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On the Existence of Asymptotic-l_p Structures in Banach Spaces

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Abstract. It is shown that if a Banach space is saturated with infinite dimensional subspaces in which all "special" n-tuples of vectors are equivalent with constants independent of n-tuples and of n, then the space contains asymptotic-l_p subspaces for some $1 \leq p \leq \infty$. This extends a result by Figiel, Frankiewicz, Komorowski and Ryll-Nardzewski.

1 Introduction

In this note we address a problem concerning asymptotic structures of infinite-dimensional Banach spaces. These structures carry information on geometric properties which are present "everywhere" and "far enough" in the space. For example, roughly speaking, in asymptotic-l_p spaces these "far enough" geometric properties resemble the ones found in classical l_p spaces (more details and precise definitions of concepts appearing in this Introduction will be given later). Strongly asymptotic-l_p spaces have, in addition, an underlying unconditional structure. Asymptotic-l_p spaces appear in connection with many important developments in the theory of infinite-dimensional Banach spaces. Tsirelson's celebrated construction in the 1970's [T] of a Banach space not containing c_0 or l_p for any $1 \leq p < \infty$ is the first non-trivial example of an asymptotic-l_1 space. The approach behind Tsirelson's construction was revisited in the early 1990's and the method gained prominence with Schlumprecht's construction [S] of an arbitrarily distortable Banach space and the solutions of the unconditional basic sequence problem by Gowers and Maurey [GM] and the distortion problem by Odell and Schlumprecht [OS].

Figiel, Frankiewicz, Komorowski and Ryll-Nardzewski [FFKR] gave necessary and sufficient conditions for finding strongly asymptotic-l_p subspaces in an arbitrary Banach space. Roughly speaking, they showed that a Banach space X contains an asymptotic-l_p basic sequence (for some fixed $1 \leq p \leq \infty$) if and only if X is saturated with sequences of subspaces of the form $X_n = X_{n1} + X_{n2} + \cdots + X_{nn}$ having the property that all n-tuples $(x_1, x_2, \ldots, x_n)$, with $x_i \in X_{ni}$ for $1 \leq i \leq n$, are uniformly equivalent to l_p^n. Our result is of the same type; but with a much weaker hypothesis we obtain the same conclusion. Namely, we consider similar decompositions for which any two n-tuples as above are uniformly equivalent to each other (with the equivalence constant independent of n). In a sense our theorem is an asymptotic version of the well-known theorem of Zippin [Z] which states that a normalized basis of a Banach space such that all normalized block bases are equivalent, must be equivalent

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to the unit vector basis of c_0 or l_p for some $1 \leq p < \infty$. The proof of our result is based on a general method of selecting basic sequences in Banach spaces satisfying certain "stabilization" properties.

Let us briefly describe the organization of this paper. In Section 2 we recall several basic concepts in Banach space theory as well as more specific results which will be used later on. Section 3 contains a standard stabilization technique that was used, for example, in [FFKR, M1, Pe]. In order to make the paper self contained we decided to include the proofs. Our main result is presented in Section 4. The proof is rather complicated, a joining of analytic and combinatorial methods, and it is divided into several steps. The main argument takes root in Maurey's proof of Gowers' dichotomy theorem for unconditional basic sequences in Banach spaces and techniques behind Ramsey theorems. Section 5 contains an extension of the main result. We consider an even weaker hypothesis and we still conclude the existence of strongly asymptotic-l_p subspaces. For the proof we also make essential use of a very recent result of Junge, Kutzarova and Odell [JKO].

2 Preliminaries

We follow [LT] for standard notation and terminology in Banach space theory. In the following, all spaces will be considered to be real, separable Banach spaces and all subspaces will be closed. We shall denote by $X, Y, \dots$ infinite dimensional Banach spaces and by $E, F, \dots$ finite dimensional Banach spaces. The sets of positive integers, rational numbers and real numbers are denoted by $\mathbb{N}$, $\mathbb{Q}$ and $\mathbb{R}$, respectively.

Let X be a Banach space and let $\{x_n\}_n$ be a non-zero sequence in X. We say that $\{x_n\}_n$ is a (*Schauder*) *basis* for X if, for each $x \in X$, there is a unique sequence of scalars $\{a_n\}_n$ such that $x = \sum_{n=1}^{\infty} a_n x_n$, where the sum converges in the norm topology. Clearly, a basis for X is linearly independent. We say that $\{x_n\}_n$ is a *basic sequence* if $\{x_n\}_n$ is a basis for the closure of its linear span. If $\|x_n\| = 1$ for any n, we say that the basic sequence $\{x_n\}$ is *normalized*.

A basis $\{x_n\}_n$ is said to be *unconditional* if for every $x \in X$ its expansion $\sum_{n=1}^{\infty} a_n x_n$ converges unconditionally. Being unconditional is equivalent to the fact that there exists a constant $C > 0$ such that for all scalars $\{a_n\}_n$ and signs $\varepsilon_n = \pm 1$, we have

$$\Big\| \sum_n \varepsilon_n a_n x_n \Big\| \leq C \Big\| \sum_n a_n x_n \Big\|.$$

The smallest C is called the unconditional basis constant of $\{x_n\}_n$.

Two sequences $\{x_n\}_n$ and $\{y_n\}_n$, possibly from different Banach spaces, are said to be *equivalent* if we can find constants C_1 and C_2 such that for all scalars $\{a_n\}_n$, we have

$$\frac{1}{C_1} \Big\| \sum_n a_n x_n \Big\| \leq \Big\| \sum_n a_n y_n \Big\| \leq C_2 \Big\| \sum_n a_n x_n \Big\|. \tag{1}$$

Let $C = C_1 C_2$. The infimum of C satisfying (1) is called the *equivalence constant* In this case we say that $\{x_n\}_n$ and $\{y_n\}_n$ are C-equivalent and sometimes we write $\| \sum a_n x_n \| \overset{C}{\sim} \| \sum a_n y_n \|$.

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Let $\{x_n\}_n$ be a basic sequence in a Banach space X. Given an increasing sequence of positive integers $p_1 < p_2 < p_3 < \cdots$, let $y_k = \sum_{i=p_k+1}^{p_{k+1}} a_i x_i$ be any non-zero vector in the span of $x_{p_k+1}, x_{p_k+1}, \ldots, x_{p_{k+1}}$. We say that $\{y_k\}_k$ is a *block basic sequence* of $\{x_n\}_n$. When $\{x_n\}_n$ is fixed, we will simply call $\{y_k\}_k$ a block basic sequence, or just a *block basis*.

A Banach space X with a basis (x_i) is called *asymptotic-l_p* [MT] if there exists $K > 0$ and an increasing function $f\colon \mathbb{N} \to \mathbb{N}$ such that for all n, if $(y_i)_{i=1}^n$ is a normalized block basis of $(x_i)_{i=f(n)}^{\infty}$, then $(y_i)_{i=1}^n$ is K-equivalent to the unit vector basis of l_p^n. In this case (x_i) is called an asymptotic-l_p basis for X.

A Banach space X with a basis (x_i) is called *strongly asymptotic-l_p* [DFKO] if there exists $K > 0$ and an increasing function $f\colon \mathbb{N} \to \mathbb{N}$ such that for all n, if $(y_i)_{i=1}^n$ is a sequence of disjointly supported vectors from $\operatorname{span}\{x_i, i \geq f(n)\}$, then $(y_i)_{i=1}^n$ is K-equivalent to the unit vector basis of l_p^n. Note that a strongly asymptotic-l_p basis is automatically unconditional. This follows immediately from the fact that for any two disjointly supported vectors y and z that start after $x_{f(2)}$ we have

$$\|y \pm z\| \approx (\|y\|^p + \|z\|^p)^{1/p}.$$

Johnson [J] constructed a so-called "modified" Tsirelson space T_M, in which the natural basis is strongly asymptotic-l_p. Casazza and Odell proved that the two definitions lead to equivalent norms [CO], hence Tsirelson's space T is actually a strongly asymptotic-l_1 space, while T^* is strongly asymptotic-l_∞. A new class of strongly asymptotic-l_p spaces, the so-called "modified mixed Tsirelson spaces", was introduced in [ADKM].

3 Stabilization Techniques

Let X be an infinite dimensional Banach space. On the set of infinite dimensional subspaces of X consider the following partial order

(2) $$Y \preccurlyeq Z \iff Y \subseteq Z + F \text{ for some finite dimensional space } F.$$

Lemma 1 *If $\{Y_n\}_n$ is a sequence of infinite dimensional subspaces of X such that $Y_{n+1} \preccurlyeq Y_n$ for each n, then there exists an infinite dimensional subspace Y of X such that $Y \preccurlyeq Y_n$ for any n.*

Proof Define for any n,

(3) $$Z_n = \bigcap_{1 \leq i \leq n} Y_i.$$

It follows easily that each Z_n is infinite dimensional, thus we can build by induction a linearly independent sequence $(y_n)_n$ such that $y_n \in Z_n$ for each n. Denote by Y the closed linear span of $(y_n)_n$. Also note that since $Z_{n+1} \subseteq Z_n$ for any n, we have that for any n and any $k \geq n$, $y_k \in Z_n$. Then it follows that for any n, $Y \subseteq \operatorname{span}\{y_1, \ldots, y_{n-1}\} + Z_n$ and from $Z_n \subseteq Y_n$ we have that $Y \preccurlyeq Y_n$. ■

Lemma 2 *Let φ be a function defined on the set of all infinite dimensional subspaces of X, taking values in $[0,\infty]$. If φ is monotone with respect to the partial order $\preccurlyeq$, then for any Y infinite dimensional subspace of X there exists Z, an infinite dimensional subspace of Y, such that for any infinite dimensional subspace Z' of Y with $Z' \preccurlyeq Z$ we have that $\varphi(Z') = \varphi(Z)$. In other words, the function φ can be stabilized by passing to a subspace.*

Proof We can assume without loss of generality that the function φ is increasing (otherwise consider $\varphi' = 1/\varphi$).

Fix an infinite dimensional subspace Y of X and assume the conclusion is false for Y. By transfinite induction and diagonalization we shall construct $\{Z_\alpha\}_{\alpha<\omega_1}$ so that

$$\beta < \alpha \Longrightarrow Z_\alpha \preccurlyeq Z_\beta \quad \text{and} \quad \varphi(Z_\alpha) < \varphi(Z_\beta). \tag{4}$$

Recall that the set $\{\alpha < \omega_1\}$ is uncountable and well ordered by "$<$" and note that relation (4) establishes a bijective order preserving correspondence between $\{\alpha < \omega_1\}$ and a subset of $[0,\infty]$ with the natural order on $\mathbb{R}$. But this is a contradiction, since $[0,\infty]$ cannot contain an uncountable subset which is well ordered with respect to the natural order on $\mathbb{R}$.

Suppose that for any subspace of Z of Y we can find another subspace Z' of Y such that $Z' \preccurlyeq Z$ and $\varphi(Z') < \varphi(Z)$. For $\alpha = 0$, let $Z_0 = Y$. Take α to be an ordinal $\alpha < \omega_1$, and assume we have defined Z_β for all $\beta < \alpha$.

If α is of the form $\beta + 1$, then from the above we can find Z_α subspace of Y such that $Z_\alpha \preccurlyeq Z_\beta$ and $\varphi(Z_\alpha) < \varphi(Z_\beta)$. Otherwise, α must be a limit ordinal and since $\alpha < \omega_1$, α is the limit of some increasing sequence of ordinal numbers $\{\alpha_n\}_n$. From the induction hypothesis we have that

$$\cdots \preccurlyeq Z_{\alpha_n} \preccurlyeq Z_{\alpha_{n-1}} \preccurlyeq \cdots \preccurlyeq Z_{\alpha_2} \preccurlyeq Z_{\alpha_1}$$

and

$$\cdots < \varphi(Z_{\alpha_n}) < \varphi(Z_{\alpha_{n-1}}) < \cdots < \varphi(Z_{\alpha_2}) < \varphi(Z_{\alpha_1}).$$

From Lemma 1 it follows that there exists Z_α infinite dimensional subspace of Y such that $Z_\alpha \preccurlyeq Z_{\alpha_n}$ for any n. Since φ is increasing we have for any n that $\varphi(Z_\alpha) \le \varphi(Z_{\alpha_n}) < \varphi(Z_{\alpha_{n-1}})$, which ends the construction. ■

The next lemma establishes that a countable family of monotone functions can be stabilized by passing to a subspace.

Lemma 3 *Let $\{\varphi_n\}_n$ be a family of functions defined on the set of all infinite dimensional subspaces of X taking values in $[0,\infty]$. If each φ_n is monotone with respect to the partial order $\preccurlyeq$, then for any infinite dimensional subspace Y of X, there exists an infinite dimensional subspace Z of Y such that for any infinite dimensional subspace Z' of Y with $Z' \preccurlyeq Z$, we have that $\varphi_n(Z') = \varphi_n(Z)$ for any n.*

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Proof Fix an infinite dimensional subspace Y of X. By applying Lemma 2 to Y and φ_1, we obtain Z_1 an infinite dimensional subspace of Y stabilizing for φ_1. Now we apply Lemma 2 to Z_1 and φ_2 to obtain Z_2 stabilizing for φ_2. Repeating this procedure, we obtain an infinite sequence $\{Z_n\}_n$ such that $Z_{n+1} \subset Z_n$ for any n. From Lemma 1 it follows that we can find an infinite dimensional subspace Z of Y such that for any n, $Z \preccurlyeq Z_n$, and since Z_n is stabilizing for φ_n we have that Z is stabilizing for φ_n. This concludes the proof. ■

Note that the previous lemmas are also true for the family of all subspaces over $\mathbb{Q}$.

4 The Main Result

In this section we prove our main structural result.

Theorem 4 *Let X be a Banach space with the following property. For any infinite dimensional subspace $Y \subseteq X$ there exists a constant M_Y such that for any n there exist infinite dimensional subspaces $U_1, U_2, \dots, U_n$ of Y such that*

$$\frac{1}{M_Y}\Big\|\sum_{i=1}^{n} a_i x_i\Big\| \le \Big\|\sum_{i=1}^{n} a_i y_i\Big\| \le M_Y \Big\|\sum_{i=1}^{n} a_i x_i\Big\| \tag{5}$$

for any collection of norm one vectors x_i, y_i in U_i, $1 \le i \le n$, and any scalars $(a_i)_{i=1}^n$.

Then there exists $p \in [1, \infty]$ such that X contains a strongly asymptotic-l_p subspace.

As we have mentioned before, our result improves on the result from [FFKR]. While the finite sequences of vectors they consider are already equivalent to a basis in a space with a norm fixed in advance (for example l_p^n), we only require that any two such sequences be equivalent.

Definition 5 A basic sequence $\{x_n\}_n$ is said to have property (P) if there is a $K < \infty$ such that for every n the following holds: for every sequence $(A_i)_{i=1}^n$ of finite mutually disjoint subsets of $\mathbb{N}$ such that $\min \bigcup_i A_i \ge n$, if $y_i, z_i \in \operatorname{span}\{x_j : j \in A_i\}$ for $i = 1, 2, \dots, n$ are two finite sequences of norm one vectors, then $\{y_i\}_1^n$ is K-equivalent to $\{z_i\}_1^n$.

We shall prove a slightly different statement from which our result follows.

Theorem 6 *Under the hypothesis of Theorem 4, the space X contains a basic sequence with property (P).*

We show first how to derive Theorem 4 from Theorem 6.

Lemma 7 *Let $\{x_n\}_n$ be a basic sequence with property (P). Then the closed span of $\{x_n\}_n$ is a strongly asymptotic-l_p space, for some $1 \le p \le \infty$.*

Proof Let $\{x_j\}_j$ be a basic sequence that has property (P) with constant K and, for a fixed (but arbitrary) n, let $\{y_i\}_{i=1}^n$ be a sequence of disjointly supported vectors from $\operatorname{span}\{x_i, i \geq n\}$. From Krivine's theorem [K] it follows that there exists $1 \leq p \leq \infty$ such that for any n, we can find normalized blocks $\{w_i\}_{i=1}^n$ of $\{x_j\}_j$ that start as far as we want, such that $\{w_i\}_{i=1}^n$ is 2-equivalent to the standard unit vector basis of l_p^n. For our fixed n we choose the block sequence $\{w_i\}_{i=1}^n$ such that it starts after x_n and has its support disjoint from $\{w_i\}_{i=1}^n$, that is, for any i and j, $\operatorname{supp} y_i \cap \operatorname{supp} w_j = \varnothing$. For any $i = 1, \dots, n$, define $A_i := \{j \in \mathbb{N} : x_j \in \operatorname{supp} y_i \cup \operatorname{supp} w_i\}$. Then $(A_i)_i$ satisfies the conditions in the definition of property (P), so it follows that $\{y_i\}_{i=1}^n$ and $\{w_i\}_{i=1}^n$ are K-equivalent, therefore $\{y_i\}_{i=1}^n$ is $2K$-equivalent to the standard unit vector basis of l_p^n. Hence the closed span of $\{x_j\}_j$ is a strongly asymptotic-l_p space. ■

Theorem 4 follows easily now. From Theorem 6 we have that we can find a basic sequence $\{x_i\}_i$ with property (P) in X and from Lemma 7 we conclude that the closed span of $\{x_i\}_i$ is a strongly asymptotic-l_p subspace of X.

Note that if a space X satisfies the hypothesis of Theorem 6, so does every infinite dimensional subspace of X. Therefore it follows that every infinite dimensional subspace contains a further stabilized asymptotic-l_p subspace, possibly for different p's.

Now it remains to prove Theorem 6. First we introduce some new notations that are convenient for the proof.

Let X be a Banach space. Denote by Δ the set of all pairs of n-tuples of vectors $\vec{x} = (x_1, \dots, x_n)$, $\vec{y} = (y_1, \dots, y_n)$ with the property that $\|x_i\| = \|y_i\|$ for any $i \leq n$ and any $n \geq 1$. If Z is a subspace of X, $\Delta(Z)$ will be the subset of Δ consisting of all pairs $(\vec{x}, \vec{y})$ of n-tuples of vectors from Z for any $n \geq 1$. Given $\vec{U} = (U_1, \dots, U_n)$ where $U_1, \dots, U_n$ are infinite dimensional subspaces of X and $\vec{u} = (u_1, \dots, u_n)$ an n-tuple of vectors, we write $\vec{u} \in \vec{U}$ if $u_i \in U_i$ for $1 \leq i \leq n$. Then set

$$\Delta(\vec{U}) = \{(\vec{u}, \vec{v}) \in \Delta : \vec{u} \in \vec{U}, \vec{v} \in \vec{U}\}.$$

This notation makes possible a more compact formulation of the hypothesis of Theorem 6, *i.e.,* for any infinite dimensional subspace Y of X there exists a constant M_Y such that for any n there exist infinite dimensional subspaces $U_1, U_2, \dots, U_n$ of Y such that

$$\frac{1}{M_Y}\Big\|\sum_{i=1}^n x_i\Big\| \leq \Big\|\sum_{i=1}^n y_i\Big\| \leq M_Y\Big\|\sum_{i=1}^n x_i\Big\| \tag{6}$$

for any $(\vec{x}, \vec{y}) \in \Delta(\vec{U})$, where $\vec{U} = (U_1, \dots, U_n)$.

It is standard in this setting to pass to vector spaces over $\mathbb{Q}$ in order to use the countable structure of such a vector space. Without loss of generality, we can assume that the Banach space X has a basis $\{e_n\}_n$. Let X_0 denote the set of all vectors of the form $\sum_{i=1}^n a_i e_i$ for $n \in \mathbb{N}$, $\{a_i\}_{i=1}^n \subseteq \mathbb{Q}$. Then X_0 is a countable vector space over $\mathbb{Q}$. Moreover, since X_0 is dense in X, it is enough to prove the conclusion of the theorem in X_0. Therefore, from this point onward, our argument will take place in X_0.

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If Y is an infinite dimensional subspace of X_0, then we denote by $\Sigma(Y)$ the set of all infinite dimensional subspaces of Y and by $\Sigma_f(Y)$ the set of all finite dimensional subspaces of Y. By "$\preccurlyeq$" we denote the partial order defined in (2) restricted to $\Sigma(X_0)$.

For any $n \geq 1$ and $\vec{E} = (E_1, E_2, \dots, E_n)$ where $E_1, E_2, \dots, E_n$ are finite dimensional subspaces of X_0 and for any $Y \in \Sigma(X_0)$, set $\varepsilon_{\vec{E},Y}$ to be the supremum of all ε for which we can find $U_1, \dots, U_n \in \Sigma(Y)$ such that for any $(u_1, \dots, u_n) \in \vec{U}$, $(v_1, \dots, v_n) \in \vec{U}$, $(e_1, \dots, e_n) \in \vec{E}$, and $(f_1, \dots, f_n) \in \vec{E}$ with the property that $(\vec{u} + \vec{e}, \vec{v} + \vec{f}) \in \Delta$, we have that

$$\varepsilon\Big\| \sum_{i=1}^{n} (u_i + e_i) \Big\| \leq \Big\| \sum_{i=1}^{n} (v_i + f_i) \Big\| \leq (1/\varepsilon) \Big\| \sum_{i=1}^{n} (u_i + e_i) \Big\|. \tag{7}$$

Note that the condition $(\vec{u}+\vec{e}, \vec{v}+\vec{f}) \in \Delta$ simply means that $\|u_i + e_i\| = \|v_i + f_i\|$ for any $1 \leq i \leq n$. For any n, by $\vec{0}_n$ we understand the n-tuple $(\{0\}, \{0\}, \dots, \{0\})$, in other words the n-tuple of finite dimensional subspaces of X_0 in which each entry is the trivial $\{0\}$ subspace. For a fixed n, comparing (7) with (6) observe that $(1/\varepsilon_{\vec{0}_n,Y})$ is simply the "best" constant M_Y appearing in (6) for this particular n.

Next, using the stabilization techniques from the previous section, we will stabilize the invariant $\varepsilon_{\vec{E},Y}$.

Since X_0 is a countable vector space and $\vec{E}$ are finite tuples with entries from $\Sigma_f(X_0)$, we have that the family $\{\varepsilon_{\vec{E},\cdot}\}$ of functions on $\Sigma(X_0)$, indexed by $\vec{E}$, is also countable. Next we show that each $\varepsilon_{\vec{E},Y}$ is increasing in Y with respect to the partial order $\preccurlyeq$ on $\Sigma(X_0)$. To this end, fix $\vec{E} = (E_1, E_2, \dots, E_n)$ and let $Y_1 \preccurlyeq Y_2$. Pick any ε that satisfies (7) for the definition of $\varepsilon_{\vec{E},Y_1}$. It follows that we can find $U_1, \dots, U_n \in \Sigma(Y_1)$ such that for any $(u_1, \dots, u_n) \in \vec{U}$, $(v_1, \dots, v_n) \in \vec{U}$, $(e_1, \dots, e_n) \in \vec{E}$, and $(f_1, \dots, f_n) \in \vec{E}$ with the property that $(\vec{u} + \vec{e}, \vec{v} + \vec{f}) \in \Delta$, relation (7) holds for ε.

For any $1 \leq i \leq n$ let $U_i' := U_i \cap Y_2$. Since U_i is a (infinite dimensional) subspace of Y_1 and $Y_1 \preccurlyeq Y_2$, we have that $U_i \preccurlyeq Y_2$, and it follows that $U_i' = U_i \cap Y_2$ is infinite dimensional. Also note that for any $1 \leq i \leq n$, U_i' is an infinite dimensional subspace of Y_2. Let $\vec{U}' = (U_1', \dots, U_n')$. Therefore, we can find $U_1', \dots, U_n' \in \Sigma(Y_2)$ such that for any $(u_1, \dots, u_n) \in \vec{U}'$, $(v_1, \dots, v_n) \in \vec{U}'$, $(e_1, \dots, e_n) \in \vec{E}$, and $(f_1, \dots, f_n) \in \vec{E}$ with the property that $(\vec{u} + \vec{e}, \vec{v} + \vec{f}) \in \Delta$, relation (7) holds for ε. But this means exactly that ε satisfies (7) for the definition of $\varepsilon_{\vec{E},Y_2}$. Taking the supremum over all these ε, it follows that $\varepsilon_{\vec{E},Y_1} \leq \varepsilon_{\vec{E},Y_2}$, hence $\varepsilon_{\vec{E},Y}$ is increasing in Y.

From Lemma 2 we have that there exists a subspace $Z \in \Sigma(X)$ stabilizing for the entire family $\{\varepsilon_{\vec{E},\cdot}\}$. In other words, we have that there exists Z such that $\varepsilon_{\vec{E},Z'} = \varepsilon_{\vec{E},Z}$ for any infinite dimensional Z' subspace of Z and any $\vec{E}$. From this moment on we proceed with the argument inside this subspace Z. Since the subspace Z is stabilizing, we can drop the subscript Z' in $\varepsilon_{\vec{E},Z'}$; the argument will take place in Z so the notation $\varepsilon_{\vec{E}}$ will be unambiguous.

From the hypothesis together with (6) and the definition of $\varepsilon_{\vec{0}_n}$, it follows that

$$\inf_n \varepsilon_{\vec{0}_n} \geq \frac{1}{M_Z} > 0$$

and $\varepsilon_{\vec{0}_n} \leq 1$ for any n.

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Pick ε_0 satisfying the following two conditions:

(i) $0 < \varepsilon_0 < \inf_n \varepsilon_{\vec{0}_n}$.

(ii) For any $\vec{E}$, $\varepsilon_0 \neq \varepsilon_{\vec{E}}$.

The following definition is very important in the logical structure of the argument. Consider the subset $A \subset \Delta(Z)$ defined by

$$(8)\qquad A := \Big\{ (\vec{x}, \vec{y}) \in \Delta(Z) : \\ \big\| \sum_i x_i \big\| < \varepsilon_0 \big\| \sum_i y_i \big\|, \text{ or } \big\| \sum_i x_i \big\| > (1/\varepsilon_0) \big\| \sum_i y_i \big\| \Big\}.$$

In other words, A consist of all $(\vec{x}, \vec{y}) \in \Delta(Z)$ which are *not* $(1/\varepsilon_0)^2$- equivalent.

We shall use the following suggestive terminology, similar to that introduced by Maurey [M2]. Let $\vec{E} = (E_1, \dots, E_n)$, where $E_i \in \Sigma_f(Z)$ for $1 \leq i \leq n$. We say that $\vec{E}$ *accepts* a subspace $Y \in \Sigma(Z)$ if and only if for any $U_1, \dots, U_n \in \Sigma(Y)$ we can find $(u_1, \dots, u_n) \in \vec{U}$, $(v_1, \dots, v_n) \in \vec{U}$, $(e_1, \dots, e_n) \in \vec{E}$, and $(f_1, \dots, f_n) \in \vec{E}$ such that $(\vec{u} + \vec{e}, \vec{v} + \vec{f}) \in A$. We say that $\vec{E}$ *rejects* Z if it does not accept any subspace Y of Z. The following lemma clarifies the dichotomy between "accepts" and "rejects".

Lemma 8 *For any $Y \in \Sigma(Z)$ we have that $\vec{E}$ accepts Y if and only if $\varepsilon_{\vec{E}} < \varepsilon_0$.*

Proof Indeed, if $\vec{E}$ accepts Y, then for any $U_1, \dots, U_n \in \Sigma(Y)$ we can find $(u_1, \dots, u_n) \in \vec{U}$, $(v_1, \dots, v_n) \in \vec{U}$, $(e_1, \dots, e_n) \in \vec{E}$, and $(f_1, \dots, f_n) \in \vec{E}$ such that

$$\big\| \sum_{i=1}^n (u_i + e_i) \big\| < \varepsilon_0 \big\| \sum_{i=1}^n (v_i + f_i) \big\| \quad \text{or} \quad \big\| \sum_{i=1}^n (u_i + e_i) \big\| > (1/\varepsilon_0) \big\| \sum_{i=1}^n (v_i + f_i) \big\|.$$

It follows that ε_0 does not satisfy the condition described in (7), hence $\varepsilon_{\vec{E},Y} \leq \varepsilon_0$. From stability and from the fact that $\varepsilon_0 \neq \varepsilon_{\vec{E}}$, we have that $\varepsilon_{\vec{E}} = \varepsilon_{\vec{E},Y} < \varepsilon_0$.

Conversely, if $\varepsilon_0 > \varepsilon_{\vec{E}} = \varepsilon_{\vec{E},Y}$, then ε_0 is not in the set of ε's from the definition of $\varepsilon_{\vec{E},Y}$. This means exactly that $\vec{E}$ accepts Y. ■

From Lemma 8 we derive the following important remark.

Remark 9 If $\vec{E}$ does not accept Z, then it does not accept any subspace of Z, hence it rejects Z. Therefore we may simply say accepts or rejects without creating confusion.

In the sequel we shall also use the following simple remarks.

Remark 10 For any $n \geq 1$, if $\vec{E} = (E_1, \dots, E_n)$ accepts (rejects), then so does $\vec{E}_\pi := (E_{\pi(1)}, \dots, E_{\pi(n)})$ where π is any permutation on $\{1, 2, \dots, n\}$. Indeed, from the definition of $\varepsilon_{\vec{E},Z}$ we can easily show that $\varepsilon_{\vec{E}} = \varepsilon_{\vec{E}_\pi}$, and the conclusion follows immediately from Lemma 9.

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Remark 11 For any $n \geq 1$, if $\vec{E} = (E_1, \ldots, E_n)$ rejects, then for any

$$\vec{e} = (e_1, \ldots, e_n) \in \vec{E} \quad \text{and} \quad \vec{f} = (f_1, \ldots, f_n) \in \vec{E}$$

with $(\vec{e}, \vec{f}) \in \Delta$ we have that $(\vec{e}, \vec{f}) \notin A$. Indeed, from the definition of "rejects" it follows that we can find $U_1, \ldots, U_n \in \Sigma(Z)$ such that for any $\vec{u} = (u_1, \ldots, u_n) \in \vec{U}$, $\vec{v} = (v_1, \ldots, v_n) \in \vec{U}$, $\vec{e} = (e_1, \ldots, e_n) \in \vec{E}$, and $\vec{f} = (f_1, \ldots, f_n) \in \vec{E}$ with $(\vec{u} + \vec{e}, \vec{v} + \vec{f}) \in \Delta$ we have that

$$\varepsilon_0 \Big\| \sum_{i=1}^{n} (u_i + e_i) \Big\| \leq \Big\| \sum_{i=1}^{n} (v_i + f_i) \Big\| \leq (1/\varepsilon_0) \Big\| \sum_{i=1}^{n} (u_i + e_i) \Big\|.$$

Our claim follows by choosing $\vec{u}$ and $\vec{v}$ as the n-tuples of null vectors.

The connection between the terminology introduced above and property (P) becomes clear in view of the following simple observation which follows immediately from the previous remark and the definition of property (P).

Remark 12 Suppose $(x_j)_j$ is a basic sequence in Z. Fix $n \geq 1$ and let $(A_i)_{i=1}^n$ be as in Definition 5. Let $E_i := \operatorname{span}\{x_j : j \in A_i\}$ for $i = 1, 2, \ldots, n$. To say that property (P) is satisfied with constant $(1/\varepsilon_0)$ is equivalent to saying that for any $n \geq 1$ any such $\vec{E} = (E_1, \ldots, E_n)$ rejects.

We shall build by induction a basic sequence $\{x_j\}_j$ that satisfies the condition equivalent to property (P), presented in Remark 12. But first we prove a key lemma for the inductive step.

Lemma 13 *Let $n \geq 2$. If $\vec{E} = (E_1, \ldots, E_n)$ rejects, then for every infinite dimensional subspace W of Z there exists an infinite dimensional subspace W' of W such that for every $w' \in W'$ we have that $(E_1 + \operatorname{span}\{w'\}, E_2, \ldots, E_n)$ rejects.*

Proof Assume that the conclusion is false. Then by Remark 9 there exists $W \in \Sigma(Z)$ such that for any $U \in \Sigma(W)$, we can find $u_0 \in U$ such that if $F_{u_0} := E_1 + \operatorname{span}\{u_0\}$, then $(F_{u_0}, E_2, \ldots, E_n)$ accepts. Thus, for any $U_2, U_3, \ldots, U_n \in \Sigma(W)$ we can find

$$\vec{u} = (u, u_2, u_3, \ldots, u_n) \in U \times U_2 \times U_3 \times \cdots \times U_n,$$
$$\vec{v} = (v, v_2, v_3, \ldots, v_n) \in U \times U_2 \times U_3 \times \cdots \times U_n$$

and

$$\vec{e} = (e_{u_0}, e_2, e_3, \ldots, e_n) \in F_{u_0} \times E_2 \times E_3 \times \cdots \times E_n,$$
$$\vec{f} = (f_{u_0}, f_2, f_3, \ldots, f_n) \in F_{u_0} \times E_2 \times E_3 \times \cdots \times E_n$$

such that $(\vec{u} + \vec{e}, \vec{v} + \vec{f}) \in A$.

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Since $e_{u_0} \in F_{u_0}$ and $f_{u_0} \in F_{u_0}$, we can write $e_{u_0} = e_1 + \alpha u_0$ and $f_{u_0} = f_1 + \beta u_0$ with $\alpha, \beta \in \mathbb{Q}$ and $e_1, f_1 \in E_1$. Hence we have that for any $(U, U_2, \dots, U_n) \in (\Sigma(W))^n$ we can find

$$(u_1, u_2, \dots, u_n) \in U \times U_2 \times \cdots \times U_n,$$
$$(v_1, v_2, \dots, v_n) \in U \times U_2 \times \cdots \times U_n,$$
$$(e_1, e_2, \dots, e_n) \in E_1 \times E_2 \times \cdots \times E_n,$$
$$(f_1, f_2, \dots, f_n) \in E_1 \times E_2 \times \cdots \times E_n$$

such that

$$((u_1, u_2, \dots, u_n) + (e_1, e_2, \dots, e_n), (v_1, v_2, \dots, v_n) + (f_1, f_2, \dots, f_n)) \in A. \tag{9}$$

Indeed, we can take $(u_2, u_3, \dots, u_n)$, $(v_2, v_3, \dots, v_n)$, $(e_1, e_2 \dots e_n)$, $(f_1, f_2 \dots f_n)$ as above and put $u_1 := u + \alpha u_0$ and $v_1 = u + \beta u_0$. Then the pair in (9) is exactly $(\vec{u} + \vec{e}, \vec{v} + \vec{f})$, and it belongs to A. This means that $(E_1, \dots, E_n)$ accepts. But this is a contradiction since $(E_1, \dots, E_n)$ rejects W. ■

Proof of Theorem 6 We shall build inductively a basic sequence $\{x_j\}_j$ having the following property:

($*$) For any $n > 1$, and for any disjoint finite subsets $A_1, A_2, \dots, A_n$ of $\{n-1, n, \dots\}$, if $E_i := \operatorname{span}\{x_j : j \in A_i\}$ for $i = 1, 2 \dots, n$ then $(E_1, E_2, \dots, E_n)$ rejects.

By convention, $\operatorname{span}\{\varnothing\} = \{0\}$. Once we build such a sequence it follows from Remark 12 that the sequence $\{x_j\}_j$ has property (P), and this will conclude the proof.

To have a better intuitive understanding of the following proof, some more explanations and clarifications are in order. First note that from Remark 10 we have that it is sufficient to check ($*$) assuming additionally that the sets $\{A_j\}_1^n$ satisfy the following two conditions: (i) if $A_i = \varnothing$, then $A_j = \varnothing$ for all $i < j \leq n$, and (ii) if $A_i \neq \varnothing$ and $A_j \neq \varnothing$ for $i < j$, then $\min A_i < \min A_j$. Another important observation is the following: we can always assume that $\min \bigcup_{i \leq n} A_i = n - 1$; indeed, otherwise if $\min \bigcup_{i \leq n} A_i := k > n - 1$, we add the empty sets $A_{n+1}, \dots, A_k, A_{k+1}$ to the existing sets $A_1, \dots, A_n$. The new family $\{A_j\}_{j=1}^{k+1}$ will satisfy the assumption, and it is a "valid" family since $\min \bigcup_{i \leq k+1} A_i \geq k$. To exemplify, instead of considering the family $A_1 = \{4, 5\}$ and $A_2 = \{8, 11, 13\}$ for $n = 2$, we consider the family $A_1 = \{4, 5\}, A_2 = \{8, 11, 13\}, A_3 = A_4 = A_5 = \varnothing$ for $n = 5$.

The fact that $\{x_n\}_n$ will be a basic sequence follows from a standard argument. At each step the choice of x_j will be from an infinite dimensional subspace. Choosing the vectors "far enough" along the basis $\{e_n\}_n$ and using the well-known gliding hump argument (see [LT]), we can obtain that the sequence $\{x_n\}_n$ is equivalent to a block basis of $\{e_n\}_n$, hence it will be itself a basic sequence. An important first remark is that from the choice of ε_0 we have that $\vec{0}_n$ rejects for any n.

Step 1: Since $\vec{0}_2$ rejects, by Lemma 13 we get $x_1 \in Z$ such that $(\operatorname{span}\{x_1\}, \{0\})$ rejects.

Step 2: Next, since $\vec{0}_3$ and the previous pair reject, we can find an infinite dimensional subspace W_0 of Z such that for any $w \in W_0$ we have $(\operatorname{span}\{x_1, w\}, \{0\})$, $(\operatorname{span}\{x_1\}, \operatorname{span}\{w\})$ and $(\operatorname{span}\{w\}, \{0\}, \{0\})$ reject (by applying Lemma 13 three times). Take as x_2 any such w, with the provision that x_2 must be also chosen according to the gliding hump procedure, as explained before. We now have that tuples

$$(\operatorname{span}\{x_1\}, \{0\}), \quad (\operatorname{span}\{x_1, x_2\}, \{0\}),$$
$$(\operatorname{span}\{x_1\}, \operatorname{span}\{x_2\}), \quad (\operatorname{span}\{x_2\}, \{0\}, \{0\}),$$

all reject.

Step 3: Since all the previous tuples and $\vec{0}_4$ reject, we can find x_3 such that by adding x_3 to any coordinate we obtain tuples $\vec{E}$ that reject. That is, in addition to the ones in Step 2, the following tuples will reject.

$$(\operatorname{span}\{x_1, x_3\}, \{0\}), \quad (\operatorname{span}\{x_1\}, \{x_3\}), \quad (\operatorname{span}\{x_1, x_2, x_3\}, \{0\})$$
$$(\operatorname{span}\{x_1, x_2\}, \{x_3\}), \quad (\operatorname{span}\{x_1, x_3\}, \operatorname{span}\{x_2\}), \quad (\operatorname{span}\{x_1\}, \operatorname{span}\{x_2, x_3\}),$$
$$(\operatorname{span}\{x_2, x_3\}, \{0\}, \{0\}), \quad (\operatorname{span}\{x_2\}, \{x_3\}, \{0\}), \quad (\operatorname{span}\{x_3\}, \{0\}, \{0\}, \{0\}).$$

The inductive idea is clear now. Suppose we have picked $x_1, x_2, \dots, x_n$ such that the inductive hypothesis holds. Let $\mathcal{S}_{n-1}$ be the set of "acceptable" tuples $\vec{E}$ built in Step $n-1$, from $x_1, x_2, \dots, x_n$. We have that for any $\vec{E} \in \mathcal{S}_{n-1}$, $\vec{E}$ rejects. We shall find a vector x_{n+1} such that any $\vec{E} \in \mathcal{S}_n$ rejects. For a vector $y \in Z$ denote by $\mathcal{S}_{n-1,y}$ the set obtained by adding y to every entry of every $\vec{E} \in \mathcal{S}_{n-1}$. Since the set $\mathcal{S}_{n-1}$ is finite and $\vec{0}_{n+1}$ rejects, by applying Lemma 13 repeatedly, we can find an infinite dimensional subspace W such that for any $w \in W$ we have that any $\vec{E} \in \mathcal{S}_{n-1,w}$ rejects and the $(n+1)$-tuple $\vec{F} = (\operatorname{span}\{w\}, \{0\}, \{0\}, \dots, \{0\})$ rejects as well. Choose any $x_{n+1} \in W$ which is "good" in the gliding hump procedure. It is easy to see now that any tuple $\vec{E} \in \mathcal{S}_n$ belongs either to $\mathcal{S}_{n-1}$ or to $\mathcal{S}_{n-1,x_{n+1}}$ or is $\vec{F}$, hence rejects. This concludes the inductive step and the proof of Theorem 6. ■

5 The Case of Equal Coefficients

In this section we investigate a stronger version of Theorem 4 where the hypothesis assumes equivalence of vectors with equal coefficients. More precisely, we prove the following.

Theorem 14 *Let X be a Banach space with the following property. For any infinite dimensional subspace $Y \subseteq X$ there exists a constant M_Y such that for any n there exist infinite dimensional subspaces $U_1, U_2, \dots, U_n$ of Y such that*

$$\frac{1}{M_Y}\Big\|\sum_{i=1}^{n} x_i\Big\| \leq \Big\|\sum_{i=1}^{n} y_i\Big\| \leq M_Y \Big\|\sum_{i=1}^{n} x_i\Big\| \tag{10}$$

for any collection of norm one vectors x_i, y_i in U_i, $1 \leq i \leq n$. Then there exists $p \in [1, \infty]$ such that X contains a strongly asymptotic-l_p subspace.

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In the first version of this paper we obtained only asymptotic- l_p subspaces. Professor E. Odell then pointed out to us how to get strongly asymptotic-l_p subspaces as well.

The proof of Theorem 14 uses the same framework as the one for Theorem 4. We start with a definition that is very similar to Definition 5 and reflects the slightly different hypothesis we have for Theorem 14.

Definition 15 A basic sequence $\{x_n\}_n$ is said to have property (P') if there is a $K < \infty$ such that for every n the following holds. For every sequence $(A_i)_{i=1}^n$ of finite mutually disjoint subsets of $\mathbb{N}$ such that $\min \bigcup_i A_i \geq n$, if $y_i, z_i \in \operatorname{span}\{x_j : j \in A_i\}$ for $i = 1, 2, \ldots, n$ are two finite sequences of norm one vectors, then

$$\Big\| \sum y_i \Big\| \overset{K}{\sim} \Big\| \sum z_i \Big\|.$$

The first main step is the following theorem which is the counterpart of Theorem 6.

Theorem 16 *Under the hypothesis of Theorem* 14*, the space X contains a basic sequence with property* (P')*.*

The proof is almost identical to that of Theorem 6, all we need to change is the definition of the set Δ. Instead of requiring the n-tuples of vectors $\vec{x} = (x_1, \ldots, x_n)$, $\vec{y} = (y_1, \ldots, y_n)$ to satisfy $\|x_i\| = \|y_i\|$ for any $i \leq n$ and any $n \geq 1$, we now require $\|x_i\| = \|y_i\| = 1$ for any $i \leq n$ and any $n \geq 1$.

Now the existence of asymptotic-l_p subspaces in X follows directly by applying a recent result of Junge, Kutzarova and Odell [JKO].

Theorem 17 ([JKO]) *Let X be a Banach space with a basis $\{x_n\}_n$. Let $1 \leq p \leq \infty$ and $K < \infty$. Assume that for all n, if $\{y_i\}_{i=1}^n$ is a normalized block basis of $\{x_i\}_{i=n}^\infty$, then $\| \sum_{i=1}^n y_i\| \overset{K}{\sim} n^{1/p}$ ($\| \sum_{i=1}^n y_i\| \overset{K}{\sim} 1$ if $p = \infty$). Then every infinite dimensional subspace of X contains an asymptotic-l_p basic sequence.*

Indeed, by Theorem 16 we can find in X a basic sequence $\{x_j\}_j$ with property (P'). Using Krivine's theorem, it can be easily shown (in a similar way as in Lemma 7) that there exist $1 \leq p \leq \infty$ such that for any n, if $\{y_i\}_{i=1}^n$ is a normalized block basis of $\{x_i\}_{i=n}^\infty$, then $\| \sum_{i=1}^n y_i\| \overset{2K}{\sim} n^{1/p}$ ($\| \sum_{i=1}^n y_i\| \overset{2K}{\sim} 1$ if $p = \infty$). From the above theorem it follows that every infinite dimensional subspace of the closed span of $\{x_j\}_j$ contains an asymptotic-l_p basic sequence.

To obtain strongly asymptotic-l_p subspaces we have to follow the argument from [JKO] for disjointly supported rather than successive vectors. Note that the basic sequence with property (P') we obtain from Theorem 16 actually satisfies a stronger property than the sequence $\{x_n\}_n$ in the above theorem. Namely, for all n, if $\{y_i\}_{i=1}^n$ is a normalized sequence of vectors disjointly supported on $\{x_i\}_{i=n}^\infty$, then $\| \sum_{i=1}^n y_i\| \overset{2K}{\sim} n^{1/p}$ ($\| \sum_{i=1}^n y_i\| \overset{2K}{\sim} 1$ if $p = \infty$). Under this stronger hypothesis it can be shown by an essentially identical argument as in [JKO] that every infinite dimensional subspace

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of X contains a strongly asymptotic-l_p basic sequence. We leave the details of the argument to the reader.

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