OPERATOR NORMS DETERMINED BY THEIR NUMERICAL RANGES

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Introduction

This paper owes its origin to the following question posed by A. M. Sinclair, "If a linear algebra with identity has two equivalent unital algebra norms, $|.|_1$ and $|.|_2$, whose corresponding numerical radii, v_1 and v_2, are equal on the whole algebra, are $|.|_1$ and $|.|_2$ related? Are they, for example, necessarily equal?" We do not give a complete answer to this question but are able to give sufficient conditions on algebras of operators for $v_1 = v_2$ to imply $|.|_1 = |.|_2$ That this implication does not hold for an arbitrary algebra with identity is demonstrated by means of a counter-example. The result for operator algebras is used to deduce some essentially non numerical range results for equivalent operator norms.

Given a normed linear space X (over $\boldsymbol{R}$ or $\boldsymbol{C}$) we shall write X' for the dual space, X_1 for the closed unit ball of X and $S(X)$ for the set $\{x \in X: \|x\| = 1\}$. We shall write $B(X)$ for the algebra of all bounded linear operators on X and $|.|$ for the operator norm on $B(X)$.

For a detailed account of the theory of numerical ranges see Bonsall and Duncan (1). We present here some of the basic definitions.

Let A be a unital normed algebra with identity e. For $a \in A$ the *numerical range* of a is defined by

$$V(A, a) = \{f(a): f \in S(A'), f(e) = 1\},$$

and the *numerical radius* of a by

$$v(a) = \sup \{|\lambda|: \lambda \in V(A, a)\}.$$

Let X be a normed linear space. $B(X)$ is a unital normed algebra with respect to $|.|$ and so each operator $T \in B(X)$ has a numerical range and a numerical radius as defined above. There is, however, an alternative definition of the numerical range of an operator. For $T \in B(X)$ the *spatial numerical range* of T is defined by

$$V(T) = \{f(Tx): x \in S(X), f \in S(X'), f(x) = 1\}.$$

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It may be shown that the closed convex hull of $V(T)$ is equal to $V(B(X), T)$ and it follows that

$$\sup\{|\lambda|: \lambda \in V(T)\} = v(T).$$

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Preliminary results

Given a convex function ϕ: $R \to R$ we shall write ϕ'_L and ϕ'_R for the left and right derivatives of ϕ respectively, these being defined on the whole of R.

1. Lemma. Let ϕ: $R \to R$ be convex. Then

$$(s-t)\phi'_L(t)+\phi(t) \leqq \phi(s) \quad (s<t).$$

Proof. This is routine.

2. Lemma. Let p and q be two norms for R^2 and A and B the corresponding closed unit balls. If $\xi A \cup \eta B$ is convex for all ξ, $\eta>0$, then there exists $K>0$ such that $p = Kq$ on R^2.

Proof. Suppose that for all ξ, $\eta>0$, $\xi A \cup \eta B$ is convex. Define ϕ and ψ: $R \to R^+$ by

$$\left.\begin{aligned} \phi(t) &= p(1, t) \\ \psi(t) &= q(1, t) \end{aligned}\right\} \quad (t \in R).$$

Clearly ϕ and ψ are continuous and strictly positive. The subadditivity of p and q implies that both ϕ and ψ are convex. Let ξ, $\eta>0$. The Minkowski functional for the convex set $\xi A \cup \eta B$ is $\min\{\xi p, \eta q\}$ and the subadditivity of this functional implies that $\min\{\xi\phi, \eta\psi\}$ is convex. We show that for all $t \in R$,

$$\frac{\phi'_L(t)}{\phi(t)} = \frac{\psi'_L(t)}{\psi(t)} \quad \text{and} \quad \frac{\phi'_R(t)}{\phi(t)} = \frac{\psi'_R(t)}{\psi(t)}.$$

Suppose that for some $t \in R$, $\dfrac{\phi'_L(t)}{\phi(t)} < \dfrac{\psi'_L(t)}{\psi(t)}$.

Let $\psi(t) = \alpha\phi(t)$. Then $\psi'_L(t)-\alpha\phi'_L(t) = \varepsilon_0$ (say)>0. Choose $\varepsilon_1 \in (0, \frac{1}{4}\varepsilon_0)$. Then there exists $\delta_1>0$ such that $0<h \leqq \delta_1$ implies

$$\left|\psi'_L(t)-\left\{\frac{\psi(t)-\psi(t-h)}{h}\right\}\right| <\varepsilon_1.$$

So in particular,

$$\psi(t-\delta_1)<\delta_1\{\varepsilon_1-\psi'_L(t)\}+\psi(t).$$

Now choose $\varepsilon_2 \in (0, 1)$ such that $\varepsilon_2 \alpha\phi(t) < \frac{1}{2}\delta_1\varepsilon_0$ and $\varepsilon_2 |\alpha\phi'_L(t)| < \frac{1}{4}\varepsilon_0$. Then we get

$$\begin{aligned}\psi(t-\delta_1) &< \delta_1\{\tfrac{1}{4}\varepsilon_0 - \tfrac{1}{2}\varepsilon_0 - \varepsilon_2\alpha\phi'_L(t) - (1-\varepsilon_2)\alpha\phi'_L(t)\} - \tfrac{1}{2}\delta_1\varepsilon_0 + \alpha\phi(t)\\ &< \delta_1\{-\tfrac{1}{4}\varepsilon_0 - \varepsilon_2\alpha\phi'_L(t)\} - \delta_1(1-\varepsilon_2)\alpha\phi'_L(t) + (1-\varepsilon_2)\alpha\phi(t)\\ &< -\delta_1(1-\varepsilon_2)\alpha\phi'_L(t) + (1-\varepsilon_2)\alpha\phi(t) \qquad (*)\\ &\leqq (1-\varepsilon_2)\alpha\phi(t-\delta_1) \quad \text{(by Lemma 1).}\end{aligned}$$

Clearly $\psi(t) > (1-\varepsilon_2)\alpha\phi(t)$. So there exists $\delta_2 \in (0, \delta_1)$ such that

$$\psi(t-\delta_2) > (1-\varepsilon_2)\alpha\phi(t-\delta_2).$$

We consider the convex function $\min\{(1-\varepsilon_2)\alpha\phi, \psi\} = \theta$ (say) and obtain a contradiction by showing that

$$\theta(t-\delta_2) > \frac{\delta_2}{\delta_1}\theta(t-\delta_1) + \frac{\delta_1-\delta_2}{\delta_1}\theta(t).$$

Observe that

$$\begin{aligned}\theta(t-\delta_1) &= \psi(t-\delta_1),\\ \theta(t-\delta_2) &= (1-\varepsilon_2)\alpha\phi(t-\delta_2),\\ \theta(t) &= (1-\varepsilon_2)\alpha\phi(t).\end{aligned}$$

Thus

$$\begin{aligned}\frac{\delta_2}{\delta_1}\theta(t-\delta_1) + \frac{\delta_1-\delta_2}{\delta_1}\theta(t) &= \frac{\delta_2}{\delta_1}\psi(t-\delta_1) + \frac{\delta_1-\delta_2}{\delta_1}(1-\varepsilon_2)\alpha\phi(t)\\ &< \frac{\delta_2}{\delta_1}\{-\delta_1(1-\varepsilon_2)\alpha\phi'_L(t)\} + (1-\varepsilon_2)\alpha\phi(t) \quad \text{(by (*) above)}\\ &\leqq (1-\varepsilon_2)\alpha\phi(t-\delta_2) \quad \text{(by Lemma 1).}\end{aligned}$$

This contradiction establishes that for all $t \in \boldsymbol{R}$,

$$\frac{\phi'_L(t)}{\phi(t)} \geqq \frac{\psi'_L(t)}{\psi(t)}.$$

It now follows by symmetry, that for all $t \in \boldsymbol{R}$,

$$\frac{\phi'_L(t)}{\phi(t)} = \frac{\psi'_L(t)}{\psi(t)} \quad \text{and} \quad \frac{\phi'_R(t)}{\phi(t)} = \frac{\psi'_R(t)}{\psi(t)}.$$

It follows from this that the continuous function ϕ/ψ: $\boldsymbol{R} \to \boldsymbol{R}$ is differentiable on $\boldsymbol{R}$ with derivative zero. Hence there exists $K > 0$ such that $\phi = K\psi$ on $\boldsymbol{R}$ and it follows that $p = Kq$ on $\boldsymbol{R}^2$.

Main results

Given a normed linear space X we shall say that a subset A of $B(X)$ is an *α-subset* of $B(X)$ if for all $x, y \in S(X)$ and all $\varepsilon > 0$ there exists $T \in A$ such that

$v(T)<1+\varepsilon$ and $\| Tx-y \|<\varepsilon$. We shall say that A is an α_0*-subset* of $B(X)$ if it is an α-subset and $T\in A$ implies that $\lambda T\in A$ for every scalar λ.

Note that any subset of $B(X)$ which contains all operators of rank 1 is an α-subset and $B(X)$ itself is an α_0-subset.

3. Theorem. *Let $\|.\|_1$ and $\|.\|_2$ be two equivalent norms for a linear space X and let v_1 and v_2 be the corresponding numerical radii on $B(X)$. If $v_1(T)=v_2(T)$ for every operator T in an α-subset of $B(X, \|.\|_1)$ then there exists $K>0$ such that $\|.\|_1 = K\|.\|_2$ on X.*

Proof. Suppose that A is an α-subset of $B(X, \|.\|_1)$ and $v_1(T)=v_2(T)$ for every $T\in A$. Further suppose that there is no $K>0$ such that $\|.\|_1 = K\|.\|_2$ on X. We may assume without loss of generality that there exist $a, b\in X$ such that $\| a \|_1<\| a \|_2$ and $\| b \|_1>\| b \|_2$. Write Y for the real linear span of $\{a, b\}$. Let $X^i = (X, \|.\|_i)$ and $Y^i = (Y, \|.\|_i)$ for $i=1, 2$. In view of Lemma 2 we may assume that $Y^1{}_1 \cup Y^2{}_1 = W$ (say) is not convex. So there exist $\bar{x}, \bar{y}\in W$ such that the segment $[\bar{x}, \bar{y}] = \{(1-t)\bar{x}+t\bar{y}: 0\leqq t\leqq 1\} \nsubseteq W$. So $\bar{x}$ and $\bar{y}\in X^1{}_1\cup X^2{}_1$ and $[\bar{x}, \bar{y}]\nsubseteq X^1{}_1\cup X^2{}_1$. Observe that $\bar{x}$, $\bar{y}$ and 0 are not collinear. Now there exist z' and z such that z' lies in the open segment

$$(\bar{x}, \bar{y}) = \{(1-t)\bar{x}+t\bar{y}: 0<t<1\}, \quad z\in(0, z') \text{ and } \| z \|_1 = \| z \|_2 = 1.$$

Let

$$x = (1-t_0)\bar{x}+t_0 z',$$

where

$$t_0 = \sup\{t\in[0, 1): \|(1-t)\bar{x}+tz'\|_1 \leqq 1\}$$

and $y=(1-s_0)\bar{y}+s_0 z'$, where

$$s_0 = \sup\{t\in[0, 1): \|(1-t)\bar{y}+tz'\|_2 \leqq 1\}.$$

Clearly $x\in[\bar{x}, z')\cap S(X^1)$ and $y\in[\bar{y}, z')\cap S(X^2)$. Now there exists

$$w\in(0, \tfrac{1}{2}(x+z'))$$

such that $\| w \|_1 = 1<\| w \|_2$. Let $\{w'\} = (w, y)\cap(0, z')$. We may choose $u'\in(w', y)$ with $\| u' \|_2>1$. Then there exists $u\in(0, u')$ such that

$$\| u \|_2 = 1<\| u \|_1.$$

Let $f\in X'$ with $\| f \|_2 = 1 = f(u)$. Then choose ε such that

$$0<\varepsilon<\frac{\| u \|_1-1}{1+\|f\|_1\| u \|_1}.$$

Then $1+\varepsilon<\| u \|_1(1-\| f \|_1\varepsilon)$. Since A is an α-subset of $B(X, \|.\|_1)$, there exists $T\in A$ such that $v_1(T)<1+\varepsilon$ and $\| T(u/\| u \|_1)-w \|_1<\varepsilon$. However,

$$\begin{aligned} 1 = f(u)<f(u') &\leqq (1-t)|f(w)|+t\,|f(y)| \\ &\qquad (\text{for some } t\in(0, 1)) \\ &\leqq (1-t)|f(w)|+t. \end{aligned}$$

So $|f(w)|>1$ and so

$$\begin{aligned}|f(Tu)| &\geqq |f(\|u\|_1 w)|-|f(Tu-\|u\|_1 w)|\\ &>\|u\|_1-\|f\|_1\|u\|_1\varepsilon\\ &>1+\varepsilon.\end{aligned}$$

Thus $v_1(T)<1+\varepsilon<v_2(T)$ and this contradiction establishes the result.

4. Theorem. *Let $\|.\|_1$ and $\|.\|_2$ be two equivalent norms for a linear space X and let U be a relative neighbourhood of* 0 *in some α_0-subset of $B(X, \|.\|_1)$. If the operator norms corresponding to $\|.\|_1$ and $\|.\|_2$ agree on $I+U$ then they agree on the whole of $B(X)$ and there exists $K>0$ such that $\|.\|_1 = K\|.\|_2$ on X.*

Proof. This follows from Theorem 3 using the fact (see Bonsall and Duncan (**1**), §2, Theorem 5) that for each $T\in B(X)$ there exists a scalar λ with modulus 1 such that

$$v_i(T)=\lim_{t\to 0+}\frac{|I+t\lambda T|_i-1}{t}\quad (i=1,2)$$

(where subscripts i denote correspondence with $\|.\|_i$).

5. Corollary. *With the notation of Theorem* 4, *if $|T|_1 = |T|_2$ for every invertible operator T in $B(X)$ then $|.|_1$ and $|.|_2$ agree on the whole of $B(X)$.*

6. Theorem. *Let $\|.\|_1$ and $\|.\|_2$ be two equivalent Banach space norms for a linear space X and let U be a relative neighbourhood of* 0 *in some α_0-subset of $B(X, \|.\|_1)$. If the operator norms corresponding to $\|.\|_1$ and $\|.\|_2$ agree on* exp (U) *then they agree on the whole of $B(X)$.*

Proof. This follows from Theorem 3 using the fact (see Bonsall and Duncan (**1**), §3, Theorem 4) that for each $T\in B(X)$ there exists a scalar λ with modulus 1 such that

$$v_i(T)=\lim_{t\to 0+}\frac{1}{t}\log|\exp(t\lambda T)|_i\quad (i=1,2)$$

(where subscripts i denote correspondence with $\|.\|_i$).

An example

We give an example of an algebra with identity, having two different unital Banach algebra norms for which the corresponding numerical ranges are identical.

We define two Banach space norms for C^2 as follows: for $(x, y)\in C^2$

$$\|(x,y)\|_1=\begin{cases}|x|\exp(\tfrac{1}{2}|y/x|\log 2) & \text{if } x\neq 0 \text{ and } |y/x|\leqq 2;\\ |y| & \text{otherwise}\end{cases}$$

$$\|(x,y)\|_2=(|x|^2+|y|^2)^{\frac{1}{2}}.$$

Subscripts 1 and 2 shall denote correspondence with $\|.\|_1$ and $\|.\|_2$ respectively.

Let S be the unilateral shift operator on C^2 given by

$$S(x, y) = (0, x) \quad ((x, y) \in C^2).$$

It may be verified (see (2)) that $V_1(S) = V_2(S) = \{\lambda \in C: |\lambda| \leqq \frac{1}{2}\}$.

Let A be the subalgebra of $B(C^2)$ generated by I and S. Then

$$A = \{\xi I + \eta S: \xi, \eta \in C\}.$$

It follows that for all $T \in A$, $V_1(T) = V_2(T)$. We show, however, that

$$|I+S|_1 < 1{\cdot}6 < |I+S|_2.$$

For $(x, y) \in C^2$, $(I+S)(x, y) = (x, x+y)$. We show first that for

$$\|(x, y)\|_1 = 1, \quad \|(x, x+y)\|_1 < 1{\cdot}55.$$

Let $(x, y) \in C^2$ with $\|(x, y)\|_1 = 1$.

(*a*) If $x = 0$ then $\|(x, x+y)\|_1 = \|(x, y)\|_1 < 1{\cdot}55$.

(*b*) If $x \neq 0$, $|y/x| > 2$ and $|(x+y)/x| > 2$ then $|y| = 1$ and

$$\begin{aligned}\|(x, x+y)\|_1 &= |x+y| \\ &< \tfrac{1}{2}|y| + |y| < 1{\cdot}55.\end{aligned}$$

(*c*) If $x \neq 0$, $|y/x| > 2$ and $|(x+y)/x| \leqq 2$ then $|y| = 1$ and

$$\begin{aligned}\|(x, x+y)\|_1 &= |x| \exp(\tfrac{1}{2}|(x+y)/x| \log 2) \\ &\leqq |x| \exp(\log 2) < 1{\cdot}55.\end{aligned}$$

(*d*) If $x \neq 0$, $|y/x| \leqq 2$ and $|(x+y)/x| > 2$ then

$$|x| \exp(\tfrac{1}{2}|y/x| \log 2) = 1 \text{ and } \|(x, x+y)\|_1 = |x+y|.$$

Let $\beta = |y/x|$. So $|x| = \exp(-\frac{1}{2}\beta \log 2)$ and

$$\|(x, x+y)\|_1 \leqq |x| + |y| = (1+\beta)\exp(-\tfrac{1}{2}\beta \log 2)$$

and from this it is routine to verify that $\|(x, x+y)\|_1 < 1{\cdot}55$.

(*e*) If $x \neq 0$, $|y/x| \leqq 2$ and $1(x+y)/x| \leqq 2$ then

$$|x| \exp(\tfrac{1}{2}|y/x| \log 2) = 1$$

and

$$\begin{aligned}\|(x, x+y)\|_1 &= |x| \exp(\tfrac{1}{2}|(x+y)/x| \log 2) \\ &\leqq \exp(-\tfrac{1}{2}|y/x| \log 2) \exp(\tfrac{1}{2}(1+|y/x|) \log 2) \\ &< 1{\cdot}55.\end{aligned}$$

It follows from (*a*), (*b*), (*c*), (*d*) and (*e*) that $|I+S|_1 < 1{\cdot}6$.

Now consider $(a, b) = (2, \sqrt{5}-1)$. It is easily verified that

$$\frac{\|(a, a+b)\|_2}{\|(a, b)\|_2} > 1{\cdot}6$$

and hence $|I+S|_2 > 1{\cdot}6$.

REFERENCES

(1) F. F. Bonsall and J. Duncan, *Numerical ranges of operators on normed spaces and of elements of normed algebras*, London Math. Soc. Lecture Note Series, No. **2** (1971).

(2) J. Duncan, C. M. McGregor, J. D. Pryce and A. J. White, The numerical index of a normed space, *J. London Math. Soc.* (2) **2** (1970), 481-488.

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