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NORMAL FAMILIES OF ZERO-FREE MEROMORPHIC FUNCTIONS

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Abstract

Let $\mathcal{F}$ be a family of zero-free meromorphic functions in a domain D, let h be a holomorphic function in D, and let k be a positive integer. If the function $f^{(k)} - h$ has at most k distinct zeros (ignoring multiplicity) in D for each $f \in \mathcal{F}$, then $\mathcal{F}$ is normal in D.

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1. Introduction

Let D be a domain in $\mathbb{C}$ and $\mathcal{F}$ be a family of meromorphic functions in D. We say that $\mathcal{F}$ is normal in D (in the sense of Montel) if each sequence $\{f_n\}$ in $\mathcal{F}$ has a subsequence $\{f_{n_j}\}$ that converges locally uniformly on D, with respect to the spherical metric, to a meromorphic function or ∞ (see Hayman [4], Schiff [9], or Yang [11]). To avoid any confusion, we point out that the spherical metric is applied to the values of the function, not to the points in D.

In 1959, Hayman [3] proved the following result.

THEOREM 1. *Let f be a nonconstant meromorphic function in $\mathbb{C}$ and k be a positive integer. Then at least one of the functions f and $f^{(k)} - 1$ has a zero. Moreover, if f is transcendental, then at least one of the functions f and $f^{(k)} - 1$ has infinitely many zeros.*

The normality corresponding to Theorem 1 was conjectured by Hayman [5] and confirmed by Gu [2].

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THEOREM 2. *Let k be a positive integer and let $\mathcal{F}$ be a family of zero-free meromorphic functions in a domain D. If, for each $f \in \mathcal{F}$, the function $f^{(k)} - 1$ has no zeros in D, then $\mathcal{F}$ is normal in D.*

In 1986, Yang [10] extended Theorem 2 as follows.

THEOREM 3. *Let $\mathcal{F}$ be a family of meromorphic functions defined in a domain D and h be a holomorphic function in D that is not identically zero. If, for each $f \in \mathcal{F}$, the functions f and $f^{(k)} - h$ have no zeros in D, then $\mathcal{F}$ is normal in D.*

Recently, Chang [1] improved Theorem 2 and proved the following result.

THEOREM 4. *Let k be a positive integer and $\mathcal{F}$ be a family of zero-free meromorphic functions in a domain D such that, for each $f \in \mathcal{F}$, the function $f^{(k)} - 1$ has at most k distinct zeros (ignoring multiplicity) in D. Then $\mathcal{F}$ is normal in D.*

Chang also gave an example to show that the condition that $f^{(k)} - 1$ has at most k distinct zeros is best possible.

It is natural to ask whether Theorem 3 remains valid if we replace the hypothesis that $f^{(k)} - h$ has no zeros with the hypothesis that $f^{(k)} - h$ has at most k distinct zeros. In this paper, we use the methods of Chang [1] and of Pang *et al.* [7] to give an affirmative answer to the question. Here is our main result.

THEOREM 5. *Let $\mathcal{F}$ be a family of zero-free meromorphic functions in a domain D, let h be a holomorphic function in D that is not identically zero, and let k be a positive integer. If the function $f^{(k)} - h$ has at most k distinct zeros (ignoring multiplicity) in D for each $f \in \mathcal{F}$, then $\mathcal{F}$ is normal in D.*

EXAMPLE 6. Suppose that $\mathcal{F} = \{f_n(z) = 1/(nz) : n = 1, 2, 3, \ldots\}$, that $D = \{z : |z| < 1\}$, and that $h(z) = 1/z^{k+1}$, where k is a positive integer. Then, for any $f_n \in \mathcal{F}$, the function $f_n^{(k)} - h$ has only one zero in D, but $\mathcal{F}$ is not normal in D. This shows that Theorem 5 is not valid if the function h is allowed to be meromorphic.

EXAMPLE 7. Suppose that $\mathcal{F} = \{f_n(z) = 1/(nz) : n \geq k!2^{(k+1)} + 1\}$, that $D = \{z : |z| < 1\}$, and that $h(z) = 1/(z - 1)^{k+1}$, where k is a positive integer. Then, for any $f_n \in \mathcal{F}$, the function $f_n^{(k)} - h$ has $k + 1$ distinct zeros in D, but $\mathcal{F}$ is not normal in D. This shows that the condition in Theorem 5 that $f^{(k)} - h$ has at most k distinct zeros (ignoring multiplicity) in D is best possible.

2. Some lemmas

For the proof of Theorem 5, we require the following results.

LEMMA 8 [8, 12]. *Let $\alpha \in \mathbb{R}$ satisfy $-1 < \alpha < \infty$ and let $\mathcal{F}$ be a family of zero-free meromorphic functions in a domain D. If $\mathcal{F}$ is not normal at $z_0 \in D$, then there exist points $z_j \in D$ tending to z_0, functions $f_j \in \mathcal{F}$, positive numbers ρ_j tending to 0, and a nonconstant zero-free meromorphic function g of order at most two such that*

$$g_n(\xi) = \rho_j^{-\alpha} f_j(z_j + \rho_j \xi) \to g(\xi)$$

locally uniformly in ξ in $\mathbb{C}$, with respect to the spherical metric.

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LEMMA 9 [7]. *Let f be a transcendental meromorphic function of finite order, a be a polynomial that is not identically zero, and k be a positive integer. If the function f has no zeros, then the function $f^{(k)} - a$ has infinitely many zeros.*

LEMMA 10 [1]. *Let f be a nonconstant zero-free rational function and k be a positive integer. Then the function $f^{(k)} - 1$ has at least $k+1$ distinct zeros (ignoring multiplicity) in $\mathbb{C}$.*

Using the method of Chang [1], we obtain the following lemma.

LEMMA 11. *Let f be a nonconstant zero-free rational function, a be a polynomial that is not identically zero, and k be a positive integer. Then the function $f^{(k)} - a$ has at least $k+1$ distinct zeros (ignoring multiplicity) in $\mathbb{C}$.*

PROOF. If $\deg a = 0$, then a is constant, and the result follows from Lemma 10.

Now we suppose that $\deg a > 0$. Since f is a nonconstant zero-free rational function, f is not a polynomial, and hence has at least one finite pole. Further, by calculation, the function $f^{(k)} - a$ has at least one zero in $\mathbb{C}$. Thus, we can write

$$a(z) = A\prod_{i=1}^{m}(z+v_i)^{m_i}, \tag{1}$$

$$f(z) = \frac{C_1}{\prod_{i=1}^{n}(z+z_i)^{n_i}}, \tag{2}$$

$$f^{(k)}(z) = a(z) + \frac{C_2\prod_{i=1}^{s}(z+w_i)^{l_i}}{\prod_{i=1}^{n}(z+z_i)^{n_i+k}}, \tag{3}$$

where A, C_1, and C_2 are nonzero constants, m, n, s, l_i, m_i, and n_i are positive integers, the v_i (when $1 \le i \le m$) are distinct complex numbers, and the w_i (when $1 \le i \le s$) and z_i (when $1 \le i \le n$) are distinct complex numbers.

Set $M = \sum_{i=1}^{m} m_i$ and $N = \sum_{i=1}^{n} n_i$. Then $\deg a = M \ge 1$. By induction, we deduce from (2) that

$$f^{(k)}(z) = \frac{P_k(z)}{\prod_{i=1}^{n}(z+z_i)^{n_i+k}}, \tag{4}$$

where P_k is a polynomial of degree $(n-1)k$. Thus, by (1), (3), and (4),

$$A\prod_{i=1}^{m}(z+v_i)^{m_i}\prod_{i=1}^{n}(z+z_i)^{n_i+k} + C_2\prod_{i=1}^{s}(z+w_i)^{l_i} = P_k(z). \tag{5}$$

It follows that

$$\sum_{i=1}^{s} l_i = \sum_{i=1}^{n}(n_i+k) + \sum_{i=1}^{m} m_i = nk + N + M$$

and $C_2 = -A$. Thus, by (5),

$$\prod_{i=1}^{m}(1+v_i t)^{m_i}\prod_{i=1}^{n}(1+z_i t)^{n_i+k} - \prod_{i=1}^{s}(1+w_i t)^{l_i} = t^{k+N+M}Q(t),$$

where $Q(t) = t^{(n-1)k} P_k(1/t)/A$. Then Q is a polynomial of degree less than $(n-1)k$, and it follows that

$$\frac{\prod_{i=1}^{m}(1+v_i t)^{m_i} \prod_{i=1}^{n}(1+z_i t)^{n_i+k}}{\prod_{i=1}^{s}(1+w_i t)^{l_i}} = 1 + \frac{t^{k+N+M} Q(t)}{\prod_{i=1}^{s}(1+w_i t)^{l_i}}. \tag{6}$$

Note that, for t near 0,

$$\frac{t^{k+N+M} Q(t)}{\prod_{i=1}^{s}(1+w_i t)^{l_i}} = t^{k+N+M}(a_0 + a_1 t + \cdots), \tag{7}$$

where $a_0 \neq 0$. Logarithmic differentiation of both sides of (6) and (7) shows that

$$\sum_{i=1}^{m} \frac{m_i v_i}{1+v_i t} + \sum_{i=1}^{n} \frac{(n_i+k) z_i}{1+z_i t} - \sum_{i=1}^{s} \frac{l_i w_i}{1+w_i t} = O(t^{k+N+M-1}) \quad \text{as } t \to 0. \tag{8}$$

Set

$$S_1 = \{v_1, v_2, \ldots, v_m\} \cap \{z_1, z_2, \ldots, z_n\}$$

and

$$S_2 = \{v_1, v_2, \ldots, v_m\} \cap \{w_1, w_2, \ldots, w_s\}.$$

We consider four cases.

Case 1: $S_1 = S_2 = \emptyset$. Let $z_{n+i} = v_i$ when $1 \le i \le m$ and

$$N_i = \begin{cases} n_i + k & \text{when } 1 \le i \le n, \\ m_{i-n} & \text{when } n+1 \le i \le n+m. \end{cases}$$

In this case, (8) may be rewritten:

$$\sum_{i=1}^{n+m} \frac{N_i z_i}{1+z_i t} - \sum_{i=1}^{s} \frac{l_i w_i}{1+w_i t} = O(t^{k+N+M-1}) \quad \text{as } t \to 0. \tag{9}$$

Comparing the coefficients of t^j when $j = 0, 1, \ldots, k+N+M-2$ in (9), we deduce that

$$\sum_{i=1}^{n+m} N_i z_i^j - \sum_{i=1}^{s} l_i w_i^j = 0 \quad \forall j \in \{1, 2, \ldots, k+N+M-1\}. \tag{10}$$

Let $z_{n+m+i} = w_i$ when $1 \le i \le s$. Noting that $\sum_{i=1}^{n+m} N_i - \sum_{i=1}^{s} l_i = 0$ and using (10), we deduce that the system of linear equations

$$\sum_{i=1}^{n+m+s} z_i^j x_i = 0, \tag{11}$$

where $0 \le j \le k+N+M-1$, has a nonzero solution

$$(x_1, \ldots, x_{n+m}, x_{n+m+1}, \ldots, x_{n+m+s}) = (N_1, \ldots, N_{n+m}, -l_1, \ldots, -l_s).$$

If $k + N + M \geq n + m + s$, then the determinant $\det(z_i^j)_{(n+m+s)\times(n+m+s)}$ of the coefficients of the system of equations (11), where $0 \leq j \leq n + m + s - 1$, is equal to zero, by Cramer's rule (see for instance [6]). However, the z_i are distinct complex numbers when $1 \leq i \leq n + m + s$, and the determinant is a Vandermonde determinant, so cannot be 0 (see [6]), which is a contradiction.

Hence, we conclude that $k + N + M < n + m + s$. It follows from this and the two inequalities $N = \sum_{i=1}^{n} n_i \geq n$ and $M = \sum_{i=1}^{m} m_i \geq m$ that $s \geq k + 1$.

Case 2: $S_1 \neq \emptyset$ and $S_2 = \emptyset$. Without loss of generality, we may and shall assume that $S_1 = \{v_1, v_2, \ldots, v_{M_1}\}$. Thus, $v_i = z_i$ when $1 \leq i \leq M_1$. Let $M_3 = m - M_1$. We consider two subcases.

Subcase 2.1: $M_3 \geq 1$. Set $z_{n+i} = v_{M_1+i}$ when $1 \leq i \leq M_3$. If $M_1 < n$, then set

$$N_i = \begin{cases} n_i + m_i + k & \text{when } 1 \leq i \leq M_1, \\ n_i + k & \text{when } M_1 + 1 \leq i \leq n, \\ m_{M_1-n+i} & \text{when } n + 1 \leq i \leq n + M_3. \end{cases}$$

If $M_1 = n$, then set

$$N_i = \begin{cases} n_i + m_i + k & \text{when } 1 \leq i \leq M_1 = n, \\ m_{M_1-n+i} & \text{when } n + 1 \leq i \leq n + M_3. \end{cases}$$

Subcase 2.2: $M_3 = 0$. If $M_1 < n$, then set

$$N_i = \begin{cases} n_i + m_i + k & \text{when } 1 \leq i \leq M_1, \\ n_i + k & \text{when } M_1 + 1 \leq i \leq n. \end{cases}$$

If $M_1 = n$, then set

$$N_i = n_i + m_i + k \quad \text{when } \leq i \leq M_1 = n.$$

In both subcases, (8) may be rewritten:

$$\sum_{i=1}^{n+M_3} \frac{N_i z_i}{1 + z_i t} - \sum_{i=1}^{s} \frac{l_i w_i}{1 + w_i t} = O(t^{k+N+M-1}) \quad \text{as } t \to 0,$$

where $0 \leq M_3 \leq m - 1$. Using the argument of Case 1, we deduce that $s \geq k + 1$.

Case 3: $S_1 = \emptyset$ and $S_2 \neq \emptyset$. Without loss of generality, we may and shall assume that $S_2 = \{v_1, v_2, \ldots, v_{M_2}\}$. Thus, $v_i = w_i$ when $1 \leq i \leq M_2$. Let $M_4 = m - M_2$. We consider two subcases.

Case 3.1: $M_4 \geq 1$. Set $w_{s+i} = v_{M_2+i}$, $1 \leq i \leq M_4$. If $M_2 < s$, then set

$$L_i = \begin{cases} l_i - m_i & \text{when } 1 \leq i \leq M_2, \\ l_i & \text{when } M_2 + 1 \leq i \leq s, \\ -m_{M_2-s+i} & \text{when } s + 1 \leq i \leq s + M_4. \end{cases}$$

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If $M_2 = s$, set

$$L_i = \begin{cases} l_i - m_i & \text{when } 1 \le i \le M_2 = s, \\ -m_{M_2 - s + i} & \text{when } s + 1 \le i \le s + M_4. \end{cases}$$

Case 3.2: $M_4 = 0$. If $M_2 < s$, then set

$$L_i = \begin{cases} l_i - m_i & \text{when } 1 \le i \le M_2, \\ l_i & \text{when } M_2 + 1 \le i \le s. \end{cases}$$

If $M_2 = s$, then set

$$L_i = l_i - m_i \quad \text{when } 1 \le i \le M_2 = s.$$

In both subcases, (8) may be rewritten:

$$\sum_{i=1}^{n} \frac{n_i z_i}{1 + z_i t} - \sum_{i=1}^{s+M_4} \frac{L_i w_i}{1 + w_i t} = O(t^{k+N+M-1}) \quad \text{as } t \to 0,$$

where $0 \le M_4 \le m - 1$. Using the argument of Case 1, we deduce that $s \ge k + 1$.

Case 4: $S_1 \ne \emptyset$ and $S_2 \ne \emptyset$. Without loss of generality, we may and shall assume that $S_1 = \{v_1, v_2, \dots, v_{M_1}\}$, $S_2 = \{w_1, w_2, \dots, w_{M_2}\}$, and $v_i = z_i$ when $1 \le i \le M_1$ and $w_i = v_{M_1+i}$ when $1 \le i \le M_2$. Set $M_5 = m - M_2 - M_1$. We consider two subcases.

Case 4.1: $M_5 \ge 1$. Set $z_{n+i} = v_{M_1+M_2+i}$, $1 \le i \le M_5$. If $M_1 < n$, then set

$$N_i = \begin{cases} n_i + m_i + k & \text{when } 1 \le i \le M_1, \\ n_i + k & \text{when } M_1 + 1 \le i \le n, \\ m_{M_1+M_2-n+i} & \text{when } n + 1 \le i \le n + M_5. \end{cases}$$

If $M_1 = n$, then set

$$N_i = \begin{cases} n_i + m_i + k & \text{when } 1 \le i \le M_1 = n, \\ m_{M_1+M_2-n+i} & \text{when } n + 1 \le i \le n + M_5. \end{cases}$$

If $M_2 < s$, then set

$$L_i = \begin{cases} l_i - m_{M_1+i} & \text{when } 1 \le i \le M_2, \\ l_i & \text{when } M_2 + 1 \le i \le s. \end{cases}$$

If $M_2 = s$, then set

$$L_i = l_i - m_{M_1+i} \quad \text{when } 1 \le i \le M_2 = s.$$

Case 4.2: $M_5 = 0$. If $M_1 < n$, then set

$$N_i = \begin{cases} n_i + m_i + k & \text{when } 1 \le i \le M_1, \\ n_i + k & \text{when } M_1 + 1 \le i \le n. \end{cases}$$

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If $M_1 = n$, then set

$$N_i = n_i + m_i + k \quad \text{when } 1 \le i \le M_1 = n.$$

And, if $M_2 < s$, set

$$L_i = \begin{cases} l_i - m_{M_1+i} & \text{when } 1 \le i \le M_2, \\ l_i & \text{when } M_2 + 1 \le i \le s. \end{cases}$$

If $M_2 = s$, then set

$$L_i = l_i - m_{M_1+i} \quad \text{when } 1 \le i \le M_2 = s.$$

In both subcases, (8) may be rewritten:

$$\sum_{i=1}^{n+M_5} \frac{N_i z_i}{1 + z_i t} - \sum_{i=1}^{s} \frac{L_i w_i}{1 + w_i t} = O(t^{k+N+M-1}) \quad \text{as } t \to 0,$$

where $0 \le M_5 \le m - 2$. Using the argument of Case 1, we deduce that $s \ge k + 1$.

This completes the proof of Lemma 11. □

3. Proof of Theorem 5

First we show that $\mathcal{F}$ is normal on the set D', defined to be $\{z \in D : h(z) \neq 0\}$. Suppose that $\mathcal{F}$ is not normal at $z_0 \in D'$. We may assume that D is the disc $\Delta(0, 1)$ with center 0 and radius 1, and that $h(z_0) = 1$. By Lemma 8, there exist points $z_j \in D$ tending to z_0, functions $f_j \in \mathcal{F}$, positive numbers ρ_j tending to 0, and a nonconstant zero-free meromorphic function g of order at most two such that

$$g_n(\xi) = \frac{f_n(z_n + \rho_n \xi)}{\rho_n^k} \to g(\xi),$$

locally uniformly in ξ in $\mathbb{C}$ with respect to the spherical metric.

We claim that the function $g^{(k)} - 1$ has at most k distinct zeros. With a view to a contradiction, suppose that $g^{(k)} - 1$ has $k + 1$ distinct zeros ξ_j when $1 \le j \le k + 1$. By Lemma 8, $g^{(k)}$ is not identically 1. By Hurwitz's theorem and because

$$g_n^{(k)}(\xi) - h(z_n + \rho_n \xi) = f_n^{(k)}(z_n + \rho_n \xi) - h(z_n + \rho_n \xi) \to g^{(k)}(\xi) - 1$$

as $n \to \infty$, there exist points $\xi_{n,j}$ when $j = 1, 2, \ldots, k + 1$ such that $\xi_{n,j} \to \xi_j$ and

$$f_n^{(k)}(z_n + \rho_n \xi_{n,j}) = h(z_n + \rho_n \xi_{n,j}).$$

However, $f_n^{(k)}(z) = h(z)$ has at most k distinct roots in D, and $z_n + \rho_n \xi_{n,j} \to z_0$, which is a contradiction, and proves our claim.

By Lemma 9, g is a rational function. But this contradicts Lemma 10, which shows that $\mathcal{F}$ is normal in D'.

We now prove that $\mathcal{F}$ is normal at points z where $h(z) = 0$. By making standard normalizations, we may assume that

$$h(z) = z^m + a_{m+1}z^{m+1} + \cdots = z^m b(z) \quad \forall z \in \Delta,$$

where $m \geq 1$, $b(0) = 1$, and $h(z) \neq 0$ when $0 < |z| < 1$. Let

$$\mathcal{F}_1 := \left\{F : F(z) = \frac{f(z)}{z^m}, f \in \mathcal{F}\right\}.$$

For all $f \in \mathcal{F}$, the function f has no zeros; hence, for all $F \in \mathcal{F}_1$, the function F has no zeros, and 0 is a pole of F with multiplicity at least m. We shall prove that $\mathcal{F}_1$ is normal at 0. Suppose otherwise: then, by Lemma 8, there exist points $z_j \in \Delta$ tending to 0, functions $F_j \in \mathcal{F}$, positive numbers ρ_j tending to 0, and a nonconstant zero-free meromorphic function g of order at most two such that

$$g_n(\xi) = \frac{F_n(z_n + \rho_n\xi)}{\rho_n^k} \to g(\xi),$$

locally uniformly on $\mathbb{C}$ with respect to the spherical metric. We distinguish two cases, following Pang *et al.* [7].

Case 1: (z_n/ρ_n) has a convergent subsequence. We still denote the subsequence by (z_n/ρ_n) and its limit by α. Let $\tilde{g}(\xi) = g(\xi - \alpha)$. Then

$$\frac{F_n(\rho_n\xi)}{\rho_n^k} = \frac{F_n(z_n + \rho_n(\xi - z_n/\rho_n))}{\rho_n^k} \to g(\xi - \alpha) = \tilde{g}(\xi),$$

the convergence being locally uniform in ξ in $\mathbb{C}$ with respect to the spherical metric, hence uniform on compact subsets of $\mathbb{C}$ disjoint from the poles of $\tilde{g}$. Clearly, $\tilde{g}$ has no zeros, and the pole of $\tilde{g}$ at 0 has order at least m. Now define $G_n(\xi) = f_n(\rho_n\xi)/\rho_n^{k+m}$ and $G(\xi) = \xi^m\tilde{g}(\xi)$. Then

$$G_n(\xi) = \frac{(\rho_n\xi)^m F_n(\rho_n\xi)}{\rho_n^{k+m}} \to \xi^m\tilde{g}(\xi) = G(\xi),$$

uniformly on compact subsets of $\mathbb{C}$ disjoint from the poles of $\tilde{g}$. Since $\tilde{g}$ has a pole of order at least m at 0, it follows that $G(0) \neq 0$; since $\tilde{g}$ has no zeros, it follows that G has no zeros. Further,

$$\lim_{n\to\infty} \frac{h(\rho_n\xi)}{\rho_n^m} = \xi^m,$$

uniformly on compact subsets of $\mathbb{C}$. So,

$$G_n^{(k)}(\xi) - \frac{h(\rho_n\xi)}{\rho_n^k} = \frac{f_n^{(k)}(\rho_n\xi) - h(\rho_n\xi)}{\rho_n^m} \to G^{(k)}(\xi) - \xi^m.$$

Since $f_n^{(k)} - h$ has at most k distinct zeros in the ball $\Delta(z_0, \delta)$ with center z_0 and radius δ, as discussed above, the equation $G^{(k)}(\xi) = \xi^m$ has at most k distinct roots in $\mathbb{C}$.

However, by Lemma 9, G is a rational function, which contradicts Lemma 11.

Case 2. (z_n/ρ_n) has a subsequence tending to ∞. We still denote the subsequence by (z_n/ρ_n). By simple calculation,

$$\begin{aligned} F^{(k)}(z) &= \frac{f^{(k)}(z)}{z^m} - \sum_{l=1}^{k} C_k^l \frac{(z^m)^{(l)} f^{(k-l)}(z)}{z^m} \\ &= \frac{f^{(k)}(z)}{z^m} - \sum_{l=1}^{k} C_l f^{(k-l)}(z) \frac{1}{z^l}, \end{aligned} \tag{12}$$

where

$$C_l = \begin{cases} C_k^l m(m-1)\cdots(m-l+1) & \text{when } l \le m, \\ 0 & \text{when } l > m. \end{cases}$$

From (12) and the identity $\rho_n^l g_n^{(k-l)}(\xi) = F_n^{(k-l)}(z_n + \rho_n \xi)$, we obtain

$$\begin{aligned} g_n^{(k)}(\xi) &= F_n^{(k)}(z_n + \rho_n \xi) \\ &= \frac{f_n^{(k)}(z_n + \rho_n \xi)}{(z_n + \rho_n \xi)^m} - \sum_{l=1}^{k} C_l F_n^{(k-l)}(z_n + \rho_n \xi) \frac{1}{(z_n + \rho_n \xi)^l} \\ &= \frac{f_n^{(k)}(z_n + \rho_n \xi)}{(z_n + \rho_n \xi)^m} - \sum_{l=1}^{k} C_l g_n^{(k-l)}(\xi) \frac{1}{(z_n/\rho_n + \xi)^l}. \end{aligned}$$

Hence,

$$\frac{f_n^{(k)}(z_n + \rho_n \xi)}{h(z_n + \rho_n \xi)} = \bigg[g_n^{(k)}(\xi) + \sum_{l=1}^{k} g_n^{(k-l)}(\xi) \frac{C_l}{(z_n/\rho_n + \xi)^l} \bigg] \frac{1}{b(z_n + \rho_n \xi)}.$$

Now $\lim_{n\to\infty} b(z_n + \rho_n \xi) = 1$ and $\lim_{n\to\infty} 1/(z_n/\rho_n + \xi) = 0$. So,

$$\frac{f_n^{(k)}(z_n + \rho_n \xi) - h(z_n + \rho_n \xi)}{h(z_n + \rho_n \xi)} \to g^{(k)}(\xi) - 1,$$

uniformly on compact subsets of $\mathbb{C}$ disjoint from the poles of g.

Since F has no zeros and $f^{(k)} - h$ has at most k distinct zeros, as discussed in Case 1, we see that g has no zeros and $g^{(k)} - 1$ has at most k distinct zeros. However, by Lemma 9, g is a rational function, and this contradicts Lemma 10.

We have thus proved that $\mathcal{F}_1$ is normal at 0. It remains to prove that $\mathcal{F}$ is normal at 0. Since $\mathcal{F}_1$ is normal at 0 and $F(0) = \infty$ for each $F \in \mathcal{F}_1$, there exists $\delta > 0$ such that $|F(z)| \ge 1$ for all $F \in \mathcal{F}_1$ and all $z \in \Delta(0, \delta)$. If $f \in \mathcal{F}$, then f has no zeros in $\Delta(0, \delta)$, so $1/f$ is analytic in $\Delta(0, \delta)$. Therefore,

$$\left| \frac{1}{f(z)} \right| = \left| \frac{1}{F(z)} \frac{1}{z^m} \right| \le \frac{2^m}{\delta^m} \quad \forall z \in \Delta\Big(0, \frac{1}{2}\delta\Big)$$

for all $f \in \mathcal{F}$. By the maximum principle and Montel's theorem, $\mathcal{F}$ is normal at 0, and thus $\mathcal{F}$ is normal in D. This completes the proof of Theorem 5.

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REMARK 12. In the proof of Theorem 5, we just use a very special case of Lemma 11, namely, when $a(z) = z^m$.

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