ON COVERING OF BALANCED INCOMPLETE BLOCK DESIGNS

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1. Introduction. Given a set E of v elements and given positive integers k ($k \leqslant v$) and λ, we understand by balanced incomplete block design (BIBD) $B[k, \lambda, v]$ a system of blocks (subsets of E) having k elements each such that every pair of elements of E is contained in exactly λ blocks.

A necessary condition for the existence of a design $B[k, \lambda, v]$ is known to be **(4)**

$$(1) \quad \lambda(v-1) \equiv 0 \pmod{k-1} \qquad \text{and} \qquad \lambda v(v-1) \equiv 0 \pmod{k(k-1)}.$$

For $k = 3$ and 4 and every λ and for $k = 5$ and $\lambda = 1, 4$, and 20 Condition (1) is also sufficient **(4)**. On the other hand, (1) is known not to be sufficient, for example, for $k = 5$, $\lambda = 2$ **(5)** and for $k = 6$ and 7, $\lambda = 1$ **(6)**.

We say that a set $F \subset E$ covers a given BIBD $B[k, \lambda, v]$ if the intersection of F with every block of $B[k, \lambda, v]$ is non-empty. A set F covering a given BIBD $B[k, \lambda, v]$ will be denoted by $F(B[k, \lambda, v])$ or briefly by $F(B)$.

H. J. Ryser raised the following problem. Given integers v, k, and λ for which some $B[k, \lambda, v]$ exists, find the greatest number $f(k, \lambda, v)$ such that for any BIBD $B[k, \lambda, v]$ every $F(B)$ has at least $f(k, \lambda, v)$ elements. A set $F(B[k, \lambda, v])$ having $f(k, \lambda, v)$ elements will be called minimal. For $k = 3$ Ryser's problem is solved here completely (see Theorem 7). In the general case partial results are obtained.

2. B-systems and T-systems. Most of the definitions and some of the propositions in this section are taken from **(4)**.

Definition 1. Given a set E of v elements, let $K = \{k_1, \ldots, k_n\}$ be a finite set of integers $3 \leqslant k_i \leqslant v$ ($i = 1, 2, \ldots, n$), and λ a positive integer. If it is possible to form a system of blocks in such a way that

(i) the number of elements in each block is some $k_i \in K$ and

(ii) every pair of elements of E is contained in exactly λ blocks,

then we shall denote such a system by $B[K, \lambda, v]$.

The class of all integers v for which systems $B[K, \lambda, v]$ exist will be denoted by $B(K, \lambda)$.

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If $K = \{k\}$ consists of one integer k only, we shall write $B[k, \lambda, v]$ and $B(k, \lambda)$ instead of $B[\{k\}, \lambda, v]$ and $B(\{k\}, \lambda)$ respectively.

The systems $B[k, \lambda, v]$ are the BIBD's introduced in Section 1.

Definition 2. Given a class of m mutually disjoint sets τ_i $(i = 0, 1, \ldots, m - 1)$ having t elements each, if it is possible to form a system of t^2 m-tuples in such a way that

(i) each m-tuple has exactly one element in common with each of the sets $\tau_i (i = 0, 1, \ldots, m - 1)$ and

(ii) every two m-tuples have at most one element in common,

then we denote this system of m-tuples by $T[m, t]$.

The class of numbers t for which systems $T[m, t]$ exist will be denoted by $T(m)$.

Definition 3. If a system $T[m, t]$ exists, and if, moreover, there are in the system at least e subsystems $(0 \leqslant e \leqslant t)$ each consisting of t mutually disjoint m-tuples, then we denote such a system by $T_e[m, t]$.

The class of all numbers t for which systems $T_e[m, t]$ exist will be denoted by $T_e(m)$.

The following propositions are proved in **(4)**.

Proposition 1. *If λ' divides λ and if $v \in B(k, \lambda')$, then also $v \in B(k, \lambda)$ and $f(k, \lambda, v) \leqslant f(k, \lambda', v)$.*

Proposition 2. *If there exists a finite projective plane of order p, then $p \in T_p(p)$ and $p \in T(p + 1)$. Moreover, any two $(p + 1)$-tuples of $T[p + 1, p]$ have non-empty intersection.*

Remark 1. The condition of Proposition 2 is satisfied if p is a power of a prime **(2**, pp. 324–328**)**.

Definition 4. Let m be a positive integer. We say that $t \gg m$, if $t = 1$ or if

$$t = \prod_{j=1}^{n} p_j^{\alpha_j},$$

where the p_j are primes and the α_j positive integers satisfying $p_j^{\alpha_j} \geqslant m$ $(j = 1, 2, \ldots, n)$.

Further, we say that $t \succ m$, if $t \gg m$ or if $t \in B(K, \lambda)$ and

$$k_i = \prod_{j=1}^{n_i} p_{i,j}^{\alpha_{i,j}} \qquad \text{for every } k_i \in K,$$

where the $p_{i,j}$ are primes and the $\alpha_{i,j}$ positive integers satisfying

$$p_{i,j}^{\alpha_{i,j}} \geqslant m \qquad (j = 1, 2, \ldots, n_i).$$

Bose and Shrikhande **(1)** proved the following propositions:

Proposition 3. *If $t \gg m$, then $t \in T_t(m)$ and $t \in T(m + 1)$.*

PROPOSITION 4. *If* $t > m$, *then* $t \in T(m)$.

3. Covering sets.

THEOREM 1. *Let* $v \in B(k, \lambda)$. *Then* $f(k, \lambda, v) \geqslant (v-1)/(k-1)$.

Proof. Suppose $f(k, \lambda, v) < (v-1)/(k-1)$. Then there exist a BIBD $B[k, \lambda, v]$ and a set $F(B)$ having fewer than $(v-1)/(k-1)$ elements. Let $a \notin F$. Consider all the blocks of $B[k, \lambda, v]$ which contain a. Their number is $\lambda(v-1)/(k-1)$. On the other hand, every element of $F(B)$ covers at most λ such blocks and consequently some of the blocks are not covered.

For $k = 3$, $\lambda = 1$, this theorem has been proved by Fulkerson and Ryser **(3)**.

In the special case when the equality $f(k, \lambda, v) = (v-1)/(k-1)$ holds, $(v-1)/(k-1)$ must be an integer and we shall use $k = q+1$, $v = qu+1$, i.e. we shall write $f(q+1, \lambda, qu+1) = u$.

THEOREM 2. *Let* $qu + 1 \in B(q+1, \lambda)$. *A necessary condition for* $f(q+1, \lambda, qu+1) = u$ *is the existence of a* BIBD $B[q+1, \lambda, u]$.

Proof. Let E be a set having $qu+1$ elements, on which a BIBD $B = B[q+1, \lambda, qu+1]$ is constructed and let $F = F(B)$. Further, let $a \in E - F$ and $G = E - (F \cup \{a\})$. By Theorem 1, $|F| \geqslant u$; here $|F|$ denotes the number of elements of F. If $|F| = u$, then every block containing a evidently contains exactly one element of F and therefore $q-1$ elements of G. Denote by $A \subset B$ the subsystem of blocks of B which contain a. Clearly $|A| = \lambda u$. The number of "mixed" pairs of elements of E, having one element in F and one in G, which appear in the blocks of B is $\lambda(q-1)u^2$, and which appear in the blocks of A is $\lambda(q-1)u$. In $B - A$ we have, accordingly, $\lambda(q-1)u(u-1)$ such "mixed" pairs. On the other hand, every block of $B - A$ has at least one element in F. Each such block which has also at least one element in G has at least q "mixed" pairs. There are, therefore, at most $\lambda(q-1)u(u-1)/q$ such blocks. The total number of blocks in $B - A$ is

$$|B| - |A| = \frac{\lambda(qu+1)qu}{(q+1)q} - \lambda u = \frac{\lambda qu(u-1)}{q+1}$$

and accordingly the number of blocks of $B - A$ included in F is at least

$$\frac{\lambda qu(u-1)}{q+1} - \frac{\lambda(q-1)u(u-1)}{q} = \frac{\lambda u(u-1)}{(q+1)q}.$$

This is, however, the exact number of blocks in $B[q+1, \lambda, u]$ and it can be attained only if such a BIBD exists.

THEOREM 3. *If there exists a projective plane of order* p, *then a necessary and sufficient condition for* $f(p+1, \lambda, pu+1) = u$ *is the existence of a* BIBD $B[p+1, \lambda, u]$.

Remark 2. The condition of Theorem 3 is satisfied if p is a power of a prime (see Remark 1).

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Proof. The necessity follows from Theorem 2. It remains to prove sufficiency. Let E be a set having $pu + 1$ elements which we denote by (i, j) $(i = 0, 1, \ldots, p - 1; j = 0, 1, \ldots, u - 1)$ and (a). We construct a BIBD

$$B[p + 1, \lambda, pu + 1] \tag{2}$$

as follows: take as blocks the sets $\{(i, j), (a): i = 0, 1, \ldots, p - 1\}$, $(j = 0, 1, \ldots, u - 1)$, λ times each. Further construct a BIBD $B[p + 1, \lambda, u]$ on the set of integers $0, \ldots, u - 1$ and for each block $\beta \in B\ [p + 1, \lambda, u]$ construct a system $T[p + 1, p]$ on the sets $\tau_j = \{(i, j): i = 0, 1, \ldots, p - 1\}$, $(j \in \beta)$ in such a way that $\{(0, j): j \in \beta\}$ is one of the $(p + 1)$-tuples in this system. The systems $T[p + 1, p]$ complete the construction of (2). By Proposition 2, the set $F = \{(0, j): j = 0, 1, \ldots, u - 1\}$ covers the BIBD (2).

Theorem 2 and the necessity of (1) for the existence of BIBD's imply the following corollary.

COROLLARY 1. *Let* $qu + 1 \in B(q + 1, \lambda)$. *A necessary condition for* $f(q + 1, \lambda, qu + 1) = u$ *is* $\lambda(u - 1) \equiv 0 \pmod q$ *and* $\lambda u(u - 1) \equiv 0 \pmod{(q + 1)q}$.

In those instances in which Condition (1) is also sufficient for the existence of the respective BIBD's **(4)**, Theorem 3 and Remark 2 imply the following corollary.

COROLLARY 2. *If* $p = 2$ *or* 3 *or if* $p = 4$ *and* $\lambda = 1, 4,$ *or* 20, *then a necessary and sufficient condition for* $f(p + 1, \lambda, pu + 1) = u$ *is* $\lambda(u - 1) \equiv 0 \pmod p$ *and* $\lambda u(u - 1) \equiv 0 \pmod{(p + 1)p}$, *and more specifically:*

$$\begin{array}{lll}
\textit{for}\ \ p = 2, & \lambda = 1 & u \equiv 1 \textit{ or } 3 \pmod 6, \\
 & \lambda = 2 & u \equiv 0 \textit{ or } 1 \pmod 3, \\
 & \lambda = 3 & u \equiv 1 \pmod 2, \\
 & \lambda = 6 & \textit{any } u, \\
\quad\ \ p = 3, & \lambda = 1 & u \equiv 1 \textit{ or } 4 \pmod{12}, \\
 & \lambda = 2 & u \equiv 1 \pmod 3, \\
 & \lambda = 3 & u \equiv 0 \textit{ or } 1 \pmod 4, \\
 & \lambda = 6 & \textit{any } u, \\
\quad\ \ p = 4, & \lambda = 1 & u \equiv 1 \textit{ or } 5 \pmod{20}, \\
 & \lambda = 4 & u \equiv 0 \textit{ or } 1 \pmod 5, \\
 & \lambda = 20 & \textit{any } u.
\end{array}$$

DEFINITION 5. Let E be a set with v elements; q and λ are positive integers; $v \in B(q + 1, \lambda)$. Let $F(B)$ be a minimal set covering a given BIBD $B[q + 1, \lambda, v]$ and having $f(q + 1, \lambda, v) = \eta$ elements. Further, let S be a subset of E having s elements and let $|S \cap F| = \sigma$. Suppose that it is possible to construct a system of blocks (subsets of E) in such a way that

(i) the number of elements in each block is $q + 1$,

(ii) every (unordered) pair of elements such that one of them is in E and the other in $E - S$ is contained in exactly λ blocks, and

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(iii) pairs of elements of S do not appear at all.

Then we shall denote such a system by $B[q + 1, \lambda, v|s]$.

If, moreover, the set F covers this system $B[q + 1, \lambda, v|s]$, then we shall write $f(q + 1, \lambda, v|s) = \eta|\sigma$.

PROPOSITION 5. *Let E be a set of v elements and S be a subset of s elements. Further, let $v \in B(q + 1, \lambda)$, $s \in B(q + 1, \lambda)$ and let $F(B[q + 1, \lambda, v])$ be a minimal set covering $B[q + 1, \lambda, v]$ and having $f(q + 1, \lambda, v) = \eta$ elements, σ of them in S. If $B[q + 1, \lambda, s] \subset B[q + 1, \lambda, v]$, then $B[q + 1, \lambda, v|s]$ exists and $f(q + 1, \lambda, v|s) = \eta|\sigma$.*

Proof. Construct BIBD $B[q + 1, \lambda, v]$ on E in such a way that its subdesign $B[q + 1, \lambda, s]$ should be on S and omit all the blocks of this subdesign.

Let p be a power of a prime and let $t \gg p$ or $t > p + 1$. Further, let u, s, and σ be non-negative integers, $\sigma \leqslant s$. Then the following theorems will be proved.

THEOREM 4. *If $B[p + 1, \lambda, pt + s|s]$ exists, $f(p + 1, \lambda, pt + s|s) = t + \sigma|\sigma$ and $u \in B(p + 1, \lambda)$, then $ptu + s \in B(p + 1, \lambda)$ and $f(p + 1, \lambda, ptu + s) \leqslant tu + \sigma$.*

THEOREM 5. *If $B[p + 1, \lambda, p^2t + s|s]$ exists, $f(p + 1, \lambda, p^2t + s|s) = pt + \sigma|\sigma$ and $u + 1 \in B(p + 1, 1)$, then $ptu + s \in B(p + 1, \lambda)$ and $f(p + 1, \lambda, ptu + s) \leqslant tu + \sigma$.*

THEOREM 6. *If $B[p + 1, \lambda, p(p + 1)t + s|s]$ exists,*

$$f(p + 1, \lambda, p(p + 1)t + s|s) = (p + 1)t + \sigma|\sigma,$$

and if, further, $u \in B(p + 1, 1)$, $u \equiv 0 \pmod{p + 1}$ and the BIBD *$B[p + 1, 1, u]$ contains a family A of $u/(p + 1)$ mutually disjoint blocks, then $ptu + s \in B(p + 1, \lambda)$ and $f(p + 1, \lambda, ptu + s) \leqslant tu + \sigma$.*

Remark 3. With respect to the conditions on u in Theorem 6 in the case $p = 2$ we note that if $u \equiv 3 \pmod 6$, then the other conditions are automatically satisfied.

Proof of Remark 3. We have to show that there exists a BIBD $B[3, 1, u]$ containing $u/3$ mutually disjoint blocks.

Let E be a set having $u = 3(2w + 1)$ elements, which we denote by (i, j) $(i = 0, 1, 2; j = 0, 1, \ldots, 2w)$. The BIBD $B[3, 1, u]$ is formed by the blocks $\{(0, j), (1, j), (2, j)\}$ $(j = 0, 1, \ldots, 2w)$ which are mutually disjoint and by the blocks

$$\{(i + 1, j), (i, j + \alpha), (i, j - \alpha)\} \qquad (i = 0, 1, 2; j = 0, 1, \ldots, 2w; \alpha = 1, 2, \ldots, w).$$

All the numbers appearing in parentheses should be taken modulo the number of values which the variable in the given place may take. In the given case the number in the first place is taken (mod 3) and in the second place (mod $2w + 1$).

Proof of Theorem 4. Let E be a set having $ptu + s$ elements which will be denoted by

$$(i, j, h) \qquad (i = 0, 1, \ldots, p - 1; j = 0, 1, \ldots, t - 1; h = 0, 1, \ldots, u - 1)$$

and (A, r) $(r = 0, 1, \ldots, s - 1)$ and let S be a subset of E composed of the elements (A, r) $(r = 0, 1, \ldots, s - 1)$. On the set E we construct a BIBD

$$B[p + 1, \lambda, ptu + s] \tag{3}$$

as follows:

(i) On the set $E_0 = S \cup \{(i, j, 0): i = 0, 1, \ldots, p - 1; j = 0, 1, \ldots, t - 1\}$ construct a BIBD $B[p + 1, \lambda, pt + s]$ in such a way that it is covered by the set $F_0 = S' \cup \{(0, j, 0): j = 0, 1, \ldots, t - 1\}$, where S' is a subset of S having σ elements.

(ii) On each of the sets

$$E_h = S \cup \{(i, j, h): i = 0, 1, \ldots, p - 1; \; j = 0, 1, \ldots, t - 1\} \qquad (h = 1, 2, \ldots, u - 1)$$

construct a system $B[p + 1, \lambda, pt + s|s]$ in such a way that it is covered by the set

$$F_h = S' \cup \{(0, j, h): j = 0, 1, \ldots, t - 1\} \qquad (h = 1, 2, \ldots, u - 1).$$

(iii) Construct BIBD $B[p + 1, \lambda, u]$ on the set of the integers $\{h\} = \{0, \ldots, u - 1\}$. For every block $\beta \in B[p + 1, \lambda, u]$ construct a system $T^*[p + 1, p]$ on the sets $\tau_h^* = \{(i, h): i = 0, 1, \ldots, p - 1\}$ $(h \in \beta)$ in such a way that the set $\{(0, h): h \in \beta\}$ should be one of the $(p + 1)$-tuples of the system. Further, for every $(p + 1)$-tuple $\gamma \in T^*[p + 1, p]$ construct a system $T[p + 1, t]$ on the sets $\tau_h = \{(i, j, h): j = 0, 1, \ldots, t - 1\}$ $((i, h) \in \gamma)$.

The BIBD (3) is obtained by taking the blocks of the design formed in (i) and of the systems formed in (ii) and the $(p + 1)$-tuples of the systems $T[p + 1, t]$ formed in (iii).

Clearly, the set $F = S' \cup \{(0, j, h): \; j = 0, 1, \ldots, t - 1; \; h = 0, 1, \ldots, u - 1\}$ covers the BIBD (3).

Proof of Theorem 5. Let E, S, and S' be the sets defined in the proof of Theorem 4.

Construct a BIBD $B[p + 1, 1, u + 1]$ on the set $\{h, a: h = 0, 1, \ldots, u - 1\}$. For every block $\beta^* \in B[p + 1, 1, u + 1]$ which contains the element a, denote $\beta' = \beta^* - \{a\}$. Let β_0' be one of the sets β'. On the set

$$S \cup \{(i, j, h): i = 0, 1, \ldots, p - 1; j = 0, 1, \ldots, t - 1; h \in \beta_0'\}$$

construct a BIBD

$$B[p + 1, \lambda, p^2 t + s] \tag{4}$$

in such a way that it is covered by the set $S' \cup \{(0, j, h): j = 0, 1, \ldots, t-1; h \in \beta_0'\}$. For every set $\beta' \neq \beta_0'$ construct in the same way a system

$$B[p+1, \lambda, p^2t + s|s] \tag{5}$$

on the set $S \cup \{(i, j, h): i = 0, 1, \ldots, p-1; j = 0, 1, \ldots, t-1; h \in \beta'\}$.

For every block $\beta \in B[p+1, 1, u+1]$ which does not contain the element a proceed as in (iii) of the proof of Theorem 4 taking $\lambda = 1$.

A BIBD $B[p+1, \lambda, ptu + s]$ consists of the blocks of BIBD (4) and systems (5) and of the $(p+1)$-tuples of the systems $T[p+1, t]$ formed in (iii) of the proof of Theorem 4, taken λ times each. This BIBD is covered by the set $F = S' \cup \{(0, j, h): j = 0, 1, \ldots, t-1; h = 0, 1, \ldots, u-1\}$.

Proof of Theorem 6. Let E, S, and S' be the sets defined in the proof of Theorem 4.

Construct a BIBD $B[p+1, 1, u]$ satisfying the conditions of the theorem. Let $\beta_0' \in A$ be a block of this BIBD. On the set $S \cup \{(i, j, h): i = 0, 1, \ldots, p-1; j = 0, 1, \ldots, t-1; h \in \beta_0'\}$ construct a BIBD

$$B[p+1, \lambda, p(p+1)t + s] \tag{6}$$

covered by the set $S' \cup \{(0, j, h): j = 0, 1, \ldots, t-1; h \in \beta_0'\}$. For every other block $\beta' \in A$, $\beta' \neq \beta_0'$ construct on the set

$$S \cup \{(i, j, h): i = 0, 1, \ldots, p-1; j = 0, 1, \ldots, t-1; h \in \beta'\},$$

a system

$$B[p+1, \lambda, p[p+1)t + s|s] \tag{7}$$

covered by the set $S' \cup \{(0, j, h): j = 0, 1, \ldots, t-1; h \in \beta'\}$.

For every block $\beta \in B[p+1, 1, u]$, $\beta \notin A$, proceed as in (iii) of the proof of Theorem 4, taking $\lambda = 1$.

A BIBD $B[p+1, \lambda, ptu + s]$ consists of the blocks of BIBD (6) and systems (7) and of the $(p+1)$-tuples of the systems $T[p+1, t]$ formed in (iii) of the proof of Theorem 4, taken λ times each. This design is covered by the set $F = S' \cup \{(0, j, h): j = 0, 1, \ldots, t-1; h = 0, 1, \ldots, u-1\}$.

4. The case p = 2. In this case, Condition (1) is known to be necessary and sufficient for the existence of corresponding BIBD's **(4)**. It follows that BIBD $B[3, \lambda, v]$ exist

for $\lambda \equiv 1$, or 5 (mod 6) if and only if $v \equiv 1$ or 3 (mod 6),
" $\lambda \equiv 2$, or 4 (mod 6) " " " " $v \equiv 0$ or 1 (mod 3),
" $\lambda \equiv 3$ (mod 6) " " " " $v \equiv 1$ (mod 2),
" $\lambda \equiv 0$ (mod 6) and for every $v > 3$.

Regarding minimal covering sets of such designs, we prove the following theorem.

THEOREM 7. *For every* λ, *if* $v \in B(3, \lambda)$, *then*

$$f(3, \lambda, v) = \begin{cases} \frac{1}{2}(v-1), & \text{if } v \equiv 3 \pmod 4, \text{ or if } v \equiv 1 \pmod 4 \\ & \qquad \text{and } \lambda \equiv 0 \pmod 2, \\ \frac{1}{2}(v+1), & \text{if } v \equiv 1 \pmod 4 \text{ and } \lambda \equiv 1 \pmod 2, \\ \frac{1}{2}v, & \text{if } v \equiv 0 \pmod 2. \end{cases}$$

Proof. It follows from Corollary 2 and Proposition 1 that $f(3, \lambda, v) = \frac{1}{2}(v-1)$ if and only if $v \equiv 3 \pmod 4$ or $v \equiv 1 \pmod 4$ and $\lambda \equiv 0 \pmod 2$. By Theorem 1, it remains to be proved that $f(3, \lambda, v) \leqslant \frac{1}{2}(v+1)$ and $f(3, \lambda, v) \leqslant \frac{1}{2}v$ for $v \equiv 1 \pmod 4$ and $v \equiv 0 \pmod 2$, respectively.

Let $v \equiv 1 \pmod 4$. For $v \in B(3, 1)$, i.e. for $v \equiv 1$ or $9 \pmod{12}$, we prove in (13)–(15) that $f(3, 1, v) \leqslant \frac{1}{2}(v+1)$. For $v \equiv 5 \pmod{12}$, i.e. $v \in B(3, 3)$, we prove in (16)–(17) that $f(3, 3, v) \leqslant \frac{1}{2}(v+1)$. On account of Proposition 1, the case $v \equiv 1 \pmod 4$ will be herewith exhausted.

We begin by proving some formulae for particular values of v.

$$(8) \qquad f(3, 3, 5) = 3.$$

Denote the elements of a set E by (i) $(i = 0, 1, 2, 3, 4)$ and form BIBD $B[3, 3, 5]$ from the blocks

$$\{(i), (i+\alpha), (i+2\alpha)\} \qquad (i = 0, 1, 2, 3, 4; \alpha = 1, 2).$$

The set $F = \{(0), (1), (2)\}$ covers this design.

$$(9) \qquad f(3, 1, 9) = 5.$$

Elements: (i, j) $\quad (i = 0, 1, 2; j = 0, 1, 2)$.
Blocks: $\{(0, j), (1, j), (2, j)\}$; $\{(i, j+1), (i, j+2), (i+1, j)\}$.
Covering set $F = \{(0, 0), (0, 1), (0, 2), (1, 0), (2, 0)\}$.

$$(10) \qquad f(3, 1, 13) = 7.$$

Elements: (i) $\quad (i = 0, 1, \ldots, 12)$.
Blocks: $\{(i), (i+1), (i+4)\}$; $\{(i), (i+2), (i+7)\}$.
Covering set $F = \{(0), (2), (4), (8), (9), (10), (12)\}$.

$$(11) \qquad B[3, 1, 21|9] \text{ exists and } f(3, 1, 21|9) = 11|5.$$

Put in Theorem 4 $s = 3$, $\sigma = 2$, $t = 3$, $\lambda = 1$, $u = 3$ and apply (9). The construction of blocks in Theorem 4 shows that $B[3, 1, 9] \subset B[3, 1, 21]$ and, by Proposition 5, (11) is proved.

$$(12) \qquad B[3, 3, 17|5] \text{ exists and } f(3, 3, 17|5) = 9|3.$$

By Proposition 5 it suffices to construct BIBD's $B[3, 3, 5]$ and $B[3, 3, 17]$ such that $B[3, 3, 5] \subset B(3, 3, 17]$ and that $F(B[3, 3, 17])$ has nine lements, three of which cover the BIBD $B[3, 3, 5]$. To construct such a $B[3, 3, 17]$, take the elements and the blocks of (8); further, take the elements (i, j, h), $(i = 0, 1;$ $j = 0, 1; h = 0, 1, 2)$ and form the blocks:

$\{(\gamma), (i, 0, h), (i, 1, h+\gamma)\}$ $(\gamma = 0, 1, 2)$ 3 times;
$\{(3+\epsilon), (0, j, h), (1, j+\epsilon, h)\}$ $(\epsilon = 0, 1)$ 3 times;
$\{(i+1, j+\epsilon, h), (i, j, h+1), (i, j, h+2)\}$ $(\epsilon = 0, 1)$;
$\{(i, j+i, h), (i+1, j, h+1), (i+1, j, h+2)\}$.
Covering set $F = \{(0, j, h), (\gamma) : j = 0, 1; h = 0, 1, 2; \gamma = 0, 1, 2\}$.

(13) *If* $v \equiv 1$ *or* $9 \pmod{24}$, *then* $f(3, 1, v) \leqslant \frac{1}{2}(v+1)$.

Put, in Theorem 5, $s = \sigma = 1$, $t = 2$, $\lambda = 1$, $u \equiv 0$ or $2 \pmod 6$ and apply (9).

(14) *If* $v \equiv 13 \pmod{24}$, *then* $f(3, 1, v) \leqslant \frac{1}{2}(v+1)$.

Put, in Theorem 6, $s = \sigma = 1$, $t = 2$, $\lambda = 1$, $u \equiv 3 \pmod 6$ and apply (10).

(15) *If* $v \equiv 21 \pmod{24}$, *then* $f(3, 1, v) \leqslant \frac{1}{2}(v+1)$.

Put, in Theorem 6, $s = 9$, $\sigma = 5$, $t = 2$, $\lambda = 1$, $u \equiv 3 \pmod 6$ and apply (11).

(16) *If* $v \equiv 5 \pmod{24}$, *then* $f(3, 3, v) \leqslant \frac{1}{2}(v+1)$.

Put, in Theorem 4, $s = \sigma = 1$, $t = 2$, $\lambda = 3$, $u \equiv 1 \pmod 6$ and apply (8).

(17) *If* $v \equiv 17 \pmod{24}$, *then* $f(3, 3, v) \leqslant \frac{1}{2}(v+1)$.

Put, in Theorem 6, $s = 5$, $\sigma = 3$, $t = 2$, $\lambda = 3$, $u \equiv 3 \pmod 6$ and apply (12).

Let $v \equiv 0 \pmod 2$. For $v \in B(3, 2)$, that is, for $v \equiv 0$ or $4 \pmod 6$ we prove in (24)–(26) that $f(3, 2, v) \leqslant \frac{1}{2}v$ and for $v \equiv 2 \pmod 6$ we prove in (27) that $f(3, 6, v) \leqslant \frac{1}{2}v$. By Proposition 1, this proves our theorem. As in the previous case, we begin by considering some special values of v.

(18) $$f(3, 2, 4) = 2.$$

Elements: (i) $(i = 0, 1, 2, 3)$.
Blocks: $\{(i), (i+1), (i+2)\}$.
Covering set $F = \{(0), (1)\}$.

(19) $$f(3, 2, 6) = 3.$$

Elements: (i, j) $(i = 0, 1; j = 0, 1, 2)$.
Blocks: $\{(0, 0), (0, 1), (0, 2)\}$;
$\{(0, j), (1, j), (1, j+\eta)\}$, $(\eta = \pm 1)$; $\{(1, j), (0, j-1), (0, j+1)\}$.
Covering set $F = \{(0, j) : j = 0, 1, 2\}$.

(20) $$f(3, 6, 8) = 4.$$

Elements: (i, j) $(i = 0, 1; j = 0, 1, 2, 3)$.
Blocks: $\{(0, j), (0, j+1), (0, j+2)\}$ twice;
$\{(i+1, j), (i, j-1), (i, j+1)\}$;
$\{(i+1, j), (i, j), (i, j+\eta)\}$ $(\eta = \pm 1)$;
$\{(0, j), (1, j), (1, j+2)\}$ twice;
$\{(0, j), (1, j+2), (1, j+\eta)\}$ $(\eta = \pm 1)$ twice.
Covering set $F = \{(0, j) : j = 0, 1, 2, 3\}$.

(21) $$f(3, 6, 14) = 7$$

Elements: (i, j) $(i = 0, 1; j = 0, 1, \ldots, 6)$.
Blocks: $\{(0, j), (0, j + 1), (0, j + 3)\}$ 5 times;
$\{(0, j), (1, j - \gamma), (1, j + \gamma)\}$ $(\gamma = 1, 2, 3)$ 4 times;
$\{(1, j), (0, j - \gamma), (0, j + \gamma)\}$ $(\gamma = 1, 2, 3)$;
$\{(0, j), (1, j), (1, j + \delta)\}$ $(\delta = 1, 2, \ldots, 6)$.
Covering set $F = \{(0, j): j = 0, 1, \ldots, 6\}$.

(22) $B[3, 6, 8|2]$ *exists and* $f(3, 6, 8|2) = 4|1$; cf. (20).

Elements: (h) $(h = 0, 1)$ and (i, j) $(i = 0, 1; j = 0, 1, 2)$.
Blocks: $\{(0), (i, j), (i, j + 1)\}$ 3 times;
$\{(1), (0, j), (1, j + \gamma)\}$ $(\gamma = 0, 1, 2)$ twice;
$\{(i + 1, j), (i, j - 1), (i, j + 1)\}$;
$\{(i + 1, j), (i, j), (i, j + \eta)\}$ $(\eta = \pm 1)$.
The pair $\{(0), (1)\}$ does not appear.
Covering set $F = \{(0), (0, j): j = 0, 1, 2\}$.

(23) $B[3, 2, 10|4]$ *exists and* $f(3, 2, 10|4) = 5|2$.

Apply Proposition 5 in the same way as in (12). Take the elements and blocks of (18). Further, take the elements (i, j) $(i = 0, 1;\ j = 0, 1, 2)$ and form the blocks:

$\{(i + 1, 0), (i, 1), (i, 2)\}$; $\{(0), (i, j), (i, j + 1)\}$;
$\{(1), (i, 0), (i, \eta)\}$ $(\eta = \pm 1)$;
$\{(1), (0, \eta), (1, \eta)\}$ $(\eta = \pm 1)$;
$\{(2), (0, 0), (1, 0)\}$ twice;
$\{(2), (i, 1), (i + 1, 2)\}$ twice;
$\{(3), (0, \eta), (1, \eta)\}$ $(\eta = \pm 1)$;
$\{(3), (i, 0), (i + 1, \eta)\}$ $(\eta = \pm 1)$.
Covering set $F = \{(0), (1), (0, j): j = 0, 1, 2\}$.

(24) *If* $v \equiv 0$ *or* 4 (mod 12), *then* $f(3, 2, v) \leqslant \frac{1}{2}v$.

Put, in Theorem 4, $s = \sigma = 0$, $t = 2$, $\lambda = 2$, $u \equiv 0$ or 1 (mod 3) and apply (18).

(25) *If* $v \equiv 6$ (mod 12), *then* $f(3, 2, v) \leqslant \frac{1}{2}v$.

Put, in Theorem 6, $s = \sigma = 0$, $t = 1$, $\lambda = 2$, $u \equiv 3$ (mod 6) and apply (19).

(26) *If* $v \equiv 10$ (mod 12), *then* $f(3, 2, v) \leqslant \frac{1}{2}v$.

Put, in Theorem 6, $s = 4$, $\sigma = 2$, $t = 1$, $\lambda = 2$, $u \equiv 3$ (mod 6) and apply (23).

(27) *If* $v \equiv 2$ (mod 6), *then* $f(3, 6, v) \leqslant \frac{1}{2}v$.

For $v = 8$, see (20); for $v = 14$, see (21). For $v > 14$ substitute in Theorem 4 $s = 2$, $\sigma = 1$, $t = 3$, $\lambda = 6$, $u \geqslant 3$ and apply (20) and (22).

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