WATER WAVES OVER A CHANNEL OF FINITE DEPTH WITH A SUBMERGED PLANE BARRIER

ALBERT E. HEINS

1. Introduction and formulation of the problem. This is the third in a series of problems in the study of surface waves which have been disturbed by the presence of a plane barrier and to which a solution may be provided. We assume as in part I,[1] that the fluid is incompressible and non-viscous, and that motion is irrotational. The differential equation to be solved is

$$\nabla^2\Phi = \Phi_{xx} + \Phi_{yy} + \Phi_{zz} = 0, \tag{1.1}$$

where Φ_{xx} denotes a second partial differentiation with respect to x, Φ_{yy} with respect to y, etc.; $\Phi(x,y,z)$ is the velocity potential of the fluid, and from it we may find the components of velocity in the fluid. On a rigid surface, the boundary condition is that there be no component of velocity normal to the surface. Translated into terms of Φ, we have $\Phi_n = 0$, where the subscript n denotes outer normal derivative. On a free surface,[2] $\Phi_n = \beta\Phi$ where β is a physical constant which is positive. The time variation which normally appears in Φ has been suppressed by the assumption that it is monochromatic. We shall assume, as in part I, that the z variation of $\Phi(x,y,z)$ is harmonic. That is, $\Phi(x,y,z) = \exp(ikz)\phi(x,y)$ so that equation (1.1) reduces to

$$\phi_{xx} + \phi_{yy} - k^2\phi = 0, \tag{1.2}$$

while the boundary conditions remain unaltered.

The geometric region over which we wish to solve equation (1.2) is a channel of finite depth a, but infinite in length. Parallel to the floor of the channel is a semi-infinite rigid barrier which is b units of length from the floor ($b < a$). In an xyz coordinate system, this figure may be described as follows.

(i) $y = 0,\ -\infty < x < \infty,\ -\infty < z < \infty$: the floor of the channel,

(ii) $y = b,\ x \geqslant 0,\ -\infty < z < \infty$: the semi-infinite rigid plane barrier, and

(iii) $y = a,\ -\infty < x < \infty,\ -\infty < z < \infty$: the free surface.

Since the z variation has been suppressed, equation (1.2) is two-dimensional and we shall show that the present boundary value problem may be formulated as an integral equation of the Wiener-Hopf type. Furthermore, the analytical conditions required for the solution of a Wiener-Hopf integral equation are

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[1]Albert E. Heins, "Water waves over a channel of finite depth with a dock", American Journal of Mathematics, vol. 70 (1948), pp. 730-748. Henceforth we shall refer to this paper as I. The second part of this series is entitled "Water waves over a channel of infinite depth with a dock" and is to be submitted for publication shortly. Reference to the physical background of this problem discussed in this paper may be found in I.

[2]The constant β is defined in Sec. **5** of this paper.

satisfied here, and hence we are in a position to solve the integral equation. We shall find that the Fourier transform of the unknown function in this integral equation provides us with interesting mathematical properties of the solution. Because we employ Fourier transform techniques here, we shall obtain the desired transform as a by-product of the work.

One feature which distinguishes this problem from the one treated in part I is that we do not have to assume the presence of a two dimensional line source to provide a travelling wave solution for $x \to -\infty$ or for $x \to \infty$, $y > b$. We shall simply require that for $x \to -\infty$, $\phi(x,y)$ be asymptotic to

$$[\alpha_1 \exp (i\kappa x) + \beta_1 \exp (-i\kappa x)] \cosh \rho_0 y/a, \quad \kappa^2 = \rho_0^2/a^2 - k^2,$$

where ρ_0 is the real positive root of the transcendental equation

$$\rho \sinh \rho - a\beta \cosh \rho = 0.$$

For $x \to \infty$, $y > b$, we shall assume that $\phi(x,y)$ is asymptotic to

$$[\alpha_2 \exp (i\kappa' x) + \beta_2 \exp (-i\kappa' x)] \cosh \rho'_0(a - y)/c,$$

where $c = a - b$, and ρ'_0 is the real positive root of[3]

$$\rho' \sinh \rho' - \beta c \cosh \rho' = 0, \qquad \kappa'^2 = \rho'^2_0/c^2 - k^2.$$

These asymptotic forms are obtained by considering first the asymptotic form of the solution of equation (1.2) when the semi-infinite barrier is not present, and second when the semi-infinite barrier extends to negative infinity. The main point here is that if we are sufficiently far away from the point $x = 0$, $y = b$, the above two asymptotic forms present themselves as the bounded solutions of two well-known potential problems. In order to insure that we obtain the bounded solutions for $x \to \infty$ and $x \to -\infty$, we require further that κ and κ' be real. The complex exponential notation describes travelling waves to the right and the left in the x direction. The convention is that exp $(i\kappa x)$ represents a travelling wave to the right while the one with the negative sign is travelling to the left. We shall find that there exist two sets of linear relations between α_1, α_2, β_1, and β_2, and thus we can find the amplitude of the reflected and transmitted waves to the left and to the right of $x = 0$.

The formulation of this problem proceeds along the lines which we described in part I. We may express $\phi(x,y)$ in the strip in terms of an appropriate Green's function $G(x,y,x',y')$ and the discontinuity of ϕ across the barrier $x \geqslant 0, y = b$. We find that we can produce the travelling wave solutions with a source free ϕ simply by demanding the mode of excitation which we described above, so that there is no difficulty in applying Green's Theorem save for $|x|$ very large. The Green's function satisfies an equation of the form (1.1) except at the point $x = x'$, $y = y'$. At this point[4]

[3]In order to formulate the integral equation, this asymptotic form need not be specified so definitely. Indeed from the Green's function which we employ, we shall find that $\phi(x,y)$ need only grow less rapidly than exp $[-\rho'_1 x/a]$ for $x \to \infty$. With the solution of the integral equation we shall find that $\phi(x,y)$ has the prescribed asymptotic form for $x \to \infty$, $y > b$.

[4]For further details see A. Sommerfeld, "Die Greensche Funktion der Schwingungsgleichung", Deutsche Mathematiker Vereinigung, vol. 21 (1912), pp. 309-353. In particular, for a discussion of the logarithmic character of the Green's function employed here, see I, p. 735.

$$\int_0^a G_x \Big|_{x=x'-0}^{x=x'+0} dy = -1 \quad \text{and} \quad \int_{-\infty}^{\infty} G_y \Big|_{y=y'-0}^{y=y'+0} dx = -1.$$

As for the boundary conditions in the Green's function, we take

$$G_y = 0 \qquad \text{when} \quad y = 0, \; -\infty < x < \infty,$$

and

$$G_y = \beta G \quad \text{when} \quad y = a, \; -\infty < x < \infty.$$

We have described this Green's function elsewhere and we simply give it here for reference. We have

$$G(x,y,x',y') = \sum_{n=1}^{\infty} \frac{(\rho_n^2+a^2\beta^2)(\cos \rho_n y/a)(\cos \rho_n y'/a) \exp[-\{\rho_n^2+a^2k^2\}^{\frac{1}{2}}|x-x'|/a]}{(\rho_n^2-a\beta+a^2\beta^2)(\rho_n^2+a^2k^2)^{\frac{1}{2}}}$$
$$- \frac{[\cosh \rho_0 y/a][\cosh \rho_0 y'/a][\rho_0^2-a^2\beta^2][\sin \kappa\,|x-x'|+\sin \kappa(x-x')]}{a\kappa(\beta a+\rho_0^2-a^2\beta^2)},$$

where ρ_0 is the real positive root of

(1.3) $$\rho \sinh \rho - \beta a \cosh \rho = 0$$

while the ρ_n are the positive imaginary roots of equation (1.3). In passing, we observe that $G(x,y,x',y') = O[\exp\{\rho_1^2+a^2k^2\}^{\frac{1}{2}}(x-x')/a]$ for $x'-x \to \infty$ and $O[\sin \kappa(x-x')]$ for $x-x' \to \infty$.

We have from Green's theorem that

$$\phi(x,y) = \int [G(P,P')\phi_{n'}(P') - \phi(P')G_{n'}(P,P')]ds',$$

where the path of integration is a rectangle with a cut along $y = b$, $x > 0$. More precisely we follow the sequence of line segments given below in the same order. (Here l and l_1 are sufficiently large positive numbers.)

	$x = -l_1$	to $x = l$,	$y = 0$;
	$y = 0$	to $y = b - 0$,	$x = l$;
	$x = l$	to $x = 0$,	$y = b - 0$;
	$x = 0$	to $x = l$,	$y = b + 0$;
	$y = b + 0$	to $y = a$,	$x = l$;
	$x = l$	to $x = -l_1$,	$y = a$;
and	$y = a$	to $y = 0$,	$x = -l_1$.

The paths below and above the line $y = b$, $x > 0$ are connected by a line segment which does not cross the rigid barrier. In view of the boundary conditions imposed on $G(P,P')$ and $\phi(P)$ we get immediately that

$$\phi(x,y) = \int_0^l [\phi(x,b+0) - \phi(x,b-0)]G_{y'}(x,y,x',b)dx'$$
$$+ [\alpha_1 \exp(i\kappa x) + \beta_1 \exp(-i\kappa x)] \cosh \rho_0 y/a$$
$$+ O[\exp(-\theta_1 l_1)] + O[\exp(-\theta l)],$$

where θ and θ_1 are two positive constants. Clearly then

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$$(1.4)\qquad \phi(x,y) = \int_0^\infty I(x')G_{y'}(x,y,x',b)\,dx' + [\alpha_1 \exp(i\kappa x) + \beta_1 \exp(-i\kappa x)] \cosh \rho_0 y/a,$$

when l and l_1 become infinite. $I(x)$ is the discontinuity of $\phi(x)$ across the barrier, that is $\phi(x,b+0) - \phi(x,b-0)$. Now from the equation (1.4) we may form the desired integral equation by noting that $\phi_y(x,b) = 0$ for $x \geqslant 0$. Hence we have the integral equation

$$(1.5)\qquad \int_0^\infty I(x')G_{yy'}(x,b,x',b)\,dx' + [\rho_0/a][\alpha_1 \exp(i\kappa x) + \beta_1 \exp(-ikx)]\sinh \rho_0 b/a = 0, \quad x > 0.$$

This is an inhomogeneous integral equation of the Wiener-Hopf type because of the limits of integration and the particular x variation of its kernel.

2. The solution of the integral equation. We first rewrite equation (1.5) so that it is defined for all x. To this end we write

$$(2.1)\qquad \int_{-\infty}^\infty I(x')G_{yy'}(x,b,x',b)dx' + \phi_0(x) = \psi(x),$$

where

$$I(x) = 0, \qquad x < 0;$$

$$\psi(x) = 0, \qquad x > 0;$$

and

$$\phi_0(x) = 0, \qquad x < 0,$$

$$= [\rho_0/a][\alpha_1 \exp(i\kappa x) + \beta_1 \exp(-i\kappa x)] \sinh \rho_0 b/a, \quad x > 0.$$

Before we attempt to apply Fourier transform techniques to equation (2.1), we investigate the nature of the growths of $I(x)$ and $\psi(x)$ for $x \to \infty$ or $x \to -\infty$ as the case may be. In the first place,

$$I(x) = O[\exp(\pm i\kappa' x)], \qquad x \to \infty.$$

As we have remarked, it is not necessary to state this so specifically. For $x \to -\infty$,

$$\psi(x) = O\left[\int_0^\infty \exp\{(\rho_1^2 + a^2k^2)^{1/2}(x - x')/a\}]\, I(x')\,dx'\right],$$

which implies the existence of the integral

$$\int_0^\infty \exp\{-(\rho_1^2 + a^2k^2)^{1/2}x'/a\}\; I(x')\,dx',$$

and this has been assured by our assumption on $I(x)$. It also implies that

$$\psi(x) = O[\exp(\rho_1^2 + a^2k^2)^{1/2}x/a], \quad x \to -\infty.$$

We assume throughout that $\psi(x)$ and $I(x)$ are integrable over any finite interval of the x axis and this, of course, is subject to verification with the solution.

Having this information, we can make some pertinent statements regarding the regions of regularity of the Fourier transforms of $I(x)$, $\psi(x)$ and $G(x,b,x',b)$.

We first examine the bilateral Fourier transform of the Green's function. We have

$$g(w,y,y') = \int_{-\infty}^{\infty} G(x,y,0,y') \exp(-iwx)\, dx$$

$$= \cosh \gamma y \frac{[\gamma \cosh \gamma(a-y') - \beta \sinh \gamma(a-y')]}{\gamma[\gamma \sinh \gamma a - \beta \cosh \gamma a]}, \quad y < y',$$

$$= \cosh \gamma y' \frac{[\gamma \cosh \gamma(a-y) - \beta \sinh \gamma(a-y)]}{\gamma[\gamma \sinh \gamma a - \beta \cosh \gamma a]}, \quad y > y',$$

where $\gamma = (k^2 + w^2)^{\frac{1}{2}}$ and $g(w,y,y')$ is regular in the strip $-(\rho_1^2+a^2k^2)^{\frac{1}{2}}/a < \operatorname{Im} w < 0$. We have already described these calculations and their justification in Part I. We now turn to the study of the transforms of $I(x)$ and $\psi(x)$.

The Fourier transform of $I(x)$ is

$$J(w) = \int_0^{\infty} \exp(-iwx)\, I(x)\, dx.$$

Subject to the verification of the integrability of $I(x)$ in $0 \leqslant x < L$ $(L > 0)$, we know that $J(w)$ is regular in the lower half plane $\operatorname{Im} w<0$, since $I(x) = O[\exp(\pm i\kappa' x)]$. On the other hand, the transform of $\psi(x)$ is

$$\Psi(w) = \int_{-\infty}^{0} \psi(x) \exp(-iwx)\, dx,$$

and this is regular in the upper half plane $\operatorname{Im} w > -(\rho_1^2 + a^2k^2)^{\frac{1}{2}}/a$, where once again we assume the integrability of $\psi(x)$ in $-L_1 < x < 0$ $(L_1 > 0)$ subject to verification. Finally, the transform of $\phi_0(x)$ is

$$\Phi_0(w) = \int_0^{\infty} \exp(-iwx)\, \phi_0(x)\, dx,$$

and this is regular in the lower half plane $\operatorname{Im} w < 0$. There is, then, a common strip of analyticity between the transforms $J(w)$, $\Psi(w)$, $\Phi_0(w)$ and $g_{yy'}(w,b,b)$, namely $-(\rho_1^2+a^2k^2)^{\frac{1}{2}}/a < \operatorname{Im} w < 0$, and we are thus permitted to take the Fourier transform of equation (2.1) to get

$$(2.2) \qquad -\frac{J(w)\, \gamma \sinh \gamma b\, [\gamma \sinh \gamma c - \beta \cosh \gamma c]}{[\gamma \sinh \gamma a - \beta \cosh \gamma a]} + \Phi_0(w) = \Psi(w).$$

The next task which confronts us is the factoring problem. That is, we ask if we can arrange equation (2.2) so that the left side is regular in the lower half plane of regularity while the right side is regular in the appropriate upper half plane of regularity. To accomplish this, we write

$$-\frac{\gamma \sinh \gamma b\, [\gamma \sinh \gamma c - \beta \cosh \gamma c]}{[\gamma \sinh \gamma a - \beta \cosh \gamma a]} = \frac{L_-(w)}{L_+(w)} = L(w),$$

where $L_-(w)$ is the factor of $L(w)$ which is regular in the lower half plane $\operatorname{Im} w < 0$, while $L_+(w)$ is that factor of $L(w)$ which is regular in the upper half plane $\operatorname{Im} w > -\theta$ where θ is the smallest of the three constants π/b, $(\rho_1^2 + a^2k^2)^{\frac{1}{2}}/a$, $(\rho'_1{}^2 + c^2k^2)^{\frac{1}{2}}/c$. Let us suppose that we have carried out this factoring explicitly. Then equation (2.2) becomes

$$(2.3)\qquad \begin{aligned} J(w)L_-(w) + [\rho_0/a][\sinh \rho_0 b/a]\left[\frac{\alpha_1 L_+(\kappa)}{i(w-\kappa)} + \frac{\beta_1 L_+(-\kappa)}{i(w+\kappa)}\right] &= L_+(w)\Psi(w) \\ + [\rho_0/a][\sinh \rho_0 b/a]\left[\alpha_1 \frac{\{L_+(\kappa) - L_+(w)\}}{i(w-\kappa)} + \beta_1 \frac{\{L_+(-\kappa) - L_+(w)\}}{i(w+\kappa)}\right]&. \end{aligned}$$

The left side of equation (2.3) is now regular in the lower half plane $\operatorname{Im} w < 0$, while the right side is regular in the upper half plane $\operatorname{Im} w > -\theta$ $(\theta > 0)$, and both sides are regular in a common strip $-\theta < \operatorname{Im} w < 0$. Hence the left side of equation (2.3) is the analytical continuation of the right side and both sides are regular everywhere. That is

$$(2.4)\qquad J(w)L_-(w) + [\rho_0/a] \sinh \rho_0 b/a \left[\frac{\alpha_1 L + (\kappa)}{i(w-\kappa)} + \frac{\beta_1 L_+(-\kappa)}{i(w+\kappa)}\right] = E(w),$$

and

$$(2.5)\qquad \begin{aligned} L_+(w)\Psi(w) + [\rho_0/a] \sinh \rho_0 b/a \Bigg[\alpha_1 &\frac{\{L_+(\kappa) - L_+(w)\}}{i(w-\kappa)} \\ &+ \beta_1 \frac{\{L_+(-\kappa) - L_+(w)\}}{i(w+\kappa)}\Bigg] = E(w), \end{aligned}$$

where $E(w)$ is an entire function which is yet to be determined.

The determination of $E(w)$ depends in part on the explicit factoring of $L(w)$. In order to do this, we write $L(w)$ in a product representation which exhibits its poles and zeros explicitly. Consider first

$$\gamma \sinh \gamma b = b(k^2 + w^2) \prod_{n=1}^{\infty} [1 + \gamma^2 b^2/n^2\pi^2].$$

For the sake of convenience we assume $k > 0$. Then

$$\begin{aligned} \gamma \sinh \gamma b &= b(w+ik)(w-ik) \prod_{n=1}^{\infty} [1 + k^2b^2/n^2\pi^2 + w^2b^2/n^2\pi^2] \\ &= b(w+ik)(w-ik) \prod_{n=1}^{\infty} [\{1 + k^2b^2/n^2\pi^2\}^{\frac{1}{2}} + iwb/n\pi] \exp(-iwb/n\pi) \\ &\times \prod_{n=1}^{\infty} [\{1 + k^2b^2/n^2\pi^2\}^{\frac{1}{2}} - iwb/n\pi] \exp(iwb/n\pi), \end{aligned}$$

where the exponential factors have been inserted to insure the absolute convergence of the infinite products. The term

$$P_1(b,w) = (w-ik) \prod_{n=1}^{\infty} [\{1 + k^2b^2/n^2\pi^2\}^{\frac{1}{2}} + iwb/n\pi] \exp(-iwb/n\pi)$$

is free of zeros in the lower half plane $\operatorname{Im} w < +k$, while the remaining factor $Q_1(b, w)$ is free of zeros in the upper half plane $\operatorname{Im} w > -k$ and $P_1(b, w)\, Q_1(b, w) = \gamma \sinh \gamma b$.

We now examine the expression $\gamma \sinh \gamma a - \beta \cosh \gamma a$. It has two real zeros $\gamma a = \pm \rho_0$ and a sequence of imaginary zeros $\gamma a = \pm i\rho_n$, $n = 1,2, \ldots$. Furthermore for n sufficiently large and positive, $\rho_n = n\pi + O\,(\beta a/n\pi)$. We find here that

$$\gamma \sinh \gamma a - \beta \cosh \gamma a = -\beta\,(1 - \gamma^2 a^2/\rho_0^2) \prod_{n=1}^{\infty} [1 + \gamma^2 a^2/\rho_n^2].$$

Upon factoring this as we did the sinh γb, we get

$$\gamma \sinh \gamma a - \beta \cosh \gamma a = P_2(a,\beta, w) Q_2(a, \beta, w)$$

where

$$P_2(a,\beta,w) = -a^2\beta(\kappa^2 - w^2)/\rho_0^2 \prod_{n=1}^{\infty} [\{1 + a^2k^2/\rho_n^2\}^{1/2} + iaw/\rho_n] \exp\,(-iaw/n\pi)$$

is free of zeros in the lower half plane $\operatorname{Im} w < 0$, while $Q_2(a, \beta, w)$ is free of zeros in the upper half plane $\operatorname{Im} w > -(\rho_1/a)\{1 + a^2k^2/\rho_1^2\}^{\frac{1}{2}}$. Finally, upon replacing a by c and therefore ρ_n by ρ'_n, we find the product decomposition for $\gamma \sinh \gamma c - \beta \cosh \gamma c$.

These individual factors enable us to write $L_-(w)$ and $L_+(w)$ explicitly. We have

$$L_-(w) = \exp\,[\chi(w)] P_1(b, w) P_2(c,\beta, w)/P_2(a,\beta,w),$$

and

$$L_+(w) = \exp\,[\chi(w)]\, Q_1(b, w)\, Q_2(c, \beta, w)/b\, Q_2(a, \beta, w).$$

The introduction of the exponential factor $\exp\,[\chi(w)]$ requires some comment. We shall presently examine $L_-(w)$ for $|w| \to \infty$, $\operatorname{Im} w < 0$, and $L_+(w)$ for $|w| \to \infty$, $\operatorname{Im} w > -\theta$. It will thus be found that $L_-(w)$ and $L_+(w)$ are of exponential order. The factor $\chi(w)$ will be chosen in such a fashion as to make them both of algebraic growth in their respective half planes of regularity as $|w| \to \infty$.

In order to find the asymptotic form of $L_-(w)$ for $|w| \to \infty$, $\operatorname{Im} w < 0$, we recall first that

$$\rho_n = n\pi + \beta a/n\pi \text{ as } n \to \infty,$$

and

$$\rho'_n = n\pi + \beta c/n\pi \text{ as } n \to \infty.$$

Furthermore, we may neglect the terms $(ka/\rho_n)^2$ relative to unity for $|w|$ sufficiently large. Hence $L_-(w)$ is of the order

$$w \exp\,[\chi(w)] \frac{\prod_{n=1}^{\infty} [1 + iwb/n\pi] \exp\,(-iwb/n\pi) \prod_{n=1}^{\infty} [1 + iwc/n\pi] \exp\,(-iwc/n\pi)}{\prod_{n=1}^{\infty} [1 + iaw/n\pi] \exp\,(-iaw/n\pi)}$$

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But

$$1/\Gamma(y) = ye^{\gamma y} \prod_{n=1}^{\infty} [1 + y/n] \exp(-y/n).$$

Hence $L_-(w)$ is of the order

$$\exp[\chi(w)]\Gamma(iaw/\pi)/\Gamma(ibw/\pi)\Gamma(icw/\pi).$$

Upon applying the Stirling expansion theorem for the gamma function we find that $L_-(w)$ is of the order

$$w^{1/2} \exp[\chi(w) + iw/\pi(a \log a - b \log b - c \log c)]$$

for $|w| \to \infty$, $\operatorname{Im} w < 0$. We then choose

$$\chi(w) = -iw/\pi(a \log a - b \log b - c \log c)$$

so that

$$L_-(w) = O(w^{\frac{1}{2}})$$

for $|w| \to \infty$, $\operatorname{Im} w < 0$. A similar calculation gives us that $L_+(w) = O(w^{-\frac{1}{2}})$ for $|w| \to \infty$, $\operatorname{Im} w > -\theta$, with the same choice for $\chi(w)$. We note, as a check, that for $|w| \to \infty$,

$$L_-(w)/L_+(w) = O(w)$$

in the strip of regularity, as it should be.

In order to determine $E(w)$, the entire function of separation, we are required to examine the asymptotic forms of equations (2.4) and (2.5). Because we anticipated our calculations and inserted $\chi(w)$ into the factoring of $L(w)$, we see that $E(w)$ is at best of algebraic order for $|w| \to \infty$, that is, a polynomial. We can say more than this about $E(w)$. For example $\Psi(w)$ approaches zero for $|w| \to \infty$, $\operatorname{Im} w > -\theta$, as a consequence of the Riemann-Lebesgue lemma. Since $L_+(w) = O(w^{-\frac{1}{2}})$ in this half plane and the remaining terms in equation (2.5) are $O(w^{-3/2})$, it follows that $E(w) = o(w^{-\frac{1}{2}})$. But since $E(w)$ is an entire function, it follows that $E(w)$ is zero in the upper half plane $\operatorname{Im} w > -\theta$, $|w| \to \infty$. We now examine equation (2.4) and find that $E(w) = o(w^{\frac{1}{2}})$, that is, $E(w)$ is constant in the lower half plane $\operatorname{Im} w < 0$, $|w| \to \infty$. From this we conclude immediately that $E(w)$ is zero. We have finally

$$J(w) = -\frac{\rho_0 \sinh \rho_0 b/a}{aiL_-(w)}\left[\frac{\alpha_1 L_+(\kappa)}{w-\kappa} + \frac{\beta_1 L_+(-\kappa)}{w+\kappa}\right]$$

which tells us that $J(w) = O(w^{-3/2})$ for $|w| \to \infty$, $\operatorname{Im} w < 0$, and hence that $\phi_+(x) - \phi_-(x) = O(x^{\frac{1}{2}})$, $x \to 0^+$, that is, $\phi_+(x) - \phi_-(x)$ is integrable in the neighbourhood of the origin. Similarly we find that $\Psi(w) = O(w^{-\frac{1}{2}})$, for $|w| \to \infty$, $\operatorname{Im} w > -\theta$, so that $\psi(x) = O(x^{-\frac{1}{2}})$, $x \to 0^-$.

3. The determination of $\phi(x, y)$. In order to determine $\phi(x, y)$, we write equation (1.4) in Fourier integral representation. Upon doing this, we get

$$\phi(x,y) = \frac{1}{2\pi}\int_\Gamma \exp(iwx)J(w)g_{y'}(w,y,b)\,dw + [\alpha_1 \exp(i\kappa x) + \beta_1 \exp(-i\kappa x)] \cosh \rho_0 y/a, \tag{3.1}$$

where Γ is a contour drawn in the strip of regularity $-\theta < \operatorname{Im} w < 0$. We have two representations for $g(w,y,b)$ depending on whether $y > b$ or $y < b$. The path Γ is closed, for $x > 0$, by a semi-circle passing between the poles on the positive imaginary axis and the radius of the circle is then allowed to become infinite. Because of the growth of $g_{y'}(w,y,b)$ and $J(w)$ in the upper half plane this is a legitimate closing of the contour Γ. These details are described adequately by many authors and we shall not pursue the matter further. For $x < 0$, a similar closing is performed in the lower half plane.

We have then three representations for $\phi(x,y)$ depending on whether $x \leqslant 0$, $0 \leqslant y \leqslant a$; $x \geqslant 0$, $0 \leqslant y \leqslant b$; or $x \geqslant 0$, $b \leqslant y \leqslant a$. We determine these from equation (3.1) by a direct evaluation of the residues and the appropriate form of the Green's function. For $x \leqslant 0$, $0 \leqslant y \leqslant a$ we have

$$\begin{aligned}\phi(x,y) &= [\alpha_1 \exp(i\kappa x) + \beta_1 \exp(-i\kappa x)] \cosh \rho_0 y/a \\ &\quad - \rho_0 \sinh \rho_0 b/a \sum_w \frac{\exp(iwx)}{L_-(w)}\left[\frac{\alpha_1 L_+(\kappa)}{w-\kappa} + \frac{\beta_1 L_+(-\kappa)}{w+\kappa}\right] \\ &\quad \times \frac{(\sin \rho_n b/a)(\cos \rho_n y/a)\rho_n(\rho_n^2 + a^2\beta^2)}{wa^3(\rho_n^2 + \beta^2 a^2 - a^2\beta^2)}.\end{aligned} \tag{3.2}$$

The summation with respect to w is over the sequence

$$w = -i(\rho_n^2 + a^2k^2)^{\frac{1}{2}}/a, \qquad (n = 1, 2, \ldots).$$

For $x \geqslant 0$, $0 \leqslant y \leqslant b$ we have

$$\begin{aligned}\phi(x,y) &= -\frac{\rho_0 \sinh \rho_0 b/a}{a}\sum_w \left[\frac{\alpha_1 L_+(\kappa)}{w-\kappa} + \frac{\beta_1 L_+(-\kappa)}{w+\kappa}\right] \\ &\quad \times \left[\frac{(\cos n\,\pi y/b)\exp(iwx)}{wb(-)^n L_+(w)}\right] \\ &\quad - \frac{\rho_0 \sinh \rho_0 b/a}{aL_+(ik)}\left[\frac{\alpha_1 L_+(\kappa)}{ik-\kappa} + \frac{\beta_1 L_+(-\kappa)}{ik+\kappa}\right]\frac{\exp(-2kx)}{2ikb}.\end{aligned} \tag{3.3}$$

In equation (3.3) the summation with respect to w is the sequence

$$w = i(k^2 + n^2\pi^2/b^2)^{\frac{1}{2}}, \qquad (n = 1, 2, \ldots).$$

Finally for $x \geqslant 0$, $b \leqslant y \leqslant a$, we have

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$$(3.4)\quad \phi(x,y) = \frac{\rho_0 \sinh \rho_0 b/a}{a}\left\{\left[\frac{\alpha_1 L_+(\kappa)}{\kappa'-\kappa} + \frac{\beta_1 L_+(-\kappa)}{\kappa'+\kappa}\right]\frac{\exp(i\kappa' x)}{L_+(\kappa')}\right.$$
$$\left. + \left[\frac{\alpha_1 L_+(\kappa)}{\kappa'+\kappa} - \frac{\beta_1 L_+(-\kappa)}{\kappa'-\kappa}\right]\frac{\exp(-i\kappa' x)}{L_+(-\kappa')}\right\}\frac{(\rho'^2_0 - \beta^2 c^2)\cosh \rho'_0(b-y)/c}{\kappa' c(\beta c - \beta^2 c^2 + \rho'^2_0)}$$
$$- \frac{\rho_0 \sinh \rho_0 b/a}{a}\sum_w \frac{\exp(iwx)}{L_+(w)}\left\{\frac{\alpha_1 L_+(\kappa)}{w-\kappa} + \frac{\beta_1 L_+(-\kappa)}{w-\kappa}\right\}$$
$$\times \frac{\{\cos \rho'_n(b-y)/c\}\{\beta^2 c^2 + \rho'^2_n\}}{cw(\rho'^2_n - \beta^2 c^2 + \beta c)},$$

where now the summation with respect to w is on the sequence

$$w = i\{k^2 + \rho'^2_n/c^2\}^{1/2} \qquad (n = 1, 2, \ldots).$$

Let us now examine the convergence of the infinite series in equations (3.2), (3.3) and (3.4). In equation (3.2), the general term of the infinite series is of the order

$$(\exp n\pi x/a)(\sin n\pi b/a)(\cos n\pi y/a)/n^{3/2}, \qquad n >> 1,$$

so that the series in (3.2) converges absolutely for $x \leqslant 0$, $0 \leqslant y \leqslant a$. For equation (3.3) we have for the order of the general term

$$(\cos n\pi y/b)\exp(-n\pi x/b)(-)^n/n^{3/2}, \qquad n >> 1,$$

while for equation (3.4) we have

$$\exp(-n\pi x/c)\cos[n\pi(y-b)/c]/n^{3/2}, \qquad n >> 1,$$

so that the infinite series converge absolutely for $x \geqslant 0$ and $0 \leqslant y \leqslant b$ or $b \leqslant y \leqslant a$ as the case may be. From the order of the general term we may deduce the behaviour of $\phi(x,y)$ in the neighbourhood of $x = 0$, $y = b$. Let us take equation (3.2) first. We write for abbreviation $r^2 = x^2 + (y-b)^2$. Then

$$\sum_{n=1}^{\infty} \exp(n\pi x/a)(\sin n\pi b/a)(\cos n\pi y/a)/n^{3/2} = O(1 + r^{1/2})$$

for $x \to 0^-$, $y \to b$. That is, $\phi(x,y)$ is bounded and its derivative with respect to r becomes infinite in such a fashion that $r\phi_r$ is finite, indeed zero, for $r \to 0$. A similar remark may be applied to the cases $x \to 0^+$, $y \to b^-$ and $x \to 0^+$, $y \to b^+$. Hence this application of Green's theorem in the neighbourhood of the point $x = 0$, $y = b$ is justified. Since the series expansions in equations (3.2), (3.3), and (3.4) converge both uniformly and absolutely in the regions given above, it is a simple matter to find their asymptotic forms. For example, for $x \to -\infty$, $0 \leqslant y \leqslant a$, $\phi(x,y)$ is asymptotic to

$$[\cosh \rho_0 y/a][\alpha_1 \exp(i\kappa x) + \beta_1 \exp(-i\kappa x)].$$

The next term in the expansion is $O[\exp\{\rho_1^2 + a^2k^2\}^{\frac{1}{2}}x/a]$. For $x \to \infty$, $0 \leqslant y \leqslant b$, $\phi(x,y) = O[\exp(-kx)]$, while for $x \to \infty$, $b \leqslant y \leqslant c$,

$$\phi(x, y) = O\,[\alpha_2 \exp(i\kappa' x) + \beta_2 \exp(-i\kappa' x)].$$

It is clear, then, that one can take the unilateral Fourier transforms of $\phi(x,y)$ for positive x, as long as $\text{Im}\, w < 0$. Finally, we can show that since the integral in equation (3.1) is uniformly convergent with respect to both x and y for $0 \leqslant y \leqslant a$, $-\infty < x < \infty$, it defines a continuous function of x and y in this region. Hence the three representations of $\phi(x,y)$ which we have are continuous across the line $x = 0$.

4. Reflection and transmission properties of the barrier. If we examine equation (3.4) for $x \to \infty$, we find that the only bounded term is

$$\frac{\rho_0}{a} \sinh \rho_0 \frac{b}{a} \left\{ \left[\frac{\alpha_1 L_+(\kappa)}{\kappa' - \kappa} + \frac{\beta_1 L_+(-\kappa)}{\kappa' + \kappa} \right] \frac{\exp(i\kappa' x)}{L_+(\kappa')} + \left[\frac{\alpha_1 L_+(\kappa)}{\kappa' + \kappa} - \frac{\beta_1 L_+(-\kappa)}{\kappa - \kappa'} \right] \frac{\exp(-i\kappa' x)}{L_+(-\kappa')} \right\} \frac{\{\cosh \rho'_0 (b-y)/c\} \{\rho'^2_0 - \beta^2 c^2\}}{c\kappa'(\rho'^2_0 - \beta^2 c^2 + \beta c)}.$$

We have here the required linear relations between α_1, α_2, β_1 and β_2. For example

$$\alpha_2 = \frac{\rho_0 (\sinh \rho_0 b/a)(\rho'^2_0 - \beta^2 c^2)}{ac\kappa'(\rho'^2_0 - \beta^2 c^2 + \beta c)} \left[\frac{\alpha_1 L_+(\kappa)}{\kappa' - \kappa} + \frac{\beta_1 L_+(-\kappa)}{\kappa' + \kappa} \right] \frac{1}{L_+(\kappa')},$$

and

$$\beta_2 = \frac{\rho_0 (\sinh \rho_0 b/a)(\rho'^2_0 - \beta^2 c^2)}{ac\kappa'(\rho'^2_0 - \beta^2 c^2 + \beta c)} \left[\frac{\alpha_1 L_+(\kappa)}{\kappa' + \kappa} + \frac{\beta_1 L_+(-\kappa)}{\kappa' - \kappa} \right] \frac{1}{L_+(-\kappa')}.$$

Now α_1 is the amplitude of the wave incident upon the barrier, so that β_1 is the amplitude of the wave reflected from the barrier for $x < 0$. Hence, if we require, for example, that no wave be incident from the right, that is $\beta_2 = 0$, then α_2 is the amplitude of the wave transmitted to the right. In this case the reflection coefficient on the left is $r_1 = \beta_1/\alpha_1$, while the transmission coefficient on the right is $t_1 = \alpha_2/\alpha_1$. On the other hand, if there is no wave reflected to the left, then $\alpha_1 = 0$, and the reflection coefficient on the right is $r_2 = \alpha_2/\beta_2$, while the transmission coefficient on the left $t_2 = \beta_1/\beta_2$.

It is not difficult to give r_1, r_2, t_1 and t_2 in terms of k, β and the ratio c/a. In the first place, since $L_+(\kappa)$ is the conjugate of $L_+(-\kappa)$, we have

$$r_1 = -\frac{(\kappa' - \kappa)}{(\kappa' + \kappa)} \exp(2i\sigma_1)$$

where $\sigma_1 = \arg L_+(\kappa)$. Similarly

$$r_2 = \frac{\kappa' - \kappa}{\kappa' + \kappa} \exp(-2i\sigma_2)$$

where $\sigma_2 = \arg L_+(\kappa')$. Furthermore

$$t_1 = \frac{(\rho_0 \sinh \rho_0 b/a)(\rho'^2_0 - \beta^2 c^2)\, L_+(\kappa)\, 4\kappa a c}{(\rho'^2_0 - \beta^2 c^2 + \beta c)\, L_+(\kappa')(\kappa' + \kappa)(a^2 \rho'^2_0 - c^2 \rho_0^2)},$$

while

$$t_2 = \frac{a(\kappa' - \kappa)c\kappa'(\rho'^2_0 - \beta^2c^2 + \beta c)\, L_+(-\kappa')}{\rho_0(\sinh \rho_0 b/a)(\rho'^2_0 - \beta^2c^2)\, L_+(-\kappa)}.$$

We are then left with the task of providing the magnitude and phase of $L_+(\pm\kappa)$ and $L_+(\pm\kappa')$. In the first place

$$|L_+(\pm\kappa)|^2 = \frac{(\beta a + \rho_0^2 - \beta^2a^2)(a^2\rho'^2_0 - \rho_0^2c^2)}{2a\rho'^2_0(\rho_0^2 - \beta^2a^2)\sinh^2 \rho_0 b/a},$$

while

$$|L_+(\pm\kappa')|^2 = \frac{2c^3(\rho'^2_0 - \beta^2c^2)}{\rho'^2_0(\rho'^2_0a^2 - \rho_0^2c^2)(\beta c + \rho'^2_0 - \beta^2c^2)}.$$

Hence

$$\left|\frac{L_+(\pm\kappa)}{L_+(\pm\kappa')}\right|^2 = \frac{(\beta c + \rho'^2_0 - \beta^2c^2)(\beta a + \rho_0^2 - \beta^2a^2)(a^2\rho'^2_0 - \rho_0^2c^2)^2}{4c^3a(\rho_0^2 - \beta^2a^2)(\rho'^2_0 - \beta^2c^2)\sinh^2 \rho_0 b/a}.$$

From this we see that

$$t_1 = \frac{\kappa\rho_0}{2}\left\{\frac{a}{c}\right\}^{1/2}\left\{\frac{\rho'^2_0 - \beta^2c^2}{\rho_0^2 - \beta^2a^2}\right\}^{1/2}\left\{\frac{\rho_0^2 + \beta a - \beta^2a^2}{\rho'^2_0 + \beta c - \beta^2c^2}\right\}^{1/2}\frac{\exp i(\sigma_1 - \sigma_2)}{\kappa + \kappa'},$$

and

$$t_2 = \frac{2\kappa'}{\rho_0}\left\{\frac{c}{a}\right\}^{1/2}\left\{\frac{\rho_0^2 - \beta^2a^2}{\rho'^2_0 - \beta^2c^2}\right\}^{1/2}\left\{\frac{\rho'^2_0 + \beta c - \beta^2c^2}{\rho_0^2 + \beta a - \beta^2a^2}\right\}^{1/2}\frac{\exp i(\sigma_1 - \sigma_2)}{\kappa + \kappa'}.$$

The phase angles σ_1 and σ_2 are given by the following infinite series

$$\begin{aligned}\sigma_1 = &\sum_{n=1}^{\infty}\left\{\arcsin \kappa a/\{\rho_n^2 - \rho_0^2\}^{1/2} - \kappa a/n\pi\right\}\\ &- \sum_{n=1}^{\infty}\left\{\arcsin \kappa c/\{\rho'^2_n - \rho_0^2\}^{1/2} - \kappa c/n\pi\right\}\\ &- \sum_{n=1}^{\infty}\left\{\arcsin \kappa b/\{n^2\pi^2 - \rho_0^2\}^{1/2} - \kappa b/n\pi\right\}\\ &- \arcsin ka/\rho_0 - \frac{\kappa}{\pi}\{a \log a - b \log b - c \log c\},\end{aligned}$$

$$\begin{aligned}\sigma_2 = &\sum_{n=1}^{\infty}\left\{\arcsin \kappa' a/\{\rho_n^2 - \rho'^2_0\}^{1/2} - \kappa' a/n\pi\right\}\\ &- \sum_{n=1}^{\infty}\left\{\arcsin \kappa' c/\{\rho'^2_n - \rho'^2_0\}^{1/2} - \kappa' c/n\pi\right\}\\ &- \sum_{n=1}^{\infty}\left\{\arcsin \kappa' b/\{n^2\pi^2 - \rho'^2_0\}^{1/2} - \kappa' b/n\pi\right\} - \arcsin kc/\rho'_0\\ &- \frac{\kappa'}{\pi}\{a \log a - b \log b - c \log c\}.\end{aligned}$$

It is clear that the nth term in any of the above infinite series is $O(1/n^3)$, so that the series all converge. It is not difficult to calculate these series as functions of k and β.

5. A reciprocity theorem. We have shown how α_1, α_2, β_1 and β_2 can be used to define the various reflection and transmission coefficients. Let us observe that $\phi(x,y)$ is complex and in order to obtain a real solution we merely have to take either the real or imaginary parts of

$$\exp\,(ikz + ift)\phi(x,y),$$

where $\beta g = f^2$ and g is the acceleration of gravity in appropriate units. Now, in complex form we may show that a relation exists between the magnitudes of α_1, α_2, β_1, and β_2. We start with Green's theorem. If $\phi^*(x,y)$ denotes the conjugate of $\phi(x,y)$, then

$$\iint[\phi\nabla^2\phi^* - \phi^*\nabla^2\phi]dA = \int[\phi\phi^*_n - \phi^*\phi_n]ds', \tag{5.1}$$

where the area is the region we have described in sec. 1. That is, it is a rectangle of length $L+L_1$ and width a with a horizontal cut parallel to the x axis as we have described. The left side of equation (5.1) vanishes since $\nabla^2\phi = k^2\phi$ and $\nabla^2\phi^* = k^2\phi^*$. Further, because of the boundary conditions on ϕ and ϕ^*, there are no contributions to the line integral along the rigid barriers or the free surface. Finally since there are no sources, we have

$$-\int_0^a[\phi\phi^*_x - \phi^*\phi_x]_{x=-L_1}dy + \int_b^a[\phi\phi^*_x - \phi^*\phi_x]_{x=L}\,dy = 0.$$

But, if we choose L and L_1 sufficiently large and positive, we have that the integral at $x = -L_1$ is

$$\begin{aligned}&2i\kappa\,[|\alpha_1|^2 - |\beta_1|^2]\int_0^a \cosh^2\,(\rho_0 y/a)dy\\ &- 2i\kappa'\,[|\alpha_2|^2 - |\beta_2|^2]\int_b^c \cosh^2\{\rho'_0(b-y)/c\}\,dy\\ &+ O\,[\exp\,(-\theta L)] + O\,[\exp\,(-\theta L_1)] = 0.\end{aligned}$$

Hence when L and $L_1 \to \infty$, we have simply that

$$\frac{\kappa a\,[|\alpha_1|^2 - |\beta_1|^2]\,[\rho_0^2 + \beta a - \beta^2 a^2]}{\rho_0^2 - \beta^2 a^2} = \frac{\kappa' c\,[|\alpha_2|^2 - |\beta_2|^2]\,[\rho'^2_0 + \beta c - \beta^2 c^2]}{\rho'^2_0 - \beta^2 c^2}.$$

Upon substituting in the expressions we found for α_2 and β_2 in terms of α_1 and α_2 we find that the above relation is identically satisfied.

The Carnegie Institute of Technology

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