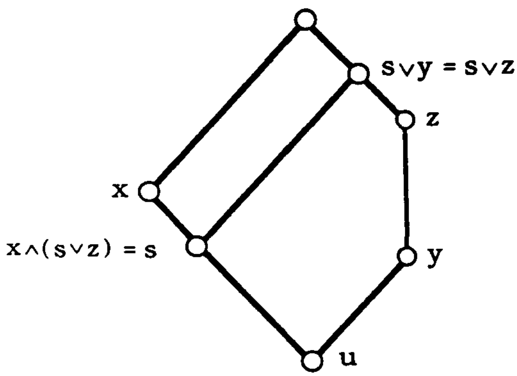

Figure 8

PROOF. We prove (i) as 18(i) was proved; (ii) and (iii) are even easier.

PROOF OF THEOREM 16. Let $L = \mathrm{Paste}(A, B, S)$. Let us assume that A and B are modular lattices, and L is nonmodular. Since L is nonmodular, it contains three elements x, y, z satisfying $N_5(x, y < z)$; let $u = x \wedge z$ and $v = x \vee y$.

We shall consider several cases. As a rule, each case will be reduced to some previous cases.

Case 1. $x, y, z \in A$ or $x, y, z \in B$. Then the N_5 is in A or B, contrary to the assumption that A and B are modular. So Case 1 cannot happen.

Case 2. $x \in S$. If $y, z \in A$ or $y, z \in B$, then Case 1 holds, a contradiction. By duality, we can assume that $y \in A - B$ and $z \in B - A$. By (Ord), there exists a $w \in S$ with $y < w < z$. Then, $N_5(x, y < w)$, contradicting Case 1. Thus Case 2 cannot occur.

From now one, we can assume that $x \in A - B$ or that $x \in B - A$. Without loss of generality, we shall assume that $x \in A - B$.

Now there are only three possibilities: $y \in B - A$ and $z \in A$; $y \in A$ and $z \in B - A$; $y \in B - A$ and $z \in B - A$. We shall further classify by specifying where u is: $A - B, B - A$, or S.

Case 3. $x \in A - B$, $y \in B - A, z \in A$.

Case 3A. $u \in S$. We show that this case cannot occur by induction on $\operatorname{rank}(x,y) = n$. Let the $\{x,y\}$-sequence $p_1, \ldots, p_n$ ($\in S$) establish the rank.

Firstly, let the sequence start below y. Set $s = p_1 \vee u$. Since $u \le s < y$, by Lemma 18, either $N_5(x,\ s < (x \vee s) \wedge z)$ which would contradict Case 1 since $x, s, (x \vee s) \wedge z \in A$, or $s = (x \vee s) \wedge z$ and $N_5(x \vee s,\ y < z)$ (see Figure 5). In the latter case, either $x \vee s \in S$, a contradiction by Case 2, or $x \vee s \notin S$ and so $x \vee s \in A - B$ (since $x \vee s \in A$). Obviously, the sequence $p_2, \ldots, p_n$ establishes that $\operatorname{rank}(x \vee s, y) \le n - 1$, completing the discussion of the case.

Secondly, let the sequence start below x. Again, set $s = p_1 \vee u$. Since $u \le s < x$, by Lemma 19, either

(i) $N_5(z,\ s < x \wedge (s \vee z))$ (see Figure 6), which would contradict Case 1 $(z, s, x \wedge (s \vee z) \in A)$,

or

(ii) $s = x \wedge (s \vee z)$ and $N_5(x,\ s \vee y < s \vee z)$ (see Figure 7); now if $s \vee y \in B - A$, then $N_5(x,\ s \vee y < s \vee z)$ satisfies the conditions of this case ($s \vee z \in A$ is obvious) and $\operatorname{rank}(x, s \vee y) < n$ (established by $p_2, \ldots, p_n$), and hence this is impossible by induction,

or

(iii) $s = x \wedge (s \vee z)$, $s \vee y = s \vee z$, and $N_5(s,\ y < z)$ (see Figure 8), a contradiction with Case 2.

Case 3B. $u \in A - B$. By (Ord), there exists an $s \in S$ with $u < s < y$. By Lemma 18, $N_5(x,\ s < (s \vee s) \wedge z)$ (see Figure 4), which would contradict Case 1 since $x, s, (x \vee s) \wedge z \in A$, or $s = (x \vee s) \wedge z$ and $N_5(x \vee s,\ y < z)$ (see Figure 5), contradicting Case 3A.

CASE 3C. $u \in B - A$. this is impossible since $x, z \in A$, hence $x \wedge z = u \in A$.

Case 3^{D}. The dual of Case 3. This case cannot hold by duality.

Case 4. $x \in A - B$, $y \in B - A$, $z \in B - A$.

Case 4A. $u \in S$. We show that this case cannot occur by induction on $\operatorname{rank}(x,y) = n$. Let the $\{x,y\}$-sequence $p_1, \ldots, p_n (\in S)$ establish the rank.

Firstly, let the sequence start below y. Set $s = p_1 \vee u$. Since $u \le s < y$, by Lemma 18, either

(i) $N_5(x,\ s < (x \vee s) \wedge z)$; now if $(x \vee s) \wedge z \in A$, then this contradicts Case 1 $(x, s, (x \vee s) \wedge z \in A)$; otherwise, $(x \vee s) \wedge z \subset B - A$ contradicting Case 3^{D}, or

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(ii) $s = (x \vee s) \wedge z$ and $N_5(x \vee s,\ y < z)$ (see Figure 5). Now if $x \vee s \in B$, then this contradicts Case 1 ($x \vee s, y, z \in B$). Otherwise, obviously, $x \vee s \in A - B$, so $N_5(x \vee s,\ y < z)$ satisfies the conditions of Case 4A, and $p_2, \dots, p_n$ establish that $\operatorname{rank}(x \vee s, y) \leq n - 1$, completing the discussion of the case.

Secondly, let the sequence start below x. Again, set $s = p_1 \vee u$. By Lemma 19, either

(i) $N_5(z,\ s < x \wedge (s \vee z))$ (see Figure 6); now if $x \wedge (s \vee z) \in B$, then we get a contradiction with Case 1 ($z, s, x \wedge (s \vee z) \in B$); otherwise, $x \wedge (s \vee z) \in A - B$, and $N_5(z,\ s < x \wedge (s \vee z))$ is covered by Case 3 using the symmetry of A and B; or

(ii) $s = x \wedge (s \vee z)$ and $N_5(x,\ s \vee y < s \vee z)$ (see Figure 7); now if $s \vee y$ or $s \vee z \in A$, then $N_5(x,\ s \vee y < s \vee z)$ is covered by a previous case; otherwise, $s \vee y,\ s \vee z \in B - A$, and so $N_5(x,\ s \vee y < s \vee z)$ satisfies the conditions of this case and $\operatorname{rank}(x, s \vee y) < n$ (established by $p_2, \dots, p_n$), and hence this is impossible by induction;

or

(iii) $s = x \wedge (s \vee z)$, $s \vee y = s \vee z$, and $N_5(s, y < z)$ (see Figure 8), a contradiction with Case 2.

Case 4B. $u \in A - B$. By (Ord), choose an $s \in S$ with $u < s < y$. By Lemma 18, $N_5(x, s < (x \vee s) \wedge z)$ (see Figure 4). Now if $(x \vee s) \wedge z \in A$, then this contradicts Case 1 ($x, s(x \vee s) \wedge z \in A$); otherwise, $(x \vee s) \wedge z \in B - A$, covered by Case 3^{D}. Or $s = (x \vee s) \wedge z$ and $N_5(x \vee s, y < z)$ (see Figure 5). If $x \vee s \in S$, then we get to Case 2; otherwise, $x \vee s \in A - B$, and $(x \vee s) \wedge z \in S$, contradicting Case 4A.

Case 4C. $u \in B - A$. We show that this case cannot occur by induction on $\operatorname{rank}(x, y) = n$. Let the $\{x, y\}$-sequence $p_1, \dots, p_n (\in S)$ establish the rank. By Lemma 19, either

(i) $N_5(z, s < x \wedge (s \vee z))$ (see Figure 6); if $x \wedge (s \vee z) \in B$, then we contradict Case 1 ($z, s < x \wedge (s \vee z) \in B$); otherwise, $x \wedge (s \vee z) \in A - B$, and we contradict Case 3^{D};

or

(ii) $s = x \wedge (s \vee z)$ and $N_5(x, s \vee y < s \vee z)$ (see Figure 7); if $s \vee y$ or $s \vee z \in A$, then $N_5(x, s \vee y < s \vee z)$ was covered in one of the previous cases (Cases 1 to 3^{D}); otherwise, $s \vee y,\ s \vee z \in B - A$, and $N_5(x, s \vee y < s \vee z)$ satisfies the conditions of Case 4C; obviously, the sequence $p_2, \dots, p_n$ establishes that $\operatorname{rank}(x, s \vee y) \leq n - 1$, completing the discussion of the case;

or

(iii) $s = x \wedge (s \vee z)$, $s \vee y = s \vee z$, and $N_5(s, y < z)$ (see Figure 8), which contradicts Case 2.

This completes the proof of the theorem.

5. Distributive lattices

In this section we prove

THEOREM 20. *The variety* **D** *of all distributive lattices is closed under pasting.*

PROOF. Let $L = \mathrm{Paste}(A, B, S)$, where A and B are distributive lattices. By Theorem 16, L is modular.

If L is not distributive, then it contains a sublattice M_3 (see Figure 9). Since A and B are distributive, we cannot have x, y, and $z \in A$ or B; without loss of generality we can assume that $x, y \in A$ and $z \in B - A$.

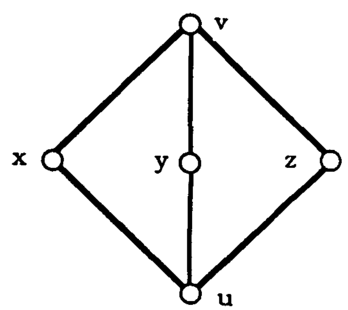

FIGURE 9

CLAIM A. *$u \in S$ and there is no $s \in A$ with $u < s < z$.*

PROOF. Since $u = x \wedge y \in A$, if $u \notin S$, then $u \in A - B$. By condition (Ord) of Theorem 6, there is an $s \in S$ with $u < s < z$. To verify Claim A, it is sufficient to show that $u < s < z$ and $s \in A$ lead to a contradiction. Indeed, since x, y, and $s \in A$ and A is distributive, we obtain that

$$s = s \wedge (x \vee y) = (s \wedge x) \vee (s \wedge y) = u \vee u = u,$$

a contradiction.

CLAIM B. $(z] - (y] \subseteq B - A$.

PROOF. Let $p \in A$ satisfy $p \leq z$ but not $p \leq y$. Then $u < p \vee u \leq z$ and $p \vee u \in A$, contradicting Claim A.

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CLAIM C. *$(y] - (z]$ contains an element of B.*

PROOF. If $(y]-(z] \subseteq A-B$, then by Claim B, $(y]$ and $(z]$ violate condition (Id) of Theorem 6.

CLAIM D. *There is an $s \in S$ with $u < s < y$.*

PROOF. By Claim C, we can take a $p \in (y], p \notin (z]$, satisfying $p \in B$. Then $u < u \vee p \leq y$ and $u \vee p \in B$. By condition (Ord) of Theorem 6, there is an $s \in S$ with $u \vee p \leq s \leq y$.

CLAIM E. *L contains an $M_3' = \{u_1, x_1, y_1, z_1, v_1\}$ with $x_1 \in A - B$, $y_1 \in S$, and $z_1 \in B - A$.*

PROOF. Let s be chosen as in Claim D. Let us define $x_1 = (x \vee z) \wedge x$, $y_1 = s$, $z_1 = (s \vee x) \wedge z$, $u_1 = x_1 \wedge y_1$ and $v_1 = x_1 \vee y_1$. It is well known that these elements form a sublattice isomorphic to M_3.

Since $z_1 \in [u, z]$, $z_1 \in B - A$ by Claim B. Also $y_1 \in S$ by the choice of s. Finally, $x_1 \in A-B$ because otherwise $M_3' \subseteq B$, contradicting the distributivity of B.

Now we have the contradiction that proves the theorem. Indeed, by Claim B applied to M_3', we have $(z_1] - (y_1] \subseteq B - A$. By interchanging A and B, Claim B applied to M_3' yields $(x_1] - (z_1] \subseteq A - B$. Hence the ideals $I = (x_1]$ and $J = (z_1]$ contradict condition (Id) of Theorem 6.

6. Concluding remarks

In the proof of Theorems 5 and 6, we only used that g_A and g_B were homomorphisms, except for the last step when we verified that h was an embedding. This proves that the following is equivalent to Definition 1.

DEFINITION 1′. Let L be a lattice. Let A, B, S be sublattices of L, $A \cap B = S$, $A \cup B = L$. Let f_A and f_B be the embeddings of A and B, respectively, into L. Then L *pastes* A and B together over S, in notation, $L = \text{Paste}(A, B, S)$, if whenever g_A and g_B are homomorphisms of A and B into a lattice K satisfying $xg_A = xg_B$ for all $x \in S$, then there is a homomorphism h of L into K satisfying $f_A h = g_A$ and $f_B h = g_B$ (see Figure 1).

We can put this in another way: let φ be a homomorphism of L onto L'; let L paste A and B together over S, and let A', B', and S' be the images of A, B, S under φ. Then L' pastes A' and B' together over S'.

Finally, we would like to mention an open problem:

PROBLEM. *Which varieties* **V** *of lattices are closed under pasting?*

In this paper we have proved that **M** and **D** are such varieties. In [5], we have discussed continuumly many varieties of modular lattices (which are known to be closed under gluing) that are closed under the pasting of finite lattices. We do not know whether these same varieties are closed under pasting in general. More generally, it would be interesting to find out whether modular varieties of lattices that are closed under gluing are also closed under pasting.

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Department of Mathematics
University of Manitoba
Winnipeg, Manitoba, R3T 2N2
Canada