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SOME NOTES ON THE METHOD OF MOVING PLANES

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In this paper, we obtain a version of the sliding plane method of Gidas, Ni and Nirenberg which applies to domains with no smoothness condition on the boundary. The method obtains results on the symmetry of positive solutions of boundary value problems for nonlinear elliptic equations. We also show how our techniques apply to some problems on half spaces.

In this note, we show how to apply the method of moving planes [6] on domains in R^m which are not at all smooth and for solutions not necessarily continuous up to the boundary. Our methods are a variant of those in Berestycki and Nirenberg [1] but are based on an idea in [4] (which was done independently of [1]). We feel that, even in the regular case, our method is a little simpler than that in [1]. (Note that our argument simplifies greatly if the solution is continuous up to the boundary.) More precisely, we are interested in symmetry and monotonicity properties of positive solutions of

$$(1) \qquad \begin{aligned} -\Delta u &= f(u) \quad \text{in} \quad \Omega \\ u &= 0 \quad \text{on} \quad \partial\Omega \end{aligned}$$

where Ω is a bounded domain in R^m. (In fact, we shall study slightly more general equations.) In [1], it is proved that the original Gidas-Ni-Nirenberg results are valid for solutions which are continuous up to the boundary without any regularity assumptions on Ω. (They also prove much more.) Here we prove similar results for solutions in $\dot{W}^{1,2}(\Omega) \cap L^\infty(\Omega)$. This seems a natural class of solutions because with no smoothness conditions on the boundary, one can frequently establish the existence of solutions in this class. (Indeed, for many problems, one can show by regularity theory that solutions of weaker types belong to this class.) This contrasts with the continuity up to the boundary requirement in [1] which one can usually only justify for domains with some regularity (regular in the sense of Wiener). Note also that it can be shown that the solutions in the sense of [1] are $\dot{W}^{1,2}(\Omega)$ solutions.

Finally, we greatly improve the result in [6] on the half space case. As a consequence we improve a result of Gidas and Spruck [8].

In Section 1, we prove our main result on bounded domains and discuss generalisations and in Section 2 we prove some technical lemmas needed in Section 1. In Section 3, we consider the problem on half spaces.

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1. The Main Result

We assume that $f : R \to R$ is a locally Lipschitz function and Ω is a bounded domain in R^m. If $\lambda \in R$, let $\Omega_\lambda = \{x \in \Omega : x_1 > \lambda\}$ and $T_\lambda = \{x \in R^m : x_1 = \lambda\}$ (where $\underline{x} = (x_1, \dots, x_m)$) and let P_λ be the reflection in the hyperplane $x_1 = \lambda$. Note that the boundary condition for a solution of (1) is given in the weak sense by the condition that $u \in \dot{W}^{1,2}(\Omega)$. Note also that, since $f(u) \in L^\infty(\Omega)$, standard local regularity result implies that u is in $W^{2,p}_{\text{loc}}(\Omega) \cap C^1_{\text{loc}}(\Omega)$.

Theorem 1. *Assume that the above conditions hold, that $u(x) > 0$ in Ω and that $P_\lambda(\Omega_\lambda) \subseteq \Omega$ for $\lambda > \lambda^*$ where T_{λ^*} intersects Ω. Then $u(P_{\lambda^*}x) \geqslant u(x)$ and $(\partial u(x)/\partial x_1) < 0$ if $x \in \Omega_{\lambda^*}$.*

Remark: As in [7], one can deduce many other results from this basic result. Before proving the result, we state three technical lemmas. We defer the proof of two of these until Section 2.

Lemma 1. *Assume that $T \subseteq R^m$ is open, b_i, $i = 1, \dots, n$ and c are measurable, $|b_i| \leqslant K$ almost everywhere on T and $|c(x)| \leqslant K$ almost everywhere on T. Then there is a $\delta > 0$ such that if $W \subseteq T$ is open and bounded, if u is a weak $(W^{1,2})$ supersolution of $-\Delta u + b.\nabla u + cu = 0$ in W, if $u^- \in \dot{W}^{1,2}(W)$, and if $\{x \in W : u(x) < 0\}$ has measure less than δ, then $u \geqslant 0$ in W.*

Remark: This is a variant of a result in [1]. Our proof seems simpler.

Lemma 2. *If $u \in W^{1,2}(\Omega)$, u is continuous on $\overline{\Omega}$ and $u \geqslant 0$ on $\partial\Omega$, then $u^- \in \dot{W}^{1,2}(\Omega)$.*

Remark: This is folklore (see Gilbarg and Trudinger [10]) but we could not locate a proof in the literature.

Lemma 3. *If W is connected and open, if $u \in W^{1,2}(W)$, if $m > 0$, if $u \geqslant m$ on a set of positive measure, and if $u \leqslant 0$ on a set of positive measure, then for each $\alpha > 0$ $\{x \in W : (m - \alpha)/2 \leqslant u(x) \leqslant u(x) \leqslant (m + \alpha)/2\}$ has positive measure.*

Results of this type are well known and reflect the "almost continuity" of u. This result follows immediately from [3, Lemma 4.4].

Proof of Theorem: Let $\widehat{n} = \sup\{x_1 : x \in \Omega\}$ and let $A = \{\lambda \in (\lambda^*, \widehat{n}) : u(P_\lambda x) > u(x)$ if $x \in \Omega_\lambda$ and $(\partial u/\partial x_1)(x) < 0$ if $x \in T_\lambda \cap \Omega\}$. (Note that, since Ω is open, $\widehat{n}$ is never achieved by points of Ω.) Let $\widetilde{A} = \{\lambda \in (\lambda^*, \widehat{n}) : (\lambda, \widehat{n}) \subseteq A\}$. We prove that $\widetilde{A}$ is open and closed in $(\lambda^*, \widehat{n})$ and that $\widetilde{A}$ is non-empty because any λ close to $\widehat{n}$ (but less than $\widehat{n}$) is in A. Hence $\widetilde{A} = (\lambda^*, \widehat{n})$ and the result will then follow easily.

First note that $\lambda \in A$ if $u(P_\lambda x) \geqslant u(x)$ on Ω_λ with equality not holding on any component of Ω_λ. To see this, note that $u(P_\lambda x)$ is a solution of (1) on Ω_λ and hence

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$w_\lambda(x) = u(P_\lambda x) - u(x)$ is a solution of $-\Delta v = f(u(P_\lambda x)) - f(u(x))$ on Ω_λ.Since f is Lipschitz, $-\Delta w_\lambda + c(x)w_\lambda = 0$ on Ω_λ where $|c(x)| \leqslant K_1$. The weak Harnack inequality (as in [**10**]) now ensures that $w_\lambda > 0$ on Ω_λ and (as in [**7**]) the boundary point version of the maximum principle (applied to w_λ)ensures that $(\partial w_\lambda/\partial x_1) > 0$ on T_λ. (Note that, if $x \in T_\lambda$, Ω_λ is smooth up to the boundary nearby. Note also that the proof of the maximum principle in [**13**] easily generalises to solutions in $W^{2,p}$.) Thus $2(\partial u/\partial x_1) < 0$ on T_λ and our claim follows.

As a first step we prove that $\lambda \in A$ if $\lambda < \widehat{n}$ but λ is close to $\widehat{n}$. In this case, $\Omega_\lambda \subseteq B\times(\lambda,\widehat{n})$ where B is a fixed ball in R^{m-1} (since Ω is bounded in R^m). Hence we see that, if λ is close to $\widehat{n}$, Ω_λ has small measure. Thus $\widetilde{W} \equiv \{x \in \Omega_\lambda : u(x) > u(P_\lambda x)\}$ has small measure. Much as before, we find that $-\Delta w_\lambda + c(x)w_\lambda = 0$ on Ω_λ where $|c(x)| \leqslant K_1$. Hence we can apply Lemma 1 if we prove that $w_\lambda^- \in \dot{W}^{1,2}(\Omega_\lambda)$. To see this, we note that, since $u \in \dot{W}^{1,2}(\Omega)$, there exists $u_n \in C_0^\infty(\Omega)$ such that $u_n \to u$ in $W^{1,2}(\Omega)$ as $n \to \infty$. Thus (see [**9**, p.81]) $|u_n| \to |u| = u$ weakly in $W^{1,2}(\Omega)$. Thus $v_n = |u_n|$ converges weakly to u in $W^{1,2}(\Omega)$, $v_n \geqslant 0$ on Ω, v_n is continuous on $\overline{\Omega}$ and v_n vanishes near $\partial\Omega$. Let $w_n(x) = u_n(P_\lambda x) - u_n(x)$. Then $w_n \to w_\lambda$ weakly on $W^{1,2}(\Omega_\lambda)$, w_n is continuous on $\overline{\Omega}_\lambda$ and $w_n \geqslant 0$ on $\partial\Omega_\lambda$ (since $u_n(x) = u_n(P_\lambda x)$ on T_λ and $u_n(P_\lambda x) \geqslant 0$ if $x \in \overline{\Omega}_\lambda \cap \partial\Omega$). Thus, by Lemma 2, $w_n^- \in \dot{W}^{1,2}(\Omega_\lambda)$. Since $w_n \to w_\lambda$ weakly in $W^{1,2}(\Omega_\lambda)$ our claim follows. This proves that $\lambda \in A$ if λ is close to $\widehat{n}$ (and hence $\lambda \in \widetilde{A}$ if λ is close to $\widehat{n}$) except we have to prove that w_λ cannot vanish on a component of Ω_λ.

If w_λ vanishes on a component Z_λ of Ω_λ then $u(P_\lambda x) \equiv u(x)$ on Z_λ. Since $\lambda > \lambda^*$ and $P_\lambda\Omega_\lambda \subseteq \Omega$ for $\lambda > \lambda^*$, a simple geometric argument ensures that $\Omega_\lambda = \{(t,x') : \lambda < t < g(x'),\ x' \in C\}$ where $C \subseteq \widehat{W} = \{x \in R^m : x_1 = 0\}$. Since Ω_λ is open, one easily sees that C is relatively open in $\widehat{W}$ and g is lower semicontinuous. It is easy to see from the lower semicontinuity of g that $Z_\lambda = \{(t,x') : \lambda < t < g(x') : x' \in \widehat{A}\}$ where $\widehat{A}$ is a component of C. Moreover, since $\lambda > \lambda^*$, it is easy to see that $P_\lambda(\partial\Omega \cap \overline{\Omega}_\lambda)$ does not intersect $\partial\Omega$. Thus, if $\widetilde{x} \in \partial\Omega \cap \overline{\Omega}_\lambda$, u is continuous at $P_\lambda\widetilde{x}$ and $u(P_\lambda\widetilde{x}) > 0$. Hence we see that, if $x' \in \widehat{A}$, $\widehat{x} = (g(x'),x') \in \overline{Z}_\lambda$ and $u(x) \geqslant m > 0$ for all x close to $P_\lambda\widehat{x}$. Since g is lower semicontinuous, we can choose $x' \in \widehat{A}$ so that g is continuous at x' (by [**11**, p.193]). Choose a neighbourhood T of x' in W so that $T \subseteq \widehat{A}$ and then choose a neighbourhood $\widehat{T}$ of $(g(x'),x')$ so that $\widehat{T} = (g(x') - \varepsilon,\ g(x') + \varepsilon) \times T$, $g(y) > g(x') - \varepsilon$ for $y \in T$ and $u(x) \geqslant m$ on $P_\lambda\widehat{T}$. Since g is continuous at x', we can choose $\widehat{T}$ so that $\{(t,x') : x' \in T,\ t > g(x') - \varepsilon,\ x \in \overline{\Omega}\} \subseteq \widehat{T}$. We now consider the set $K = \{(t,x') : x' \in T,\ t > g(x') - \varepsilon\}$. Now, since $u \in \dot{W}^{1,2}(\Omega)$, the function $\widehat{u}(x)$ defined by $\widehat{u}(x) = \begin{cases} u(x) & \text{if } x \in \Omega \\ 0 & \text{otherwise} \end{cases}$ is in $W^{1,2}(R^m)$. Hence $\widehat{u}|_K$ is in $W^{1,2}(K)$. Since $u(x) = u(P_\lambda x)$ on Z_λ, we eventually see that the function

$\widetilde{u}(x) = \begin{cases} u(P_\lambda x) & \text{if } x \in K \\ 0 & \text{otherwise} \end{cases}$ is in $W^{1,2}(K)$. This is impossible by Lemma 3 since our construction ensures that $\widetilde{u}(x) \geqslant m$ if $x \in K \cap \Omega$ and is zero on $K \setminus \Omega$. Hence w_λ cannot vanish identically. For future reference note that this argument does not use that λ is close to $\widehat{n}$.

As a second step, we prove that A is open. Assume that $\mu \in A$. Thus $u(P_\mu x) > u(x)$ if $x \in \Omega_\mu$. Since u is continuous on Ω_μ, it follows by continuity that $u(P_\lambda x) > u(x)$ if λ is near μ and $x \in \Omega_\mu$ unless x is close to $x_1 = \mu$ or to $\partial\Omega$. Hence we see that $u(P_\lambda x) > u(x)$ on Ω_λ for λ near μ unless x_1 is close to μ or x is close to $\partial\Omega$. Let $\Omega_\varepsilon = \{x \in \overline{\Omega} : d(x, \partial\Omega) \leqslant \varepsilon\}$. By countable additivity $m(\Omega_\varepsilon) \to m(\partial\Omega)$ as $\varepsilon \to 0$ and hence $m\{x \in \Omega : d(x, \partial\Omega) \leqslant \varepsilon\} \to 0$ as $\varepsilon \to 0$. Here m denotes Lebésque measure. Since the set of points of Ω where x_1 is near μ has small measure (by a similar argument to that in the first step), we see that, if λ is near μ, $\widetilde{Z}_\lambda = \{x \in \Omega_\lambda : u(P_\lambda x) < u(x)\}$ has small measure, that is $\{x \in \Omega_\lambda : w_\lambda(x) < 0\}$ has small measure. As in the first step, $w_\lambda^- \in \dot{W}^{1,2}(\Omega_\lambda)$ and $-\Delta w_\lambda + c w_\lambda = 0$ on Ω_λ where $|c(x)| \leqslant K_1$. Hence, by Lemma 1, $w_\lambda \geqslant 0$ on Ω_λ as required. We need to prove strict inequality. As before, the only way that this can fail is that w_λ vanishes on a component of Ω_λ. This is impossible by our earlier arguments. Thus A is open and hence $\widetilde{A}$ is open.

It remains to prove that $\widetilde{A}$ is closed. If $\lambda_n \in A$ and $\lambda_n \to \lambda$ as $n \to \infty$, it follows by the continuity of u on Ω that $u(P_\lambda x) \geqslant u(x)$ on Ω_λ. (Note that, if $x \in \Omega_\lambda$, $x \in \Omega_{\lambda_n}$ for large n.) We can then prove that strict inequality holds everywhere by a similar proof to the proof that w_λ cannot vanish on Z_λ. By our earlier arguments, it follows that $(\partial u/\partial x_1) < 0$ on T_λ. Thus $\lambda \in A$. Hence A is closed and thus $\widetilde{A}$ is closed. Hence by connectedness $\widetilde{A} = (\lambda, \widehat{n})$ as required.

By the same limiting argument as in the previous paragraph, $u(P_{\lambda^*} x) \geqslant u(x)$ on Ω_{λ^*}. This completes the proof. □

REMARKS:

1. The proof can be simplified a great deal if u is continuous on $\overline{\Omega}$.

2. The result is still true if in the statement of the theorem we replace Ω_λ by a component of Ω_λ. The proof is essentially the same. This is sometimes useful as in [5].

3. Our techniques can easily be extended to cover cases where f depends on ∇u provided that f is even in $(\partial u/\partial x_1)$, and f is Lipschitz in ∇u. (We need only assume that f is locally Lipschitz on ∇u if ∇u is bounded on Ω.) Note that our result here is a little weaker than that in [1] or [7] in this case as they allow some linear dependence on $(\partial u/\partial x_1)$. As in [1] or [7], we could allow f to depend on x in a suitable way. Note that in [1] much more strongly nonlinear equations are studied.

4. Our proof can be modified to apply to the periodic boundary value problem.

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That is, the problem

$$\frac{\partial u}{\partial t} = \Delta u + f(t,u) \quad \text{in} \quad \Omega \times [0,T]$$
$$u = 0 \quad \text{on} \quad \partial\Omega \times [0,T]$$
$$u \quad \text{is } T \quad \text{periodic in } t$$

(where f is T periodic in t.) Here we assume the same conditions on Ω and f as in the statement of the theorem and obtain analogous conclusions on the behaviour of u in x. Here we look for solutions in $L^\infty(\Omega \times [0,T]) \cap W^{1,2}(\Omega \times [0,T])$ such that $u(\ ,t) \in \dot{W}^{1,2}(\Omega)$ for almost all t. The proof needs some modifications. We need an analogue of Lemma 1 for the parabolic problem where we assume $u \in W^{1,2}(\Omega \times [0,T])$, u is T periodic in t, $u(\ ,t) \in \dot{W}^{1,2}(\Omega)$ for almost all t and $m\{x : (x,t) \in W, u(x,t) < 0\}$ is small for all t. (Note that, if we use the moving plane argument, the set where w_λ is negative will have small measure for each t.) This remark is joint work with P. Hess. Details will appear elsewhere.

5. The argument in part of our proof shows that if $u \in \dot{W}^{1,2}(\Omega)$, $z \in \partial\Omega$ and if $\{x \in R^m \setminus \Omega,\ \|x - z\| < \varepsilon\}$ has positive measure for every $\varepsilon > 0$, then $\{x \in \Omega : \|x - z\| < \varepsilon : |u(x)| < \varepsilon\}$ has positive measure for every $\varepsilon > 0$.

2. Proof of technical results

Proof of Lemma 1: If $\phi \in \dot{W}^{1,2}(W)$ and ϕ is non-negative

$$\int_W \nabla u \nabla \phi + b.\nabla u \phi + cu\phi \geqslant 0.$$

Thus, if $\phi = -u^-$

$$\int_W |\nabla u^-|^2 \leqslant -\int_W b.\nabla u^- u^- + c(u^-)^2 \tag{2}$$
$$\leqslant K \|\nabla u^-\|_2 \|u^-\|_2 + K \|u^-\|_2^2$$

(by the Cauchy-Schwartz inequality and our assumptions on b and c)

$$\leqslant \frac{1}{2} \|\nabla u^-\|_2^2 + (2K^2 + K) \|u^-\|_2^2$$

Hence
$$\|\nabla u^-\|_2^2 \leqslant (4K^2 + 2K) \|u^-\|^2.$$

Note that the norm are calculated on W. Now the *proof* of the Poincare inequality in [**10**, p.157] shows that

$$\|v\|_2 \leqslant (K_2 m(Z))^{1/2} \|\nabla v\|_2$$

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for all $v \in \dot{W}^{1,2}(W)$, where $Z = \{x \in W : v(x) \neq 0\}$. If we apply this to u^- we see that we contradict (2) when δ is small unless $u^-|_W = 0$. Thus $u \geqslant 0$ on W as required.

PROOF OF LEMMA 2: First note that $u \in \dot{W}^{1,2}(\Omega)$ if $u \in W^{1,2}(\Omega)$ and u vanishes close to $\partial\Omega$ (by mollifiers). If u is continuous on $\overline{\Omega}$ and $u \geqslant 0$ on $\partial\Omega$, then u^- vanishes on $\partial\Omega$. Hence it suffices to prove the result when u vanishes on $\partial\Omega$. Now $(u+\varepsilon)^-$ vanishes near $\partial\Omega$ (since $u = 0$ on $\partial\Omega$) and hence $(u+\varepsilon)^- \in \dot{W}^{1,2}(\Omega)$ by our comments above. Thus it suffices to prove that $(u+\varepsilon)^- \to u$ weakly in $W^{1,2}(\Omega)$ as $\varepsilon \to 0$. This follows easily by a similar argument to that in [**12**, p.93]. ∎

3. THE HALF SPACE CASE

In this section we show that positive bounded solutions of

$$(3) \qquad \begin{aligned} -\Delta u &= f(u) \quad \text{in} \quad T \\ u &= 0 \quad \text{on} \quad \partial T \end{aligned}$$

are increasing in x_1. Here $T = \{x \in R^m : x_1 > 0\}$. We improve considerably an earlier result of ours [**6**] and a result in [**8**]. Note that T is a half space in R^m (rather than R^n as in [**6**]). This enables us to avoid an unfortunate choice of notation in [**6**]. Note also that, as in [**6**] or [**8**], half space results are frequently needed in "blowing up" arguments.

THEOREM 2. *Assume that f is C^1, with $f(0) > 0$ or both $f(0) = 0$ and $f'(0) \geqslant 0$, and that $\widetilde{u}$ is a non-trivial, non- negative bounded solution of (3) on T with $\widetilde{u} = 0$ on ∂T. Then $(\partial\widetilde{u}/\partial x_1) > 0$ if $x_1 \geqslant 0$.*

PROOF: This follows from [**6**, Proposition 4] unless $f(0) = f'(0) = 0$. In this case, we must modify the argument in [**6**] by using some of the ideas here. Note that, since $f(0) = 0$, the maximum principle ensures that $\widetilde{u}(x) > 0$ on T. If $x_1 = 0$, the result follows trivially from the maximum principle.

Let $\mathcal{L}$ denote the non-trivial non-negative bounded solutions u of (3) such that $\|u\|_\infty \leqslant \|\widetilde{u}\|_\infty$. We say that $\lambda \in \theta \subset (0,\infty)$ if $u(\lambda + x_1, z) \geqslant u(\lambda - x_1, z)$ for every $x_1 \in [0,\lambda]$, $z \in R^{m-1}$, $u \in \mathcal{L}$. Let $\widetilde{\theta} = \{\lambda \in (0,\infty) : (0,\lambda) \subseteq \theta\}$. If $u \in \mathcal{L}$ and $\lambda \in \theta$, it follows by applying the maximum principle to $u(\lambda + x_1, z) - u(\lambda - x_1, z)$ that

$$u(\lambda + x_1, z) > u(\lambda - x_1, z) \quad \text{for} \quad 0 < x_1 \leqslant \lambda$$

and

$$(4) \qquad \frac{\partial u}{\partial x_1}(\lambda, z) > 0.$$

Hence we see that it suffices to prove that $\widetilde{\theta}$ is a non- empty open and closed subset of $(0,\infty)$. (Connectedness then implies that $\widetilde{\theta} = (0,\infty)$ and the result follows by (4) above. It is easy to see that $\widetilde{\theta}$ is closed in $(0,\infty)$.)

We first prove that there is a $k > 0$ such that $(0,k) \subseteq \theta$ (and hence $(0,k) \subseteq \widetilde{\theta}$). As in [**6**], standard local $W^{2,p}$ estimates implies that ∇u is bounded on T and the bound holds uniformly for u in $\mathcal{L}$. Hence, given $\delta > 0$, there is a $k_1 > 0$ such that $u(x_1,z) \leqslant \delta$ if $0 \leqslant x_1 \leqslant k_1$, $z \in R^{m-1}$, $u \in \mathcal{L}$. If $\lambda > 0$ and $u \in \mathcal{L}$; let $w_\lambda(x_1,z) = u(\lambda + x_1, z) - u(\lambda - x_1, z)$. Then

$$\begin{aligned} -\Delta w_\lambda &= f(u(\lambda + x_1, z)) - f(u(\lambda - x_1, z)) \\ &= f'(u(s,z))w_\lambda, \end{aligned}$$

where s is between $\lambda + x_1$ and $\lambda - x_1$. Since $f'(0) = 0$, we see that, if $\varepsilon > 0$, we can choose $\lambda_0 > 0$ such that $|f'(u(s,z))| \leqslant \varepsilon$ if $0 \leqslant \lambda \leqslant \lambda_0$, $0 \leqslant x_1 \leqslant \lambda$, $z \in R^{m-1}$, $u \in \mathcal{L}$. Thus

$$-\Delta(-w_\lambda) \leqslant \varepsilon(-w_\lambda) \tag{5}$$

on $Z_\lambda = \{(x_1, z) \in (0,\lambda) \times R^{m-1} : w_\lambda(x_1,z) < 0\}$ if $\lambda < \lambda_0$. We prove that Z_λ is empty for $\lambda < \lambda_0$ if we choose λ_0 suitably (where λ_0 can be chosen uniformly for $u \in \mathcal{L}$). It suffices to choose λ_0 such that $\lambda_0^{-2}\pi^2 > \varepsilon$. It then follows as we shall see below that, if $0 < \lambda < \lambda_0$, there is a positive function h on $[0,\lambda_0] \times R^{m-1}$ such that $-\Delta h = \varepsilon h$ and $h(x) \to \infty$ as $\|x\| \to \infty$. Then, since w_λ is bounded, $w_\lambda(x)/h(x) \to 0$ as $\|x\| \to \infty$, $x \in \overline{Z}_\lambda$ and thus $-w_\lambda/h$ has a maximum value on $\overline{Z}_\lambda$. On ∂Z_λ, $w_\lambda = 0$ (because $u(x_1,z) > 0$ if $x_1 = 2\lambda$) and thus the maximum occurs at an interior point of Z_λ. However, by (5) and by [**13**, Chapter 2, Theorem 10], $-w_\lambda/h$ can not have a non-negative maximum in Z_λ and hence we have a contradication unless Z_λ is empty. It remains to construct h. Choose λ_1 such that $\lambda_1 > \lambda_0$ and $\lambda_1^{-2}\pi^2 > \varepsilon$. We then look for a solution h of $-\Delta h = \varepsilon h$ of the form $\sin\left(\pi\lambda_1^{-1}x_1 - \delta\right)V(R)$ where δ is small and positive, $R = \|z\|$ and V is the solution of

$$\begin{gathered} -\frac{1}{R^{m-2}}\frac{d}{dR}\left(R^{m-2}\frac{dV}{dR}\right) = \left(\varepsilon - \lambda_1^{-2}\pi^2\right)V \\ V(0) = 1,\ V'(0) = 0. \end{gathered}$$

It is easy to see that this is the required h. (Since $\varepsilon - \lambda_1^{-2}\pi^2 < 0$, it is easy to prove that V is increasing and $\lim\limits_{R\to\infty} V(R) = \infty$ as required.) Hence we have shown that $\widetilde{\theta}$ is non-empty.

It remains to prove $\widetilde{\theta}$ is open. Suppose not. Then there exist $\lambda \in \theta$ and λ_n decreasing to λ such that $\lambda_n \notin \theta$. Hence there exist $u_n \in \mathcal{L}$, $x_1^n \in (0,\lambda_n)$ and

$z_n \in R^{m-1}$ such that

(6) $$u_n(\lambda_n + x_1^n, z_n) < u_n(\lambda_n - x_1^n, z_n).$$

Since $u_n(x_1, z+k)$ is also a solution of our equation if $k \in R^{m-1}$, $\widehat{u}_n(x_1,z) \equiv u_n(x_1, z - z_n) \in \mathcal{L}$. Thus we may assume $z_n = 0$ for all n. By choosing subsequences if necessary, we can assume that $x_1^n \to a \in [0,\lambda]$ as $n \to \infty$ and $\widehat{u}_n$ converges uniformly on compact subsets of $\overline{T}$ to a non-negative solution w of (3) with $\|w\|_\infty \leqslant \|\widetilde{u}\|_\infty$. By (6), $w(\lambda + a, 0) \leqslant w(\lambda - a, 0)$. Since $\lambda \in \theta$, either w vanishes identically or $w \in \mathcal{L}$ and $a = 0$. In the latter case, since $\lambda \in \theta$, our earlier comments imply that $(\partial w/\partial x_1)(\lambda, 0) > 0$ (see (4)). Thus, since we can use standard regularity theory to ensure that $\widehat{u}_n$ converges to w in the C^1 norm uniformly on compact sets in $\overline{T}$, we see that $(\partial \widehat{u}_n/\partial x_1)(x_1, 0) > 0$ if x_1 is near λ_n and n is large (where what is meant by near is independent of n). Hence $\widehat{u}_n(\lambda_n + x_1, 0) > \widehat{u}_n(\lambda_n - x_1, 0)$ if x_1 is small and n is large. This contradicts (6). (Remember we have shown that $x_1^n \to 0$ as $n \to \infty$). Hence $w \equiv 0$ and thus $\widehat{u}_n$ converges to zero uniformly on compact subsets of $\overline{T}$. Hence u_n $(t_n, z_n) \to 0$ as $n \to \infty$ if $\lambda_n - x_1^n \leqslant t_n \leqslant \lambda_n + x_1^n$. We have proved that, if $u_n \in \mathcal{L}$ and $(x_1^n, z_n) \in Z_{\lambda_n}$, then $u_n(t_n, z_n) \to 0$ as $n \to \infty$ provided that $\lambda_n - x_1^n \leqslant t_n \leqslant \lambda_n + x_1^n$. Note that Z_{λ_n} depends on u_n. Hence if λ_n is close to λ and $u_n \in \mathcal{L}$, $u_n(t_n, z)$ is uniformly small for $(x_1, z) \in Z_{\lambda_n}$. Here, as before, $\lambda_n - x_1 \leqslant t_n \leqslant \lambda_n + x_1$. As earlier, the mean value theorem implies that $-\Delta w_n(x_1, z) = f'(u_n(t_n, z)) w_{\lambda_n}(x_1, z)$ where $\lambda_n - x_1 \leqslant t_n \leqslant \lambda_n + x_1$. Since $f'(0) = 0$ and since $u_n(t_n, z)$ is uniformly small if $(x_1, z) \in Z_{\lambda_n}$ and n is large, it follows that, if $\varepsilon > 0$, $-\Delta(-w_{\lambda_n}) \leqslant \varepsilon(-w_{\lambda_n})$ on Z_{λ_n} provided that $u_n \in \mathcal{L}$ and n is large. In particular, if we choose $\varepsilon < \pi^2 \lambda^{-2}$, we can argue by using the maximum principle as in the previous paragraph that this leads to a contradiction for large n if Z_{λ_n} is non-empty. Hence $\widetilde{\theta}$ is open as required. This completes the proof. ∎

REMARKS: 1. The result implies that there is no positive solution $\widetilde{u}$ of (1) with $\widetilde{u}(x_1, z) \to 0$ as $x_1 \to \infty$. In particular, this implies condition (ii) can be entirely removed from [**6**, Theorems 3 and 4]. The two theorems concern the positive solutions of $-\Delta u = \lambda f(u)$ on a bounded domain Ω with Dirichlet boundary conditions when λ is large.

2. As in [**6**], it follows easily from the theorem that $\lim_{x_1 \to \infty} \widetilde{u}(x_1, z)$ exists and is a bounded positive solution of $-\Delta u = f(u)$ on R^{m-1}. (To prove that it is a solution of this equation we use a test function $\phi_1(x_1 + n)\phi_2(z)$ where ϕ_1 and ϕ_2 have compact support and let n tend to infinity.) This gives a necessary condition for the existence of a solution of (3). For example, it follows from Theorem 2 here and [**9**, Theorem 6.1] that the problem $-\Delta u = u^\alpha$ in T, $u = 0$ on ∂T have no bounded positive solution if $1 < \alpha < (m+1)/(m-3)$ ($\alpha < \infty$ if $m \leqslant 3$). This considerably improves in Gidas and Spruck [**8**, Theorem 1.3].

3. If $f(0) > 0$ or if $f(0) = 0$ and $f'(0) > 0$, there is a converse to the result of the previous paragraph. If there is a bounded positive solution $\widehat{v}$ of $-\Delta u = f(u)$ on R^m (in particular if there is a solution of this equation on R^{m-1}), then there is a bounded positive solution of (3). We use the method of sub and supersolutions. If $f(0) > 0$, we use zero is a subsolution and $\widehat{v}$ is a supersolution. (Technically we use the method of sub and supersolutions on $\{x \in \overline{T} : \|x\| \leqslant n\}$ with Dirichlet boundary conditions and let n tend to infinity.) If $f(0) = 0$ and $f'(0) > 0$, the argument is very similar but we have to construct the subsolution differently. We choose a ball $B \subseteq T$ of sufficiently large radius such that $f'(0)\lambda_1(B) < 1$ where $\lambda_1(B)$ denotes the smallest eigenvalue of $-\Delta$ on B for Dirichlet boundary conditions. Let $\phi_1(x)$ denote the positive eigenfunction corresponding to $\lambda_1(B)$. Then it is easy but tedious to use [**2**, Lemma I.1] to prove that, if ε is small and positive, $h_\varepsilon(x) = \varepsilon\phi_1(x)$ if $x \in B$ and zero otherwise is a subsolution on T. Note that the result at the end of the last remark can be used to show that there need not be a positive solution of (3) when there is a positive bounded solution of $-\Delta u = f(u)$ on R^m in the case where $f(0) = f'(0) = 0$.

4. Our techniques can be used in a number of other cases. For example, they could be used to obtain similar results on quarter spaces (that is $\{x \in R^m : x_1 > 0,\ x_2 > 0\}$).

5. It is possible to give a proof of Theorem 2 without using the proof in [**6**]. The idea is to obtain lower estimates for solutions if $f(0) > 0$ or $f'(0) > 0$ by constructing families of subsolutions of compact support and by using a variant of [**6**, Proposition 1].

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