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ON CERTAIN PAIRS OF AUTOMORPHISMS OF RINGS

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Abstract

In this paper we prove algebraic generalizations of some results of C. J. K. Batty and A. B. Thaheem, concerned with the identity $\alpha+\alpha^{-1}=\beta+\beta^{-1}$ where α and β are automorphisms of a C^*-algebra. The main result asserts that if automorphisms α, β of a semiprime ring R satisfy $\alpha+\alpha^{-1}=\beta+\beta^{-1}$ then there exist invariant ideals U_1, U_2 and U_3 of R such that $U_i \cap U_j = 0$, $i \neq j$, $U_1 \oplus U_2 \oplus U_3$ is an essential ideal, $\alpha=\beta$ on U_1, $\alpha=\beta^{-1}$ on U_2, and $\alpha^2=\beta^2=\alpha^{-2}$ on U_3. Furthermore, if the annihilator of any ideal in R is a direct summand (in particular, if R is a von Neumann algebra), then $U_1 \oplus U_2 \oplus U_3 = R$.

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Introduction

Over the last ten years a lot of work has been done on the operator equation

$$(*) \qquad \alpha+\alpha^{-1}=\beta+\beta^{-1}$$

where α and β are $*$-automorphisms of a von Neumann algebra. We refer to some recent papers [1, 5] for a detailed discussion on this equation and a more comprehensive bibliography.

It seems that the culminating results in the series of papers concerning the equation $(*)$ can be found in the paper [1] of Batty, where the treatment of this problem was extended from von Neumann algebras to C^*-algebras. The main result in [1] gives a condition which is both necessary and sufficient for solution of the equation in C^*-algebras, and which produces as corollaries the necessary conditions which we will establish purely algebraically.

Our work had been motivated by the following two results of Batty [1; Corollary 3.2, Corollary 3.5]:

THEOREM A. *Suppose that* $*$*-automorphisms* α *and* β *of a* C^**-algebra* R *satisfy* $(*)$. *Then there exist ideals* I_1, I_2 *and* I_3 *of* R, *each invariant under* α, β, α^{-1} *and* β^{-1}, *such that* $I_1 \cap I_2 \cap I_3 = 0$, $I_3 \subseteq I_1 + I_2$ *and, for every* x *in* R, $\beta(x) - \alpha(x) \in I_1$, $\beta(x) - \alpha^{-1}(x) \in I_2$, $\beta^2(x) - \alpha^2(x) \in I_3$, $\beta^2(x) - \alpha^{-2}(x) \in I_3$.

The next theorem was also proved by Thaheem in [4, 5].

THEOREM B. *Suppose that* $*$*-automorphisms* α *and* β *of a von Neumann algebra* R *satisfy* $(*)$. *Then* $R = U_1 \oplus U_2 \oplus U_3$, *where* U_1, U_2 *and* U_3 *are von Neumann subalgebras of* R, *invariant under* α *and* β, *such that* $\alpha = \beta$ *on* U_1, $\alpha = \beta^{-1}$ *on* U_2, *and* $\alpha^2 = \beta^2 = \alpha^{-2}$ *on* U_3.

In this paper we will generalize both Theorems A and B. Our methods are much more elementary than those employed by the other authors. Roughly speaking, we will show that the presence of analysis in the study of equation $(*)$ is sometimes superfluous. We will see that Theorem A remains true if R is an arbitrary semiprime ring, and that Theorem B holds if R is a semiprime ring in which the annihilator of any ideal is a direct summand. Moreover, if we no longer insist that $U_1 \oplus U_2 \oplus U_3$ is R but rather just a "large piece" of R (more precisely, an essential ideal), then Theorem B holds in any semiprime ring R.

In particular, our results imply that the assumption that α and β preserve adjoints, which is required in Theorems A and B, can be removed.

We remark that the study of equation $(*)$ is much simpler if one assumes that α and β commute. It turns out that in this case the presence of the ideal U_3 in Theorem B is not necessary (see, for example, [3, 6]). An algebraic generalization of this result is presented in our forthcoming paper [2].

Preliminaries

We recall a few definitions and easy results. Let R be a ring. Then R is said to be prime if $aRb = 0$ implies $a = 0$ or $b = 0$. A von Neumann algebra is prime if and only if it is a factor (that is, its center consists of

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scalar multiples of the identity). If $aRa = 0$ implies $a = 0$, then R is called semiprime. Every C^*-algebra R is semiprime (for $0 \neq aa^*a \in aRa$ if $a \neq 0$).

REMARK A. Let R be semiprime. Suppose that $a, b \in R$ satisfy $aRb = 0$. Then we also have $(bRa)R(bRa) = 0$, $abRab = 0$, $baRba = 0$, and therefore $bRa = 0$, $ab = 0$, $ba = 0$ by the semiprimeness of R. Observe that the left and the right annihilators of an ideal U of R coincide. It will be denoted by $\mathrm{Ann}(U)$. Note that $U \cap \mathrm{Ann}(U) = 0$, and that $U \oplus \mathrm{Ann}(U)$ is an essential ideal.

We will be especially concerned with semiprime rings R in which the annihilator of any ideal is a direct summand; that is, $\mathrm{Ann}(U) \oplus \mathrm{Ann}(\mathrm{Ann}(U)) = R$ for any ideal U of R. Note that every von Neumann algebra has this property; namely, the annihilator of any ideal is of the form pR for some central projection p in R. More generally, the same is true for AW^*-algebras.

REMARK B. Let α be an automorphism of a ring R. Suppose that the ideal I of R is invariant under α and α^{-1}, that is, α maps I onto itself. One can easily verify that in this case the two-sided annihilator Ann(I) of I is also invariant under α and α^{-1}.

The results

We begin our investigation of the equation $(*)$ by considering a somewhat more general situation where automorphisms α, β and γ satisfy $\alpha + \gamma = \beta + 1$.

LEMMA 1. *Let α, β, γ be automorphisms of a ring R. If $\alpha + \gamma = \beta + 1$ then*
(1) $(\alpha - 1)(x)R((\beta + 1)(\alpha - \beta))(w)R(\alpha - \beta)(z) = 0$,
(2) $(\alpha - 1)(x)R((\beta + 1)(\alpha - 1))(w)R(\alpha - \beta)(z) = 0$ *for all* $x, w, z \in R$.

PROOF. From $\alpha - \beta = 1 - \gamma$ it follows that

$$\begin{aligned}(\alpha - \beta)(x)\alpha(y) &+ \beta(x)(\alpha - \beta)(y)\\ &= (\alpha - \beta)(xy) = (1 - \gamma)(xy)\\ &= (1 - \gamma)(x)y + \gamma(x)(1 - \gamma)(y)\\ &= (\alpha - \beta)(x)y + \gamma(x)(\alpha - \beta)(y).\end{aligned}$$

Thus $(\alpha - \beta)(x)(\alpha - 1)(y) + (\beta - \gamma)(x)(\alpha - \beta)(y) = 0$. That is,

(3) $\quad (\alpha - \beta)(x)(\alpha - 1)(y) + (\alpha - 1)(x)(\alpha - \beta)(y) = 0 \quad$ for all $x, y \in R$

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since $\beta - \gamma = \alpha - 1$ by assumption. Replacing y by yz in (3) we obtain

$$(\alpha-\beta)(x)(\alpha-1)(y)\alpha(z) + (\alpha-\beta)(x)y(\alpha-1)(z) \\ + (\alpha-1)(x)(\alpha-\beta)(y)\alpha(z) + (\alpha-1)(x)\beta(y)(\alpha-\beta)(z) = 0.$$

By (3) this relation reduces to

$$(4) \qquad (\alpha-\beta)(x)y(\alpha-1)(z) \\ + (\alpha-1)(x)\beta(y)(\alpha-\beta)(z) = 0 \quad \text{for all } x, y, z \in R.$$

Replacing y by $y(\alpha-\beta)(w)u$ in (4) we get

$$(\alpha-\beta)(x)y(\alpha-\beta)(w)u(\alpha-1)(z) \\ = -(\alpha-1)(x)\beta(y)(\beta(\alpha-\beta))(w)\beta(u)(\alpha-\beta)(z).$$

But on the other hand, using (4) twice we obtain

$$(\alpha-\beta)(x)y\{(\alpha-\beta)(w)u(\alpha-1)(z)\} \\ = -\{(\alpha-\beta)(x)y(\alpha-1)(w)\}\beta(u)(\alpha-\beta)(z) \\ = (\alpha-1)(x)\beta(y)(\alpha-\beta)(w)\beta(u)(\alpha-\beta)(z).$$

Comparing the last two relations we obtain (1). In a similar fashion, by substituting $y(\alpha-1)(w)u$ for y in (4), one shows that (2) holds.

Corollary 1. *Suppose that automorphisms α, β, γ of a prime ring R satisfy $\alpha + \gamma = \beta + 1$. If $\alpha \neq \beta$ and $\alpha \neq 1$ then $\alpha = \beta\gamma$, $\gamma = \beta\alpha$, and $\beta^2 = 1$.*

Proof. From (1) it follows immediately that $\beta(\alpha-\beta) = -(\alpha-\beta)$. By assumption, $\alpha - \beta = 1 - \gamma$, therefore this relation yields $\beta - \alpha = \beta(1-\gamma) = \beta - \beta\gamma$ which means that $\alpha = \beta\gamma$. Similarly, by (2) we have $\beta(\alpha-1) = -(\alpha-1)$; since $1 - \alpha = \gamma - \beta$ it follows that $\gamma = \beta\alpha$. According to both identities, $\alpha = \beta\gamma$ and $\gamma = \beta\alpha$, we are forced to conclude that $\beta^2 = 1$.

Remark 1. The next simple example illustrates Corollary 1 (compare with [1; Proposition 2.1]). Let R be an algebra with unit element e, and let b in R be such that $b^2 = e$. Define the inner automorphism β by $\beta(x) = bxb$. Let λ be any scalar different from 1 and -1 and define the inner automorphism γ by $\gamma(x) = (1-\lambda^2)^{-1}(e+\lambda b)x(e-\lambda b)$. Note that $\beta\gamma + \gamma = \beta + 1$.

As a special case of Corollary 1 we obtain an extension of [1; Corollary 3.3].

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COROLLARY 2. *Suppose that automorphisms* α *and* β *of a prime ring* R *satisfy* $\alpha + \alpha^{-1} = \beta + \beta^{-1}$. *If* $\alpha \neq \beta$ *and* $\alpha \neq \beta^{-1}$ *then* $\alpha^2 = \beta^2 = \alpha^{-2}$.

PROOF. We have $\alpha\beta + \alpha^{-1}\beta = \beta^2 + 1$; now apply Corollary 1.

LEMMA 2. *If automorphisms* α *and* β *of a semiprime ring* R *satisfy* $\alpha + \alpha^{-1} = \beta + \beta^{-1}$, *then*

(5) $$(\alpha - \beta^{-1})(x)R(\alpha^2 - \beta^2)(y) = 0 \quad \textit{for all } x, y \in R,$$

(6) $$(\alpha - \beta)(x)R(\alpha^2 - \beta^{-2})(y) = 0 \quad \textit{for all } x, y \in R.$$

PROOF. We have $\alpha\beta + \alpha^{-1}\beta = \beta^2 + 1$. Therefore Lemma 1 implies that

$$(\alpha\beta - 1)(x)R((\beta^2 + 1)(\alpha\beta - \beta^2))(w)R(\alpha\beta - \beta^2)(z) = 0$$

for all $x, w, z \in R$. Since β is onto we then also have

(7) $(\alpha - \beta^{-1})(x)R((\beta^2 + 1)(\alpha - \beta))(w)R(\alpha - \beta)(z) = 0$ for all $x, w, z \in R$.

We have

$$\beta^2(\alpha - \beta) + (\alpha - \beta) = \beta^2(\beta^{-1} - \alpha^{-1}) + (\alpha - \beta) = \alpha - \beta^2\alpha^{-1} = (\alpha^2 - \beta^2)\alpha^{-1},$$

therefore it follows from (7) that

(8) $$(\alpha - \beta^{-1})(x)R(\alpha^2 - \beta^2)(y)R(\alpha - \beta)(z) = 0 \quad \text{for all } x, y, z \in R.$$

The range of $\alpha^2 - \beta^2$ is contained in the range of $\alpha - \beta$; indeed, we have $\alpha^2 - \beta^2 = \alpha(\alpha + \alpha^{-1}) - \beta(\beta + \beta^{-1}) = (\alpha - \beta)(\alpha + \alpha^{-1})$. Hence (8) yields

$$(\alpha - \beta^{-1})(x)R(\alpha^2 - \beta^2)(y)R(\alpha^2 - \beta^2)(z) = 0 \quad \text{for all } x, y, z \in R.$$

But then (5) holds by the semiprimeness of R. Noting that $\alpha^2 + \alpha^{-2} = \beta^2 + \beta^{-2}$, and then using the analogous approach as in the proof of (5), one proves (6).

LEMMA 3. *If automorphisms* α *and* β *of a semiprime ring* R *satisfy* $\alpha + \alpha^{-1} = \beta + \beta^{-1}$, *then* α *commutes with* β^2 *and* β *commutes with* α^2.

PROOF. Let us show that α commutes with β^2. The initial hypothesis yields

(9) $$\alpha(\alpha - \beta) = (\alpha - \beta)\beta^{-1}.$$

By (6) it follows that

$$(\alpha(\alpha - \beta))(x)\alpha(R)(\alpha(\alpha^2 - \beta^{-2}))(y) = 0 \quad \text{for all } x, y \in R.$$

In view of (9) this relation implies that

$$(\alpha-\beta)(x)R(\alpha^3-\alpha\beta^{-2})(y)=0 \quad \text{for all } x, y \in R.$$

Substituting $\alpha(y)$ for y in (6) we obtain

$$(\alpha-\beta)(x)R(\alpha^3-\beta^{-2}\alpha)(y)=0 \quad \text{for all } x, y \in R.$$

Comparing the last two relations we get

$$(10) \qquad (\alpha-\beta)(x)R(\alpha\beta^{-2}-\beta^{-2}\alpha)(y)=0 \quad \text{for all } x, y \in R.$$

Multiply the identity $\alpha+\alpha^{-1}=\beta+\beta^{-1}$ from the left by β and from the right by α; then we get $\beta\alpha^2-\beta^2\alpha=\alpha-\beta$. Since $\alpha-\beta=\beta^{-1}-\alpha^{-1}$, and since $\beta^2=\alpha^2+\alpha^{-2}-\beta^{-2}$, it follows that $\beta^{-1}+\alpha^3-\beta\alpha^2=\beta^{-2}\alpha$. Consequently

$$\alpha\beta^{-2}-\beta^{-2}\alpha=(\alpha-\beta)(\beta^{-2}-\alpha^2).$$

Therefore (10) implies that

$$(\alpha\beta^{-2}-\beta^{-2}\alpha)(y)R(\alpha\beta^{-2}-\beta^{-2}\alpha)(y)=0$$

for every $y \in R$. But then $\alpha\beta^{-2}=\beta^{-2}\alpha$ since R is semiprime. Thus α and β^2 commute. For the sake of symmetry we omit the proof of the commutativity of α^2 and β.

Corollary 3. *Let R be a semiprime ring with unit element and containing the element $1/2$. If inner automorphisms α, β of R satisfy $\alpha+\alpha^{-1}=\beta+\beta^{-1}$, then they commute.*

Proof. Let $a, b \in R$ be such that $\alpha(x)=axa^{-1}$ and $\beta(x)=bxb^{-1}$. By assumption, $axa^{-1}+a^{-1}xa=bxb^{-1}+b^{-1}xb$ for all $x \in R$. In particular, $2a=bab^{-1}+b^{-1}ab$. Multiplying from the right by b we obtain

$$(11) \qquad 2ab=ba+b^{-1}ab^2.$$

By Lemma 3, α and β^2 commute. Hence $ab^2=cb^2a$ for some c in the center of R. By (11) we then have $2ab=(1+c)ba$. Since R contains the element $1/2$ it follows that $ab=c_1ba$, where c_1 is an invertible element in the center of R. But then α and β commute.

Remark 2. The case where commuting automorphisms α, β of a semiprime ring R satisfy $\alpha+\alpha^{-1}=\beta+\beta^{-1}$ is considered in our paper [2]. In particular, it was shown that if R is prime of characteristic not 2 then either $\alpha=\beta$ or $\alpha=\beta^{-1}$. Combining this with Corollary 3 we obtain the following result which generalizes [3; Corollary 2.1]: Let R be a prime ring with unit element and containing the element $1/2$. If inner automorphisms α, β of R satisfy $\alpha + \alpha^{-1} = \beta + \beta^{-1}$ then either $\alpha = \beta$ or $\alpha = \beta^{-1}$.

We now come to the main result of this paper.

THEOREM 1. *Let α and β be automorphisms of a semiprime ring R such that $\alpha + \alpha^{-1} = \beta + \beta^{-1}$. Then there exists ideals U_1, U_2 and U_3 of R such that*

(i) *$U_i \cap U_j = 0$, $i \neq j$, and $U_1 \oplus U_2 \oplus U_3$ is an essential ideal of R. Moreover, if the annihilator of any ideal in R is a direct summand (in particular, if R is a von Neumann algebra), then $U_1 \oplus U_2 \oplus U_3 = R$,*

(ii) *U_i are invariant under α, β, α^{-1} and β^{-1},*

(iii) *$\alpha = \beta$ on U_1,*

(iv) *$\alpha = \beta^{-1}$ on U_2,*

(v) *$\alpha^2 = \beta^2 = \alpha^{-2}$ on U_3.*

REMARK 3. In [5] Thaheem constructed an example of automorphisms α and β satisfying $\alpha + \alpha^{-1} = \beta + \beta^{-1}$ on a von Neumann algebra R but there is no decomposition of R for which $\alpha = \beta$ on the one part and $\alpha = \beta^{-1}$ on the other part. Thus the presence of an ideal U_3 in Theorem 1 is really necessary. In Thaheem's example the algebra R was not prime. We do not know whether the equation $\alpha + \alpha^{-1} = \beta + \beta^{-1}$ has any nontrivial solutions in prime rings (in the sense that $\alpha \neq \beta$ and $\alpha \neq \beta^{-1}$). In order to find such a solution one can assume that $\alpha^2 = \beta^2 = \alpha^{-2}$ (Corollary 2) and that α, β are not both inner (Remark 2).

PROOF OF THEOREM 1. Let U_0 be an ideal of R generated by all $(\alpha^2 - \beta^{-2})(x)$, $x \in R$. We set $V = \mathrm{Ann}(U_0)$ and $U_1 = \mathrm{Ann}(V)$. By Remark A we have $U_1 \cap V = 0$ and $U_1 \oplus V$ is an essential ideal. From Lemma 3 we see that the mapping $\alpha^2 - \beta^{-2}$ commutes with α, β, α^{-1} and β^{-1}. Simple calculations show that this implies that U_0 is invariant under α, β, α^{-1} and β^{-1}. But then, by Remark B, the same is true for ideals V and U_1.

Take $u_1 \in U_1$. Since U_1 is invariant under α and β, $(\alpha - \beta)(u_1)$ lies in U_1. However, from Lemma 2 (and Remark A) it follows that the range of $\alpha - \beta$ lies in $\mathrm{Ann}(U_0) = V$. Since $U_1 \cap V = 0$ we then have $\alpha(u_1) = \beta(u_1)$. Thus we have proved (iii).

Let V_1 be an ideal of R generated by all $(\beta^2 - \beta^{-2})(v)$, $v \in V$. Of course, $V_1 \subseteq V$. We define $U_3 = \mathrm{Ann}(V_1) \cap V$ and $U_2 = \mathrm{Ann}(U_3) \cap V$. Since $U_2 \subseteq \mathrm{Ann}(U_3)$, we have $U_2 \cap U_3 = 0$. Next, since U_2 and U_3 are

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contained in V, we also have $U_2 \cap U_1 = 0$ and $U_3 \cap U_1 = 0$. Let us show that the ideal $W = U_1 \oplus U_2 \oplus U_3$ is essential. We have to show that

$$\mathrm{Ann}(W) = \mathrm{Ann}(U_1) \cap \mathrm{Ann}(U_2) \cap \mathrm{Ann}(U_3)$$

is equal to zero. Since $U_2 = \mathrm{Ann}(U_3) \cap V$, we have

$$\mathrm{Ann}(W) \cap V = \mathrm{Ann}(U_1) \cap \mathrm{Ann}(U_2) \cap U_2 = 0.$$

Hence $\mathrm{Ann}(W) \subseteq \mathrm{Ann}(V)$. But on the other hand we have $\mathrm{Ann}(W) \subseteq \mathrm{Ann}(U_1)$. Since $\mathrm{Ann}(V) \cap \mathrm{Ann}(U_1) = \mathrm{Ann}(V \oplus U_1) = 0$ (namely, the ideal $V \oplus U_1$ is essential) it follows that $\mathrm{Ann}(W) = 0$. That is, W is essential.

Assume that the annihilator of any ideal in R is a direct summand. Then $U_1 \oplus V = R$. We want to show that $U_2 \oplus U_3 = V$. By assumption, $\mathrm{Ann}(V_1)$ is a direct summand. Thus $R = \mathrm{Ann}(V_1) \oplus Z$ for some ideal Z of R. Since $V_1 \subseteq V$, we have

$$Z = \mathrm{Ann}(\mathrm{Ann}(V_1)) \subseteq \mathrm{Ann}(\mathrm{Ann}(V)) = \mathrm{Ann}(U_1) = V.$$

Thus Z is contained in V. Pick $v \in V$. There exist elements $w \in \mathrm{Ann}(V_1)$ and $z \in Z$ such that $v = w + z$. We claim that $w \in U_3$ and $z \in U_2$. Since $z \in Z \subseteq V$, we also have $w \in V$. Thus $w \in \mathrm{Ann}(V_1) \cap V = U_3$. The ideal U_3 is contained in $\mathrm{Ann}(V_1)$, therefore $Z = \mathrm{Ann}(\mathrm{Ann}(V_1)) \subseteq \mathrm{Ann}(U_3)$. Hence $z \in \mathrm{Ann}(U_3) \cap V = U_2$. With this we have proved that $U_1 \oplus U_2 = V$, and, therefore, $U_1 \oplus U_2 \oplus U_3 = R$. The proof of (i) is thus complete.

Since α, β, α^{-1} and β^{-1} commute with $\beta^2 - \beta^{-2}$ (Lemma 3), and since all these automorphisms leave V invariant, it follows easily that V_1 is also invariant under α, β, α^{-1} and β^{-1}. Using Remark B we see the same is true for the ideal $U_3 = \mathrm{Ann}(V_1) \cap V$. Similarly we argue about the ideal U_2. Thus (ii) is proved.

Let us prove (iv). Given $v \in V$, we have $\alpha^2(v) - \beta^{-2}(v) \in V$ since V is invariant under α^2 and β^{-2}. But on the other hand, $(\alpha^2 - \beta^{-2})(v)$ is contained in U_0. Since $U_0 \cap V = 0$ it follows that $\alpha^2(v) = \beta^{-2}(v)$. Lemma 2 then yields

$$(\alpha - \beta^{-1})(x)R(\beta^2 - \beta^{-2})(v) = 0 \quad \text{for all } x \in R,\, v \in V.$$

This means that the range of $\alpha - \beta^{-1}$ is contained in $\mathrm{Ann}(V_1)$. Since $(\alpha - \beta^{-1})(u_2) \in U_2$ if $u_2 \in U_2$, and since $U_2 \cap \mathrm{Ann}(V_1) = U_2 \cap \mathrm{Ann}(V_1) \cap V = U_2 \cap U_3 = 0$, it follows that $\alpha(u_2) = \beta^{-1}(u_2)$.

It remains to prove (v). Pick $u_3 \in U_3$. On the one hand we have $(\beta^2 - \beta^{-2})(u_3) \in U_3$, and on the other hand, by the definition of V_1, $(\beta^2 - \beta^{-2})(u_3) \in V_1$. However, $U_3 \cap V_1 = V \cap \mathrm{Ann}(V_1) \cap V_1 = 0$, and hence

$\beta^2(u_3) = \beta^{-2}(u_3)$. We have proved that $\alpha^2 = \beta^{-2}$ on V, and therefore also on $U_3 \subseteq V$. With this (v) is proved.

In the case that the annihilator of any ideal of R is a direct summand, we see from Theorem 1 that the range of $\alpha - \beta$ is contained in the ideal $\mathrm{I}_1 = U_2 \oplus U_3$, the range of $\alpha - \beta^{-1}$ is contained in $\mathrm{I}_2 = U_1 \oplus U_3$, and the union of the ranges of $\alpha^2 - \beta^2$ and $\alpha^2 - \beta^{-2}$ is contained in $\mathrm{I}_3 = U_1 \oplus U_2$. This result is in accordance with Theorem A. With the aid of Lemmas 2 and 3, even if we do not assume that the annihilator of any ideal of R is a direct summand, it is not difficult to prove the following generalization of Theorem A.

THEOREM 2. *Let α and β be automorphisms of a semiprime ring R such that $\alpha + \alpha^{-1} = \beta + \beta^{-1}$. Then there exists ideals I_1, I_2 and I_3 of R, each invariant under α, α^{-1}, β and β^{-1}, such that $\mathrm{I}_1 \cap \mathrm{I}_2 \cap \mathrm{I}_3 = 0$, $\mathrm{I}_3 \subseteq \mathrm{I}_1 \cap \mathrm{I}_2$ and, for each x in R,*

$$\beta(x) - \alpha(x) \in \mathrm{I}_1, \qquad \beta(x) - \alpha^{-1}(x) \in \mathrm{I}_2,$$

$$\beta^2(x) - \alpha^2(x) \in \mathrm{I}_3, \qquad \beta^2(x) - \alpha^{-2}(x) \in \mathrm{I}_3.$$

PROOF. Let J be an ideal of R generated by all $(\alpha^2 - \beta^{-2})(x)$, $x \in R$, and set $\mathrm{I}_1 = \mathrm{Ann}(J)$. From Lemma 2 we see that $(\alpha - \beta)(x) \in \mathrm{I}_1$ for every $x \in R$. Since the mapping $\alpha^2 - \beta^{-2}$ commutes with α, β, α^{-1} and β^{-1} (Lemma 3) it follows that J is invariant under α, β, α^{-1} and β^{-1}. But then Remark B tells us that the same is true for the ideal I_1.

We introduce L to be an ideal of R generated by all $(\alpha^2 - \beta^2)(x)$, $x \in R$, and let $\mathrm{I}_2 = \mathrm{Ann}(L)$. Similarly as above one deduces that the range of $\alpha - \beta^{-1}$ is contained in I_2, and that L and I_2 are invariant under α, β, α^{-1} and β^{-1}.

The union of the ranges of $\alpha^2 - \beta^{-2}$ and $\alpha^2 - \beta^2$ is certainly contained in the ideal $\mathrm{I}_3 = J + L$. Of course, I_3 is also invariant under α, β, α^{-1} and β^{-1}. Next,

$$\mathrm{I}_1 \cap \mathrm{I}_2 \cap \mathrm{I}_3 = \mathrm{Ann}(J) \cap \mathrm{Ann}(L) \cap (J + L) = \mathrm{Ann}(J + L) \cap (J + L) = 0$$

by Remark A. From $\alpha^2 - \beta^2 = \alpha(\alpha + \alpha^{-1}) - \beta(\beta + \beta^{-1}) = (\alpha - \beta)(\beta + \beta^{-1})$ we see that the range of $\alpha^2 - \beta^2$ is contained in the range of $\alpha - \beta$. Therefore, L is contained in I_1. Similarly we see that J is contained in I_2. Consequently I_3 is contained in $\mathrm{I}_1 + \mathrm{I}_2$. The proof of the theorem is complete.

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