EXTENSIONS OF VANDERMONDE TYPE CONVOLUTIONS WITH SEVERAL SUMMATIONS AND THEIR APPLICATIONS - I

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(received June 4, 1968)

1. Summary. In an earlier paper [8], one of the authors has established some Vandermonde type convolution identities involving multinomial coefficients with several summations which evidently are generalizations of identities in [1] with one summation. In this paper similar identities are derived for coefficients (defined below) of a general type, in the line of the results in [2] and [3]. Furthermore, in a series of papers [4], [5], [6], Gould has obtained results on inversion of series and on classical polynomials by an extensive use of these identities with one summation. The purpose of this paper is to extend the results of Gould in the light of new convolution identities with several summations.

2. Introduction. In what follows we write for brevity Σ in the place of $\sum_{i=1}^{k}$ and Π in place of $\prod_{i=1}^{k}$. Let, for any a_i and b_i, and for non-negative integral values of the n_i

$$(1) \qquad \frac{\Sigma a_i}{\Sigma(a_i + b_i n_i)} \left(\Sigma (a_i + b_i n_i)\right)_{n_1 + n_2 + \dots + n_k} \Big/ \Pi n_i! \ ,$$

with $(a)_n = a(a-1)\dots(a-n+1)$, be denoted by $P(a_1, \dots, a_k;\ b_1, \dots, b_k;\ n_1, \dots, n_k)$ or briefly as $P(a_i, b_i, n_i; k)$ with the convention that

$$(2) \qquad P(a_i, b_i, n_i; k) = \begin{cases} 1 & \text{for } n_1 = \dots = n_k = 0, \\ 0 & \text{for } a_1 = \dots = a_k = 0, \text{ but not all } n_i \text{ are zero.} \end{cases}$$

Note that the coefficient (1) is slightly different from that in [8]. In this terminology, the main results in [8] can be stated as

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Canad. Math. Bull. vol. 12, no. 1, 1969

https://doi.org/10.4153/CMB-1969-006-4 Published online by Cambridge University Press

$$(3)\qquad \sum_{j_k=0}^{n_k} \cdots \sum_{j_1=0}^{n_1} P(a_i, b_i, j_i; k)\; P(c_i, b_i, n_i - j_i;\; k) = P(a_i + c_i, b_i, n_i; k)\,,$$

$$(4)\qquad \sum_{j_k=0}^{n_k} \cdots \sum_{j_1=0}^{n_1} P(a_i, b_i, j_i; k)\; \frac{\Sigma (c_i + b_i(n_i - j_i))}{\Sigma\, c_i}\; P(c_i, b_i, n_i - j_i; k)$$

$$= \frac{\Sigma (a_i + c_i + b_i n_i)}{\Sigma (a_i + c_i)}\; P(a_i + c_i, b_i, n_i; k)\,,$$

and

$$(5)\qquad \sum_{j_k=0}^{n_k} \cdots \sum_{j_1=0}^{n_1} \{\Sigma (p_i + q_i j_i)\}\; P(a_i, b_i, j_i; k)\; P(c_i, b_i, n_i - j_i; k)$$

$$= \frac{(\Sigma p_i)\, \Sigma (a_i + c_i) + (\Sigma a_i)\, \Sigma q_i n_i}{\Sigma (a_i + c_i)}\; P(a_i + c_i, b_i, n_i; k)\,,$$

which are based on

$$(6)\qquad \sum_{j_k=0}^{\infty} \cdots \sum_{j_1=0}^{\infty} P(a_i, b_i, j_i;\; k)\; \Pi\, s_i^{j_i} = z^{\Sigma a_i}\,,$$

and

$$(7)\qquad \sum_{j_k=0}^{\infty} \cdots \sum_{j_1=0}^{\infty} \frac{\Sigma (a_i + b_i j_i)}{\Sigma\, a_i}\; P(a_i, b_i, j_i; k)\; \Pi\, s_i^{j_i}$$

$$= \frac{z^{\Sigma a_i}}{1 - \Sigma s_i b_i z^{b_i - 1}}\,,$$

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where

(8) $$\Sigma s_i z^{b_i} = z - 1 .$$

We further define

(9) $$B(a_i, b_i, n_i; k) = B(a_1, \ldots, a_k; b_1, \ldots, b_k; n_1, \ldots, n_k)$$

$$= \frac{\Sigma a_i}{\Sigma(a_i + b_i n_i)} \circ \frac{(\Sigma(a_i + b_i n_i))^{n_1 + n_2 + \ldots + n_k}}{\Pi n_i !} ,$$

for any a_i, b_i and non-negative integral values of n_i, with the convention that

(10) $$B(a_i, b_i, n_i; k) = \begin{cases} 1 & \text{for } n_1 = \ldots = n_k = 0 , \\ 0 & \text{for } a_1 = \ldots = a_k = 0 , \end{cases}$$

but not all n_i are zero.

Expression (9), when $k = 1$, reduces to the coefficients in the Abel series (see [2]). Following verbatim the proofs in [8] which essentially use the technique in [1], we are led to the results for the coefficient $B(a_i, b_i, n_i; k)$ stated below:

(11) $$\sum_{j_k=0}^{\infty} \cdots \sum_{j_1=0}^{\infty} B(a_i, b_i, j_i; k) \, \Pi s_i^{j_i} = z^{\Sigma a_i} ,$$

(12) $$\sum_{j_k=0}^{\infty} \cdots \sum_{j_1=0}^{\infty} \frac{\Sigma(a_i + b_i j_i)}{\Sigma a_i} B(a_i, b_i, j_i; k) \, \Pi s_i^{j_i}$$

$$= \frac{z^{\Sigma a_i}}{1 - \Sigma s_i b_i z^{b_i}} ,$$

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where

(13) $$\Sigma s_i z^{b_i} = \log z ,$$

which corresponds to (6), (7), and (8) respectively. Then the usual procedure shows that convolutions (3), (4) and (5) hold good if $P(a_i, b_i, j_i; k)$ is replaced by $B(a_i, b_i, j_i; k)$ everywhere.

The above discussion motivates consideration of the coefficient $C(a_i, b_i, j_i; k) = C(a_1, \ldots, a_k; b_1, \ldots, b_k; n_1, \ldots, n_k)$, which will be called C_k-coefficient, satisfying the following absolutely convergent power series:

(14) $$\sum_{j_k=0}^{\infty} \cdots \sum_{j_1=0}^{\infty} C(a_i, b_i, j_i; k) \; \Pi \, s_i^{j_i} = z^{\Sigma a_i} ,$$

(15) $$\sum_{j_k=0}^{\infty} \cdots \sum_{j_1=0}^{\infty} G(a_i, b_i, j_i; k) \; \Pi \, s_i^{j_i} = z^{\Sigma a_i} g(z; b_1, \ldots, b_k) ,$$

where

(16) $$G(a_i, b_i, n_i; k) = G(a_1, \ldots, a_k; b_1, \ldots, b_k; n_1, \ldots, n_k)$$

$$= \frac{\Sigma (a_i + b_i n_i)}{\Sigma a_i} \; C(a_i, b_i, n_i; k) ,$$

(17) $$\Sigma s_i z^{b_i} = f(z) ,$$

and $g(z; b_1, \ldots, b_k)$ is a function of z, independent of a_i.

These lead to the convolution identities

(18) $$\sum_{j_k=0}^{n_k} \cdots \sum_{j_1=0}^{n_1} C(a_i, b_i, j_i; k) \quad C(c_i, b_i, n_i - j_i; k)$$

$$= \quad C(a_i + c_i, b_i, n_i; k),$$

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$$(19)\qquad \sum_{j_k=0}^{n_k} \cdots \sum_{j_1=0}^{n_1} C(a_i, b_i, j_i;\ k)\ G(c_i, b_i, n_i - j_i; k)$$

$$= \quad G(a_i + c_i, b_i, n_i; k),$$

and

$$(20)\qquad \sum_{j_k=0}^{n_k} \cdots \sum_{j_1=0}^{n_1} \{\Sigma(p_i + q_i j_i)\}\quad C(a_i, b_i, j_i;\ k)$$

$$\times\ C(c_i, b_i, n_i - j_i; k) = \frac{(\Sigma p_i)\,\Sigma\,(a_i + c_i) + (\Sigma a_i)\ \Sigma q_i n_i}{\Sigma(a_i + c_i)}$$

$$\times\quad C(a_i + c_i, b_i, n_i; k)\ .$$

Treatment of such coefficients for $k = 1$ **already** exists in the literature [2], [3]. In concluding this section, we offer the following remarks.

(i) $P(a_i, b_i, n_i; k)$ and $B(a_i, b_i, n_i; k)$ are special cases of C_k-coefficient which result from a particular choice of $f(z)$ viz. $(z - 1)$ and $\log z$ respectively.

(ii) The identities (18), (19) and (20) are readily established once the expressions for generating functions of the coefficients are known in the forms of (14) and (15).

(iii) The convolution identities can be obtained either by following the procedure suggested by Gould or with the help of the extended Lagrange inversion formula for power series (ref. Skalsky [9]). The essential feature of Gould's method is to obtain the generating function of the coefficients under consideration (e.g. (14), (15)), from which the relation (17) would follow. On the other hand, Lagrange's power series expansion method assumes the knowledge of a relation of the type (17) but not the coefficients and would subsequently yield the coefficients. These two procedures in a way complement each other.

For completeness the extension of Lagrange's formula for two complex variables is stated below from [7], the generalized formula for several variables being obvious.

Consider two simultaneous equations.

$$(21)\qquad P(x, y) = x - a - s_1\, \psi(x, y) = 0, \qquad Q(x, y) = y - b - s_2\, \phi(x, y) = 0,$$

where x, y are complex variables and $\psi(x, y)$ and $\phi(x, y)$ are analytic in the neighbourhood of (a, b). Then (21) has a unique solution (ξ, η) analytic in some neighbourhood of (a, b), and any function $F(\xi, \eta)$ which is analytic in that neighbourhood can be expanded as a double power series in S_1 and S_2 as follows:

$$(22)\qquad \frac{F(\xi, \eta)}{\left[\frac{D(P, Q)}{D(x, y)}\right]_{\substack{x=\xi \\ y=\eta}}} = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{s_1^m}{m!} \frac{s_2^n}{n!} \frac{\partial^{m+n}}{\partial a^m \partial b^n} \{F(a, b)\, \psi^m(a, b) . \phi^n(a, b)\},$$

where

$$\frac{D(P, Q)}{D(x, y)} = \begin{vmatrix} \frac{\partial P}{\partial x} & \frac{\partial P}{\partial y} \\ \frac{\partial Q}{\partial x} & \frac{\partial Q}{\partial y} \end{vmatrix} .$$

3. Some new convolution identities. In this section we prove an important convolution formula which generalizes (5.5) and (6.9) in [5]. The variables and numbers under discussion are complex.

THEOREM 1.

(a)

$$(23)\qquad \sum_{j_k=0}^{n_k} \cdots \sum_{j_1=0}^{n_1} (-1)^{\Sigma j_i} P(c_i, t_i b_i, j_i; k)\, P(a_i + b_i j_i - j_i, (1-t_i)b_i, n_i - j_i; k)$$

$$= P(a_i - c_i, (1-t_i)b_i, n_i; k) ,$$

and

(b)

$$(24)\qquad \sum_{j_k=0}^{n_k} \cdots \sum_{j_1=0}^{n_1} (-1)^{\Sigma j_i} B(c_i, t_i b_i, j_i; k)\, B(a_i + b_i j_i, (1-t_i)b_i, n_i - j_i; k)$$

$$= B(a_i - c_i, (1-t_i)b_i, n_i; k) ,$$

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$t_1, \ldots, t_k$ being any complex numbers.

Proof. Because of the cumbersome nature of the expressions for general k, the proof is demonstrated for $k = 2$. By repeating similar steps as in [5], we can establish the theorem. However, we shall apply the extended Lagrange formula (22) for the proof. Since the nature of the argument remains the same for (a) and (b), we only discuss the proof for (b).

If we write $b_i t_i$ for b_i in (13) with $k = 2$ we get

$$\log z = s_1 z^{b_1 t_1} + s_2 z^{b_2 t_2} .$$

Set

(25) $$\log z = x + y \quad \text{where} \quad x = s_1 z^{b_1 t_1}$$

$$\text{and} \quad y = s_2 z^{b_2 t_2} .$$

Note that (25) is in the form of (21). The Lagrange expansion of $\exp\{(c_1 + c_2)(x + y)\}$ with the help of (22) yields

(26) $$\exp\{(c_1 + c_2)(x + y)\} = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{s_1^m}{m!} \cdot \frac{s_2^n}{n!}$$

$$\times \frac{\partial^{m+n}}{\partial x^m \partial y^n} [\{1 - b_1 t_1 s_1 \exp(b_1 t_1 (x+y)) - b_2 t_2 s_2 \exp(b_2 t_2 (x+y))\}$$

$$\times \exp\{(c_1 + c_2 + b_1 t_1 m + b_2 t_2 n)(x+y)\}] \Big|_{\substack{x=0 \\ y=0}}$$

$$= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} B(c_1, c_2; t_1 b_1, t_2 b_2; m, n)\, s_1^m s_2^n .$$

In (26) if we replace x and y by $-x$ and $-y$ respectively and multiply by $\exp\{(a_1 + a_2)(x + y)\}$ on both sides, we get

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(27) $$\exp\{(a_1 + a_2 - c_1 - c_2)(x+y)\} = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} (-1)^{m+n}$$

$$\times\ B(c_1, c_2; t_1 b_1, t_2 b_2; m, n)\ z_1^{n_1} z_2^{n_2} \exp\{(a_1 + a_2 + b_1 m + b_2 n)(x+y)\}$$

with

$$z_1 = x / \exp\{(1 - t_1) b_1 (x+y)\}$$

and

$$z_2 = y / \exp\{(1 - t_2) b_2 (x+y)\} .$$

Again, replacing b_1 and b_2 in (13) with $k = 2$, by $(1 - t_1) b_1$ and $(1 - t_2) b_2$ respectively, we would end with the expressions

(28) $$\exp\{(a_1 + a_2 + b_1 m + b_2 n)(x+y)\}$$

$$= \sum_{n_2=0}^{\infty} \sum_{n_1=0}^{\infty} B(a_1 + b_1 m, a_2 + b_2 n; (1 - t_1) b_1, (1 - t_2) b_2; n_1, n_2) z_1^{n_1} z_2^{n_2},$$

and

(29) $$\exp\{(a_1 + a_2 - c_1 - c_2)(x+y)\}$$

$$\sum_{n_2=0}^{\infty} \sum_{n_1=0}^{\infty} B(a_1 - c_1, a_2 - c_2; (1 - t_1) b_1, (1 - t_2) b_2; n_1, n_2) z_1^{n_1} z_2^{n_2},$$

by similar steps as above.

The right hand side of (25) after substitution of the expression for $\exp\{(a_1 + a_2 + b_1 m + b_2 n)(x+y)\}$ from (28) can be written as

$$\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} (-1)^{m+n} B(c_1, c_2; t_1 b_1, t_2 b_2; m, n)$$

$$\times \; B(a_1 + b_1 m, a_2 + b_2 n; (1-t_1)b_1, (1-t_2)b_2; n_1, n_2) \, z_1^{n_1+m} \, z_2^{n_2+n}$$

$$= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \sum_{n_1=m}^{\infty} \sum_{n_2=n}^{\infty} (-1)^{m+n} B(c_1, c_2; t_1 b_1, t_2 b_2; m, n)$$

$$\times \; B(a_1 + b_1 m, a_2 + b_2 n; (1-t_1)b_1, (1-t_2)b_2; n_1 - m, n_2 - n) \, z_1^{n_1} z_2^{n_2}$$

$$= \sum_{n_2=0}^{\infty} \sum_{n_1=0}^{\infty} z_1^{n_1} z_2^{n_2} \sum_{m=0}^{n_1} \sum_{n=0}^{n_2} (-1)^{m+n} B(c_1, c_2; t_1 b_1, t_2 b_2; m, n)$$

$$\times \; B(a_1 + b_1 m, a_2 + b_2 n; (1-t_1)b_1, (1-t_2)b_2; n_1 - m, n_2 - n).$$

On the other hand the expression on the left hand side is given by (29). By comparing the coefficients of $z_1^{n_1} z_2^{n_2}$ on both sides of (27), the proof of (b) is complete. For part (a), we consider (8) in place of (13) and proceed as above.

Lastly, we give the extension of Jensen's convolution formula for the coefficient $B(a_i, b_i, n_i; k)$ as

(30)
$$S(a_1, \ldots, a_k; c_1, \ldots, c_k; n_1, \ldots, n_k)$$

$$= \sum_{j_k=0}^{n_k} \cdots \sum_{j_1=0}^{n_1} \frac{\Sigma(a_i + b_i j_i)}{\Sigma a_i} B(a_i, b_i, j_i; k)$$

$$\times \; \frac{\Sigma(c_i - b_i j_i)}{\Sigma(c_i - b_i n_i)} B(c_i - b_i n_i, b_i, n_i - j_i; k)$$

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$$= \sum_{j_k=0}^{n_k} \cdots \sum_{j_1=0}^{n_k} \frac{\Sigma j_i}{j_1! \ldots j_k!} \; B(a_i + c_i, 0, n_i - j_i; k) \; \Pi \, b_i^{j_i}$$

The proof is on the same lines as that of (2.3) in [3]. Here we use the recurrence relation

$$(31) \qquad S(a_1, \ldots, a_k; b_1, \ldots, b_k; n_1, \ldots, n_k)$$

$$- \Sigma b_i \; S(a_1, \ldots, a_i + b_i, \ldots, a_k; c_1, \ldots, c_i - b_i, \ldots, c_k; n_1, \ldots, n_i - 1, \ldots, n_k)$$

$$= \frac{\{\Sigma(a_i + c_i)\}^{n_1 + \ldots + n_k}}{\Pi \, n_i!}$$

repeatedly. The extended formula for $P(a_i, b_i, n_i; k)$ is already given in [8].

4. Orthogonal relations and inversion of series. Substituting $a_i = c_i (i = 1, \ldots, k)$ in (23) and (24), we have the orthogonal relations

$$(32) \qquad \sum_{j_k=0}^{n_k} \cdots \sum_{j_1=0}^{n_1} (-1)^{\Sigma j_i} \; P(a_i, t_i b_i, j_i; k) \, P(a_i + b_i j_i - j_i, (1 - t_i) b_i, n_i - j_i; k)$$

$$= \delta(n_1, \ldots, n_k) ,$$

and

$$(33) \qquad \sum_{j_k=0}^{n_k} \cdots \sum_{j_1=0}^{n_1} (-1)^{\Sigma j_i} \; B(a_i, t_i b_i, j_i; k) \, B(a_i + b_i j_i, (1 - t_i) b_i, n_i - j_i; k)$$

$$= \delta(n_1, \ldots, n_k) ,$$

respectively, where we define

$$\delta(n_1, \ldots, n_k) = \begin{cases} 1 & \text{if } n_1 = \ldots = n_k = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Applications of (32) and (33) yield the following inverse series relations for functions of several variables.

THEOREM 2.

(a)

(34) $$F(a_1, \ldots, a_k) = \sum_{j_k=0}^{m_k} \cdots \sum_{j_1=0}^{m_1} (-1)^{\Sigma j_i} P(a_i, t_i b_i, j_i; k) \times f(a_1 + b_1 j_1 - j_1, \ldots, a_k + b_k j_k - j_k)$$

if and only if

(35) $$f(a_1, \ldots, a_k) = \sum_{j_k=0}^{m_k} \cdots \sum_{j_1=0}^{m_1} P(a_i, (1 - t_i)b_i, j_i; k) \times F(a_1 + b_1 j_1 - j_1, \ldots, a_k + b_k j_k - j_k) ;$$

(b)

(36) $$F^*(a_1, \ldots, a_k) = \sum_{j_k=0}^{m_k} \cdots \sum_{j_1=0}^{m_1} (-1)^{\Sigma j_1} B(a_i, t_i b_i, j_i; k) \times f^*(a_1 + b_1 j_1, \ldots, a_k + b_k j_k)$$

if and only if

(37) $$f^*(a_1, \ldots, a_k) = \sum_{j_k=0}^{m_k} \cdots \sum_{j_1=0}^{m_1} B(a_i, (1 - t_i)b_i, j_i; k) \times F^*(a_1 + b_1 j_1, \ldots, a_k + b_k j_k) ;$$

where the m_i are non-negative integers and depend upon whether we consider finite or infinite series.

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Proof. The proof is on the same lines as in [5]. We shall deal only with the proof of (a). The theorem is true for $m_i = \infty$, $i = 1, \ldots, k$, which can be verified by direct substitution and the use of (32) and (33). If $[a_i/(1-b_i)] > 0$, we can discuss a finite series case by setting $m_i = [a_i/(1-b_i)]$, where $[p]$ is as usual the greatest integer less than or equal to p. In that case the right hand side of (34) after substituting the expression for $f(a_1+b_1j_1-j_1, \ldots, a_k+b_kj_k-j_k)$ from (35) becomes

$$\sum_{j_k=0}^{m_k} \cdots \sum_{j_1=0}^{m_1} (-1)^{\Sigma j_i} P(a_i, t_ib_i, j_i; k)$$

$$\times \sum_{r_k=0}^{[(a_k+b_kj_k-j_k)/(1-b_k)]} \cdots \sum_{r_1=0}^{[(a_1+b_1j_1-j_1)/(1-b_1)]} P(a_i+b_ij_i-j_i, (1-t_i)b_i, r_i; k)$$

$$\times F(a_1+b_1j_1-j_1+r_1b_1-r_1, \ldots, a_k+b_kj_k-j_k+r_kb_k-r_k)$$

$$= \sum_{j_k=0}^{m_k} \cdots \sum_{j_1=0}^{m_1} \sum_{r_k=j_k}^{m_k} \cdots \sum_{r_1=j_1}^{m_1} (-1)^{\Sigma j_i} P(a_i, t_ib_i, j_i; k)$$

$$\times P(a_i+b_ij_i-j_i, (1-t_i)b_i, r_i-j_i; k) \; F(a_1+b_1r_1-r_1, \ldots, a_k+b_kr_k-r_k)$$

$$= \sum_{r_k=0}^{m_k} \cdots \sum_{r_1=0}^{m_1} \delta(r_1, \ldots, r_k) \; F(a_1+b_1r_1-r_1, \ldots, a_k+b_kr_k-r_k)$$

$$= F(a_1, \ldots, a_k).$$

The converse is similarly established.

In the part (b) a finite case can be considered by setting $m_i = [-a_i/b_i] > 0$, $i = 1, \ldots, k$.

Following [4] even if all the particular cases of the theorem can be written, we shall only present a few interesting ones.

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Substituting $t_i = 1$, $b_i = -1$, $i = 1, \ldots, k$, and hence $m_i = [a_i/2]$, $i = 1, \ldots, k$, in (34) and (35) we get the relations:

$$(38) \qquad F(a_1, \ldots, a_k) = \sum_{j_k=0}^{[a_k/2]} \cdots \sum_{j_1=0}^{[a_1/2]} (-1)^{\Sigma j_i} \frac{\Sigma a_i}{\Sigma(a_i - j_i)} \times \frac{(\Sigma(a_i - j_i))_{j_1 + \ldots + j_k}}{\Pi j_i!} f(a_1 - 2j_1, \ldots, a_k - 2j_k)$$

if and only if

$$(39) \qquad f(a_1, \ldots, a_k) = \sum_{j_k=0}^{[a_k/2]} \cdots \sum_{j_1=0}^{[a_1/2]} \frac{(\Sigma a_i)_{j_1 + \ldots + j_k}}{\Pi j_i!} \times F(a_1 - 2j_1, \ldots, a_k - 2j_k) .$$

Define a polynomial T, in k variables $x_1, \ldots, x_k$ in the following manner:

Let

$$(40) \qquad H(x_1, \ldots, x_k; n_1, \ldots, n_k) = \sum_{j_k=0}^{[n_k/2]} \cdots \sum_{j_1=0}^{[n_1/2]} (-1)^{\Sigma j_i} \times \frac{\Sigma n_i}{\Sigma(n_i - j_i)} \cdot \frac{(\Sigma(n_i - j_i))_{j_1 + \ldots + j_k}}{\Pi j_i!} \Pi (2x_i)^{n_i - j_i} ,$$

$$T(x_1, \ldots, x_k; 0, \ldots, 0) = H(x_1, \ldots, x_k; 0, \ldots, 0) ,$$

and

$$T(x_1, \ldots, x_k; n_1, \ldots n_k) = \tfrac{1}{2} H(x_1, \ldots, x_k; n_1, \ldots, n_k) .$$

Inversion of (40) by (38) and (39) is given by

$$(41) \qquad \prod_{i=1}^{k} (2x_i)^{n_i} = \sum_{j_k=0}^{[n_k/2]} \cdots \sum_{j_1=0}^{[n_1/2]} \frac{(\Sigma n_i)_{j_1+\cdots+j_k}}{\Pi\, j_i!}$$

$$\times \; H(x_1, \ldots, x_k; n_1 - 2j_1, \ldots, n_k - 2j_k) \; .$$

It can be observed that

$$T(x_1, \ldots, x_k; 0, \ldots 0, n_i, 0, \ldots, 0), \; i = 1, \ldots, k,$$

is the Tchebychev polynomial of degree n_i, in the variable x_i.

Again, substituting $b_i = c_i + 1$ and $t_i = c_i/(c_i+1)$, $i = 1, \ldots, k$, in (34) and (35), we have for $m_i = \infty$, $i = 1, \ldots, k$, the following set of relations:

$$(42) \qquad F(a_1, \ldots, a_k) = \sum_{j_k=0}^{\infty} \cdots \sum_{j_1=0}^{\infty} (-1)^{\Sigma j_i} P(a_i, c_i, j_i; k)$$

$$\times \; f(a_1 + c_1 j_1, \ldots, a_k + c_k j_k)$$

if and only if

$$(43) \qquad f(a_1, \ldots, a_k) = \sum_{j_k=0}^{\infty} \cdots \sum_{j_1=0}^{\infty} P(a_i, 1, j_i; k)\, F(a_1 + c_1 j_1, \ldots, a_k + c_k j_k) \; .$$

Let us define the J-function in k variables $t_1, \ldots, t_k$ by

$$(44) \qquad J(a_i, b_i, t_i; k) = J(a_1, \ldots, a_k; b_1, \ldots, b_k; t_1, \ldots, t_k)$$

$$= \sum_{j_k=0}^{\infty} \cdots \sum_{j_1=0}^{\infty} (-1)^{\Sigma j_i} \frac{\Pi (t_i/2)^{a_i + b_i j_i}}{\Pi j_i!\, \Gamma(\Sigma(a_i + b_i j_i - j_i) + 1)} \, ,$$

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which for $k = 1$ reduces to the generalized Bessel function defined by Bateman as discussed in [5]. When $t_i = 2x_i$ (44) becomes

$$(45) \qquad (\Sigma a_i)\, J\,(a_i, b_i, 2x_i; k) = \sum_{j_k=0}^{\infty} \cdots \sum_{j_1=0}^{\infty} (-1)^{\Sigma j_i} P(a_i, b_i, j_i; k) \times \frac{\Pi\, x_i^{a_i + b_i j_i}}{\Gamma(\Sigma(a_i + b_i j_i))} .$$

With the help of (42) and (43), (45) can be inverted as

$$(46) \qquad \prod_{i=1}^{k} x_i^{a_i} = \sum_{j_k=0}^{\infty} \cdots \sum_{j_1=0}^{\infty} \frac{\Gamma(\Sigma(a_i + b_i j_i))}{\Pi\, j_i!} \Sigma(a_i + b_i j_i)\, J\,(a_i + b_i j_i, b_i, 2x_i; k) .$$

Lastly, in this section, we state a generalization of Theorem 1 and of Theorem 2 in [4] as follows:

$$(47) \qquad F^{**}(n_1, \ldots, n_k) = \sum_{j_k=0}^{n_k} \cdots \sum_{j_1=0}^{n_1} (-1)^{\Sigma j_i} \frac{(\Sigma(a_i + b_i j_i - j_i))_{n_1 + \ldots + n_k - j_1 - \ldots - j_k}}{\Pi\,(n_i - j_i)!} \times \frac{(\Sigma(a_i + b_i j_i))_{j_1 + \ldots + j_k}}{\Pi\, j_i!} f^{**}(j_1, \ldots, j_k)$$

then (a)

$$(48) \qquad \sum_{j_k=0}^{\infty} \cdots \sum_{j_1=0}^{\infty} \frac{(\Sigma(a_i + b_i j_i))_{j_1 + \ldots + j_k}}{\Pi\, j_i!} f^{**}(j_1, \ldots, j_k)\, \Pi\, s_i^{j_i}$$

$$= v^{\Sigma a_i} \sum_{r_k=0}^{\infty} \cdots \sum_{r_1=0}^{\infty} (-1)^{\Sigma r_i}\, \Pi\, u_i^{r_i}\, F^{**}(r_1, \ldots, r_k) ,$$

where $\qquad s_i = u_i(1 - \Sigma u_i)^{b_i - 1} , \quad i = 1, \ldots, k ,$

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and

$$\nu = 1/(1 - \Sigma u_i) ,$$

and (b)

$$(49) \qquad \frac{(\Sigma(a_i+b_i n_i))_{j_1+\ldots+j_k}}{\Pi\, n_i !} f^{**}(n_1, \ldots, n_k)$$

$$= \sum_{j_k=0}^{n_k} \ldots \sum_{j_1=0}^{n_1} (-1)^{\Sigma j_i} \frac{\Sigma(a_i + b_i j_i - j_i)}{\Sigma(a_i + b_i n_i - j_i)}$$

$$\times \frac{(\Sigma(a_i + b_i n_i - j_i))_{n_1+\ldots+n_k - j_1 - \ldots - j_k}}{\Pi(n_i - j_i)!} F^{**}(j_1, \ldots, j_k) .$$

Proofs are on the same lines as that in [4]. Observe that (47) and (49) can be considered as another set of inversion formulas.

5. <u>Further generalizations of Bateman integral formula</u>. Consider two general forms of the J-function defined in (44) and (45) as

$$(50) \qquad J_G(a_i, b_i, t_i; k) = \sum_{j_k=0}^{\infty} \ldots \sum_{j_1=0}^{\infty} (-1)^{\Sigma j_i} \Pi(t_i/2)^{a_i+b_i j_i}$$

$$\times \frac{G(a_i, b_i, j_i; k)}{\Gamma(\Sigma(a_i+b_i j_i)+1)} ,$$

and

$$(51) \qquad J_{\mathcal{C}}(a_i, b_i, t_i; k) = \sum_{j_k=0}^{\infty} \ldots \sum_{j_1=0}^{\infty} (-1)^{\Sigma j_i} \Pi(t_i/2)^{a_i+b_i j_i}$$

$$\times \frac{\mathcal{C}(a_i, b_i, j_i; k)}{\Gamma(\Sigma(a_i+b_i j_i)+1)} ,$$

where $G(a_i, b_i, j_i; k)$ and $C(a_i, b_i, j_i; k)$ are general C_k-coefficients defined in Section 2. In [6], Gould proves (see Relation (2.7)) the following generalization of the Bateman integral formula, in our terminology:

$$(52)\quad J_G(a_1 + c_1; b_1, t_1) = c_1 \int_0^1 J_G(a_1; b_1; (1-u)t_1)\, J_G(c_1; b_1; ut_1) \frac{du}{u} ,$$

with the help of the convolution formula

$$\sum_{k=0}^{n_1} G(a_1; b_1; k)\, C(c_1; b_1; n_1 - k) = G(a_1 + c_1; b_1; n_1) .$$

Analogously, we can prove

$$(53)\qquad J_G(a_i + c_i, b_i, t_i; k) = \Sigma c_i \int_0^1 J_G(a_i, b_i, (1-u)t_i; k)$$

$$\times\ J_G(c_i, b_i, ut_i; k) \frac{du}{u} .$$

Similar generalizations for (3.3) and (3.5) in [6] are

$$(54)\qquad \int_0^1 J_C(a_i, b_i, (1-u)t_i; k)\ J_C(c_i, b_i, ut_i; k)\ du$$

$$= \left\{ \frac{1}{s} \int_0^s J_C(a_i + c_i, b_i, st_i; k)\, ds \right\}_{s=1} ,$$

and

$$(55)\qquad \int_0^1 J_G(a_i, b_i, (1-u)t_i; k)\ J_G(c_i, b_i, ut_i; k)\ u^{\Sigma e_i} (1-u)^{\Sigma d_i}\ du$$

$$\sum_{j_k=0}^{\infty} \cdots \sum_{j_1=0}^{\infty} (-1)^{\Sigma j_i}\ \Pi (t_i/2)^{a_i + c_i + b_i j_i} \frac{K(j_1, \ldots, j_k)}{\Gamma(\Sigma(a_i + d_i + c_i + e_i + b_i j_i) + 1)} ,$$

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where

$$K(j_1, \ldots, j_k) = \sum_{r_k=0}^{j_k} \cdots \sum_{r_1=0}^{j_1} G(a_i, b_i, r_i; k)\ G\ (c_i, b_i, j_i - r_i; k)$$

$$\times \frac{\Gamma\{\Sigma(a_i + d_i + b_i r_i) + 1\}\ \Gamma\{\Sigma(c_i + e_i + b_i(j_i - r_i)) + 1\}}{\Gamma\{\Sigma(a_i + b_i r_i) + 1\}\ \Gamma\{\Sigma(c_i + b_i(j_i - r_i)) + 1\}} .$$

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