ON THE MINIMAL LIPSCHITZ CONSTANT

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In this paper we give necessary and sufficient conditions that a continuous transformation $f: A \to A$ of a metric space A with the metric r should be a contraction with respect to an equivalent metric s. This is the solution of a problem stated by J. S. W. Wong [2].

Let E_r be the set of all metrics equivalent to r (i.e. $S \in E_r$ if and only if s generates the same topology as r) and let E_r^* be a subset of E_r consisting of all bounded metrics. Denote

$$\theta(f, r) = \sup \left[\frac{r(fx, fy)}{r(x, y)} : x, y \in A, \ x \neq y \right] .$$

Finally let $d_r(X)$ mean the diameter of $X \subset A$ with respect to the metric r.

THEOREM 1.

$$\inf [\theta(f, s) : s \in E_r] \leq 1 .$$

THEOREM 2.

$$\inf [\theta(f, s) : s \in E_r^*] < 1$$

if and only if there exists a metric $\hat{s} \in E_r^*$ such that:

$$\limsup_{n \to \infty} \sqrt[n]{d_{\hat{s}}(f^n(A))} < 1 .$$

THEOREM 3.

$$\inf [\theta(f, s) : s \in E_r] < 1$$

if and only if there exists a metric $\hat{s} \in E_r$, a constant $q < 1$

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and a sequence of spheres $K_1 \subset K_2 \subset K_3 \subset \dots$ such that : $\bigcup_{i=1}^{\infty} K_i = A$ and

$$(1) \qquad \limsup_{n\to\infty} \sqrt[n]{d_{\hat{s}}(f^n(Ki))} \le q , \qquad i = 1, 2, \dots$$

Proof. Suppose the metric s satisfies the conditions of Theorem 3. Consider the power series

$$s_\tau(x, y) = s(x, y) + \sum_{n=1}^{\infty} s(f_x^n, f_y^n)\tau^n, \qquad \tau > 0.$$

By (1) this series is uniformly convergent on an arbitrary sphere K_i, $i = 1, 2, \dots$ and its radius of convergence is equal at least $\frac{1}{q}$. In view of $s \le s_\tau$ and by continuity of s_τ it follows that $s_\tau \in E_r$. Moreover, we have

$$s_\tau(fx, fy) = s(fx, fy) + \sum_{n=1}^{\infty} s(f_x^{n+1}, f_y^{n+1})\tau^n$$

$$\le \frac{1}{\tau}\{s(x, y) + \sum_{n=1}^{\infty} s(f_x^n, f_y^n)\tau^n\} = \frac{1}{\tau} s_\tau(x, y) .$$

Hence $\theta(f, s_\tau) \le \frac{1}{\tau}$ and $\inf [\theta(f, s_\tau): \tau < \frac{1}{q}] \le q < 1$.

On the other hand, if $\theta(f, s) = q < 1$ for some metric $s \in E_r$ then (1) holds for an arbitrary sequence of spheres K_i.

Theorem 2 can be proved quite similarly. Theorem 1 follows immediately from the fact that the set E_r^* is non empty and for $s \in E_r^*$ the radius of convergence of the power series (1) is equal at least 1.

Let us now consider a stronger equivalence relation between metrics. We assume that $r \sim s$ if there exist two constants $\alpha > 0$, $\beta > 0$ such that $\alpha r(x, y) \le s(x, y) \le \beta r(x, y)$

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for arbitrary $x, y \in A$. In this case every transformation which is Lipschitzian in one metric is also Lipschitzian with respect to any equivalent metric.

THEOREM 4.

$$\inf [\theta(f, s) : s \sim r] = \lim_{n\to\infty} \sqrt[n]{\theta(f^n, s)} = \inf_{n=1} \sqrt[n]{\theta(f^n, s)}.$$

($\lim_{n\to\infty} \sqrt[n]{\theta(f^n, s)}$ does not depend on the choice of $s \sim r$).

For the proof cf [1].

REFERENCES

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