Proceedings of the Edinburgh Mathematical Society (2008) **51**, 581–607 ©
DOI:10.1017/S0013091506001234 Printed in the United Kingdom

SPECTRAL MULTIPLIERS FOR LAPLACIANS ASSOCIATED WITH SOME DIRICHLET FORMS

ANDREA CARBONARO[1], GIANCARLO MAUCERI[1] AND STEFANO MEDA[2]

[1]*Dipartimento di Matematica, Università di Genova, via Dodecaneso 35, 16146 Genova, Italy* (mauceri@dima.unige.it)
[2]*Dipartimento di Matematica e Applicazioni, Università di Milano-Bicocca, via Roberto Cozzi 53, 20125 Milano, Italy*

(Received 19 October 2006)

Abstract Let $\mathcal{L}^\varphi = -\Delta - \varphi^{-1}\nabla\varphi \cdot \nabla$ be the self-adjoint operator associated with the Dirichlet form

$$Q^\varphi(f) = \int_{\mathbb{R}^d} |\nabla f(x)|^2 \, \mathrm{d}\lambda^\varphi(x),$$

where φ is a positive C^2 function, $\mathrm{d}\lambda^\varphi = \varphi \, \mathrm{d}\lambda$ and λ denotes Lebesgue measure on $\mathbb{R}^d$. We study the boundedness on $L^p(\lambda^\varphi)$ of spectral multipliers of $\mathcal{L}^\varphi$. We prove that if φ grows or decays at most exponentially at infinity and satisfies a suitable 'curvature condition', then functions which are bounded and holomorphic in the intersection of a parabolic region and a sector and satisfy Mihlin-type conditions at infinity are spectral multipliers of $L^p(\lambda^\varphi)$. The parabolic region depends on φ, on p and on the infimum of the essential spectrum of the operator $\mathcal{L}^\varphi$ on $L^2(\lambda^\varphi)$. The sector depends on the angle of holomorphy of the semigroup generated by $\mathcal{L}^\varphi$ on $L^p(\lambda^\varphi)$.

Keywords: functional calculus; spectral multiplier; Dirichlet form

2000 *Mathematics subject classification:* Primary 47A60; 42B15; 47D03; 60G15

1. Introduction

Let φ be a positive function on $\mathbb{R}^d$. We shall denote by λ^φ the measure such that $\mathrm{d}\lambda^\varphi = \varphi \, \mathrm{d}\lambda$, where λ denotes Lebesgue measure on $\mathbb{R}^d$. Let Q^φ denote the Dirichlet form defined by

$$Q^\varphi(f) = \int_{\mathbb{R}^d} |\nabla f|^2 \, \mathrm{d}\lambda^\varphi$$

for all f in the form domain. The form Q^φ determines a unique non-negative self-adjoint operator $\mathcal{L}^\varphi$ on $L^2(\lambda^\varphi)$ such that

$$\langle \mathcal{L}^\varphi f, f \rangle = Q^\varphi(f) \quad \text{for all } f \in \mathrm{Dom}(\mathcal{L}^\varphi). \tag{1.1}$$

If the function φ is differentiable, a simple integration by parts shows that

$$\mathcal{L}^\varphi f = -(\Delta + \varphi^{-1}\nabla\varphi \cdot \nabla) f \quad \text{for all } f \in C_c^\infty(\mathbb{R}^d).$$

Let $\sigma(\mathcal{L}^\varphi)$ denote the spectrum of $\mathcal{L}^\varphi$ on $L^2(\lambda^\varphi)$. We shall denote by $\sigma_{\mathrm{d}}(\mathcal{L}^\varphi)$ the discrete spectrum, i.e. the set of all isolated eigenvalues with finite multiplicity, and by $\sigma_{\mathrm{e}}(\mathcal{L}^\varphi) = \sigma(\mathcal{L}^\varphi) \setminus \sigma_{\mathrm{d}}(\mathcal{L}^\varphi)$ the essential spectrum of $\mathcal{L}^\varphi$. Set $b_{\mathrm{e}} = \inf \sigma_{\mathrm{e}}(\mathcal{L}^\varphi)$. Then $\sigma_{\mathrm{d}}(\mathcal{L}^\varphi) \cap [0, b_{\mathrm{e}})$ is a countable set which has at most b_{e} as an accumulation point. Let $\{\mathcal{P}_s\}$ be the spectral resolution of $\mathcal{L}^\varphi$ for which

$$\mathcal{L}^\varphi f = \int_b^\infty s \,\mathrm{d}\mathcal{P}_s f \quad \text{for all } f \in \mathrm{Dom}(\mathcal{L}^\varphi),$$

where b denotes the bottom of $\sigma(\mathcal{L}^\varphi)$. If m is a bounded Borel function on $\sigma(\mathcal{L}^\varphi)$, then the operator $m(\mathcal{L}^\varphi)$ spectrally defined by

$$m(\mathcal{L}^\varphi) f = \int_b^\infty m(s) \,\mathrm{d}\mathcal{P}_s f \quad \text{for all } f \in L^2(\lambda^\varphi)$$

is bounded on $L^2(\lambda^\varphi)$. We call $m(\mathcal{L}^\varphi)$ the *spectral operator* associated with the *spectral multiplier* m. If $m(\mathcal{L}^\varphi)$ extends from $L^2(\lambda^\varphi) \cap L^p(\lambda^\varphi)$ to a bounded operator on $L^p(\lambda^\varphi)$ for some p in $[1, \infty)$, we say that m is a $L^p(\lambda^\varphi)$ spectral multiplier for $\mathcal{L}^\varphi$. The spectral multiplier problem for $\mathcal{L}^\varphi$ consists in finding conditions, necessary or sufficient, which imply that m is an $L^p(\lambda^\varphi)$ spectral multiplier for $\mathcal{L}^\varphi$.

We assume throughout that φ is in $C^2(\mathbb{R}^d)$ and satisfies

$$\frac{\varphi(x)}{\varphi(y)} \leqslant C\mathrm{e}^{\beta|x-y|} \quad \text{for all } x, y \in \mathbb{R}^d \tag{1.2}$$

and

$$\frac{1}{2}\frac{\Delta\varphi(x)}{\varphi(x)} - \frac{1}{4}\frac{|\nabla\varphi(x)|^2}{\varphi(x)^2} \geqslant -\kappa \quad \text{for all } x \in \mathbb{R}^d \tag{1.3}$$

for suitable positive constants C, β and κ.

Assumptions (1.2) and (1.3) may be regarded as weak bounded-curvature conditions. Indeed, suppose that φ is smooth. Then the operator $\mathcal{L}^\varphi$ may be viewed as the restriction to rotationally invariant functions of the Laplace–Beltrami operator on the 'warped product' manifold $M = \mathbb{R}^d \times_{\varphi^2} \mathbb{T}$, endowed with the metric $\mathrm{d}s^2 = \mathrm{d}x^2 + \varphi(x)^2\,\mathrm{d}\theta^2$. Recall that a Riemannian manifold has bounded curvature up to order k if all the covariant derivatives of the curvature tensor up to order k are bounded. Now, in the coordinate system (x, θ), the vector fields $X_j = \partial_{x_j}$ and $\Theta = \varphi^{-1}\partial_\theta$ are an orthonormal frame. It is easy to see that the non-vanishing components of the curvature tensor of M are

$$R_{X_i\Theta}(X_j) = -\frac{\partial^2_{x_i x_j}\varphi}{\varphi}\Theta, \qquad R_{X_i\Theta}(\Theta) = \sum_{k=1}^d \frac{\partial^2_{x_i x_k}\varphi}{\varphi} X_k.$$

Thus, the manifold M has bounded curvature up to order k if and only if the derivatives of $\log(\varphi)$ of order $j \in \{2, \dots, k+2\}$ are bounded. Now, assumption (1.2) follows from the boundedness of $|\nabla \log(\varphi)|$ and assumption (1.3) is equivalent to $d(d+1)S + |D_\Theta\Theta|^2 \leqslant \kappa$, where $S = -(2/d(d+1))\varphi^{-1}\Delta\varphi$ is the scalar curvature of M and $D_\Theta\Theta = -\varphi^{-1}\nabla\varphi$ is the covariant derivative of Θ with respect to itself. This explains why we may consider (1.2) and (1.3) as weak bounded-curvature assumptions.

In this paper we prove that $\mathcal{L}^\varphi$ admits an holomorphic $L^p(\lambda^\varphi)$ functional calculus on suitable pencil-like regions of the complex plane. Each such region is the intersection of a parabolic region and a sector, and depends on p, on β and on the bottom of the essential spectrum of $\mathcal{L}^\varphi$. To describe our result we need more notation. For each positive w, denote by $\boldsymbol{P}_w$ the parabolic region, which is the image of the strip $\{z \in \mathbb{C} : |\operatorname{Im} z| < w\}$ under the map $z \mapsto z^2 + b_{\mathrm{e}}$. For each θ in $(0, \pi)$, let $\boldsymbol{S}_\theta$ denote the sector $\{z \in \mathbb{C} \setminus \{0\} : |\arg z| < \theta\}$. Denote by $\boldsymbol{R}_{w,\theta}$ the intersection of $\boldsymbol{P}_w$ and $\boldsymbol{S}_\theta$.

Suppose that p is in $[1, \infty)$ and denote by ϕ_p^* the angle $\arcsin|2/p - 1|$. Suppose also that w is a number greater than $\beta|1/p - 1/2|$ and that $\theta > \phi_p^*$. It may be that, for some values of the parameters, $\boldsymbol{R}_{w,\theta}$ is just the parabolic region $\boldsymbol{P}_w$. Indeed, if $b_{\mathrm{e}} > \frac{1}{4}\beta^2$, then $\sqrt{b_{\mathrm{e}}}|2/p - 1| > \beta|1/p - 1/2|$. Thus, for any w in the interval $(\beta|1/p - 1/2|, \sqrt{b_{\mathrm{e}}}|2/p - 1|)$, the parabolic region $\boldsymbol{P}_w$ is contained in the sector $\boldsymbol{S}_{\phi_p^*}$. However, if $b_{\mathrm{e}} \leqslant \frac{1}{4}\beta^2$, then $\boldsymbol{P}_w$ is not contained in the sector $\boldsymbol{S}_{\phi_p^*}$. By a result in [**3**] (applied to the manifold $\mathbb{R}^d \times_{\varphi^2} \mathbb{T}$ considered above), this happens when the measure λ^φ is infinite. In particular, the vertex of the parabola might lie on the negative semi-axis.

Our main result (see Theorem 4.2 below) is the following. If p is in $(1, \infty)$, $w > \beta|1/p - 1/2|$, $\theta > \phi_p^*$, b_{e} is positive, m is defined on the spectrum of $\mathcal{L}^\varphi$ and extends to a function holomorphic and bounded in $\boldsymbol{R}_{w,\theta}$, and satisfies Mihlin-type conditions of the form

$$|D^j m(\zeta)| \leqslant C|\zeta^{-j}| \quad \text{for all } j \in \{0, 1, \dots, L\}$$

for all sufficiently large ζ in $\boldsymbol{R}_{w,\theta}$ and L in $\mathbb{N}$, then $m(\mathcal{L}^\varphi)$ is bounded on $L^p(\lambda^\varphi)$.

It is natural to speculate on the optimality of our result. In particular, one would like to know whether the pencil-like domains are the 'right ones' in this setting or if there are even more natural, smaller domains which still work. Unfortunately, at this stage of our investigation we have not been able to answer this question.

Note that the angle ϕ_p^* is related to the angle of holomorphy on L^p of all symmetric diffusion semigroups. Indeed, by a result in [**12**], such semigroups are bounded and holomorphic on L^p at least in the sector $S_{\pi/2-\phi_p^*}$. It is well known that generators of symmetric diffusion semigroups have bounded holomorphic functional calculus on L^p in sectors (see [**5, 11**]). It is also known that some of them have L^p functional calculus precisely on $\boldsymbol{S}_{\phi_p^*}$ (see [**4, 7, 13**]). The optimal angle ψ for which all generators of symmetric diffusion semigroups have bounded holomorphic functional calculus on $\boldsymbol{S}_\psi$ is not known. We observe that the operator $-\mathcal{L}^\varphi$ generates a symmetric diffusion semigroup on $(\mathbb{R}^d, \lambda^\varphi)$.

As a consequence of Theorem 4.2, we show that for each $\theta > \phi_p^*$ the $L^p(\lambda^\varphi)$ norm of the imaginary powers $m_u(\mathcal{L}^\varphi)$ (see Remark 4.4 below for the precise definition) of $\mathcal{L}^\varphi$ is controlled by $C\mathrm{e}^{\theta|u|}$, where C is independent of u in $\mathbb{R}$.

The proof of Theorem 4.2 is in two stages. First we show that if m is holomorphic and bounded in $\boldsymbol{P}_w$ and its boundary values satisfy suitable differential inequalities, then $m(\mathcal{L}^\varphi)$ is bounded on $L^p(\lambda^\varphi)$ for p in $(1, \infty)$, and satisfies a weak type 1 inequality when

$p = 1$ (see Theorem 3.1 below). Next we show that if m is defined on $\boldsymbol{R}_{w,\theta}$ and satisfies the assumptions of Theorem 4.2, then m may be written as the sum of m_0, which is holomorphic in the sector $\boldsymbol{S}_\theta$ and m_∞, which is holomorphic in the parabolic region $\boldsymbol{P}_w$ and satisfies the assumptions of Theorem 3.1.

To prove Theorem 3.1 we use a variant of a well-known method of Taylor [**16**]. To apply Taylor's method in our setting we must overcome a difficulty. Indeed, to prove a more general multiplier theorem, instead of translating the operator with the bottom b of the L^2 spectrum as Taylor does, we translate with the bottom of the *essential spectrum* b_{e}. This enables us to require holomorphy of the multiplier in the parabolic region $\boldsymbol{P}_w$, which is smaller then the region $\boldsymbol{P}_w + b - b_{\mathrm{e}}$ considered by Taylor. Since the region $\boldsymbol{P}_w$ does not contain all the discrete spectrum of $\mathcal{L}^\varphi$, to prove that $m(\mathcal{L}^\varphi)$ is bounded on $L^p(\lambda^\varphi)$ we must estimate separately the contribution to $m(\mathcal{L}^\varphi)$ from the eigenvalues $E_1, \dots, E_N$ that fall outside the region of holomorphy of the multiplier. Let $\mathit{\Pi}_{E_j}$ denote the corresponding spectral projections. Since the difference $\mathcal{L}^\varphi - \sum_j E_j \mathit{\Pi}_{E_j}$ does not have the finite propagation speed property, the application of Taylor's method is not so straightforward. This explains why the argument in the proof of Theorem 3.1 is more technical than Taylor's.

In our discussion of the geometric meaning of the curvature conditions on φ we have already remarked that the operator $\mathcal{L}^\varphi$ is the restriction to the subspace of rotationally invariant functions of the Laplace–Beltrami operator on a manifold M (at least when φ is smooth). Thus, it is natural to speculate whether Theorem 3.1 may be deduced from Taylor's results for the Laplace–Beltrami operator on Riemannian manifolds of bounded geometry [**16**].

First, note that for general φ the manifold M has no positive injectivity radius, because the infimum of φ may be zero. Thus, [**16**, Theorem 1.1] is not applicable to M.

Second, note that M is a quotient of its simply connected covering $\tilde{M} = \mathbb{R}^d \times_{\varphi^2} \mathbb{R}$, which indeed has positive injectivity radius. Then, in the case where all the derivatives of $\log(\varphi)$ are bounded, we may apply [**16**, Proposition 3.2] and we find that if p is in $(1, \infty)$, m is holomorphic and bounded in $\boldsymbol{P}_w + b - b_{\mathrm{e}}$, decays at infinity and satisfies suitable differential estimates on the boundary of $\boldsymbol{P}_w + b - b_{\mathrm{e}}$, then $m(\mathcal{L}^\varphi)$ is bounded on $L^p(M)$.

We emphasize the fact that our main result does not require any decay of m at infinity, that holomorphy of m is required only in the parabolic region $\boldsymbol{P}_w$, which is smaller than $\boldsymbol{P}_w + b - b_{\mathrm{e}}$ whenever $b < b_{\mathrm{e}}$, and finally that the assumptions on φ are fairly weak.

Our paper is organized as follows. Section 2 contains some background material and preliminary results. Section 3 is devoted to the proof of a multiplier theorem in the parabolic region $\boldsymbol{P}_w$. In § 4 we prove the multiplier theorem in the region $\boldsymbol{R}_{w,\theta}$, apart from a technical lemma, Lemma 4.5, whose proof is deferred to § 5.

The letter C will always denote a positive constant, which may not be the same at different occurrences, and may depend on any quantifier written, implicitly or explicitly, before the relevant formula. If $\mathcal{T}$ is a bounded operator from a normed space X to a normed space Y, we denote by $|||\mathcal{T}|||_{X;Y}$ its operator norm from X to Y. In the case where $X = Y$, we write simply $|||\mathcal{T}|||_X$ instead of $|||\mathcal{T}|||_{X;X}$.

https://doi.org/10.1017/S0013091506001234 Published online by Cambridge University Press

2. Background material and preliminary results

Taylor [**16**] has invented a powerful method for proving multiplier results for a given operator, $\mathcal{T}$. His method has two basic ingredients: the finite propagation speed of $\mathcal{T}$, and the existence of a uniformly bounded local L^2-parametrix for $\mathcal{T}$. In this section we illustrate the prerequisites and the modifications of Taylor's method needed in our case.

Classical energy estimates for the wave equation in $\mathbb{R}^d$ (see, for example, [**15**]) imply that the operator $\mathcal{L}^\varphi - b_{\mathrm{e}}$ has finite propagation speed, i.e. for all functions $f_1, f_2 \in C_c(X)$ with $\operatorname{supp} f_i \subset B(x_i, r_i)$,

$$\langle \cos(t\sqrt{\mathcal{L}^\varphi - b_{\mathrm{e}}}) f_1, f_2 \rangle = 0 \quad \text{for all } t < |x_1 - x_2| - r_1 - r_2.$$

Note that, since the cosine is an even entire function, the wave propagator $\cos(t\sqrt{\mathcal{L}^\varphi - b_{\mathrm{e}}})$ is well defined even though the spectrum of the operator $\mathcal{L}^\varphi - b_{\mathrm{e}}$ has non-empty intersection with the negative semi-axis.

A noteworthy consequence of the finite propagation speed that will be of importance to us is that if f is an even bounded Borel function such that the support of its Fourier transform $\hat{f}$ is contained in $[-h, h]$, then the support of the Schwartz kernel of $f(\sqrt{\mathcal{L}^\varphi})$ is contained in $\{(x, y) \in \mathbb{R}^d \times \mathbb{R}^d : |x - y| \leqslant h\}$.

Now we concentrate on the second ingredient of Taylor's method, i.e. the existence of the local L^2-parametrix for $\mathcal{L}^\varphi$. The existence of a local parametrix which is *uniformly bounded* in $L^2(\lambda^\varphi)$ is equivalent to ultracontractivity estimates for the semigroup generated by $-\mathcal{L}^\varphi$. It may be worth observing that the semigroup generated by $-\mathcal{L}^\varphi$ need not be ultracontractive, as simple examples show (just take $\varphi(x) = \mathrm{e}^x$ on the real line). To overcome this difficulty, it is convenient to reduce the analysis of $\mathcal{L}^\varphi$ to that of a Schrödinger operator. This idea was introduced in [**6**, § 4.7] to obtain Gaussian estimates of the heat kernel.

Let $\mathcal{U}_2 : L^2(\lambda) \to L^2(\lambda^\varphi)$ be the isometry defined by

$$\mathcal{U}_2 f = \varphi^{-1/2} f.$$

A simple computation shows that

$$\mathcal{U}_2^{-1} \mathcal{L}^\varphi \mathcal{U}_2 = \mathcal{H},$$

where

$$\mathcal{H} = -\Delta + V^\varphi$$

is the Schrödinger operator with potential

$$V^\varphi = \frac{1}{2}\frac{\Delta\varphi}{\varphi} - \frac{1}{4}\frac{|\nabla\varphi|^2}{\varphi^2}.$$

Note that, by assumption (1.3), the potential V^φ is bounded from below by $-\kappa$.

Clearly, the spectra of $\mathcal{L}^\varphi$ on $L^2(\lambda^\varphi)$ and of $\mathcal{H}$ on $L^2(\lambda)$ coincide. Moreover, if m is a bounded Borel function on $\sigma(\mathcal{L}^\varphi)$, then $m(\mathcal{L}^\varphi) = \mathcal{U}_2 m(\mathcal{H}) \mathcal{U}_2^{-1}$.

Note also that the operator $\mathcal{H}$ has the finite speed of propagation property [**15**].

Now we prove the existence of a local L^2-parametrix for the operator $\mathcal{H}$.

Proposition 2.1. *Let ℓ be the least even integer greater than $\frac{1}{4}d$. There then exist two even functions F_1 and F_2 and a constant C such that*

(i) $I = F_1(\mathcal{H})\mathcal{H}^\ell + F_2(\mathcal{H})$,

(ii) *the operators $F_j(\mathcal{H})$ are integral operators with kernels $k_j(x,y)$ with respect to the Lebesgue measure which are supported in $\{(x,y) : |x-y| \leqslant 1\}$ and satisfy the inequalities*

$$\sup_x \|k_j(x,\cdot)\|_{L^2(\lambda)} \leqslant C, \quad j = 1,2,$$

(iii) *if the operator $(I+\mathcal{H})^\ell m(\mathcal{H})$ is bounded on $L^2(\lambda)$, then $m(\mathcal{H})$ is an integral operator whose kernel with respect to the Lebesgue measure λ is*

$$\mathcal{H}^\ell m(\mathcal{H})k_1(x,\cdot) + m(\mathcal{H})k_2(x,\cdot).$$

Proof. Let ψ be an even function in $C_c^\infty(\mathbb{R})$ supported in $[-\ell^{-1},\ell^{-1}]$, such that $\int \psi(t)\,\mathrm{d}t = 1$. Denote by $\check{\psi}$ its inverse Fourier transform. The identity

$$\begin{aligned} 1 &= (1-\check{\psi}(\lambda)+\check{\psi}(\lambda))^\ell \\ &= \left(\frac{1-\check{\psi}(\lambda)}{\lambda}\right)^\ell \lambda^\ell + \sum_{j=1}^{\ell} C_{j,\ell}\check{\psi}(\lambda)^j \\ &= F_1(\lambda)\lambda^\ell + F_2(\lambda) \end{aligned}$$

holds for suitable constants $C_{j,\ell}$. The functions F_j, $j=1,2$ are even entire functions and satisfy

$$(1+|\lambda|)^\ell |F_j(\lambda)| \leqslant C \quad \text{for all } \lambda \in \mathbb{R},\ j=1,2. \tag{2.1}$$

Moreover, their Fourier transforms $\hat{F}_j$ are supported in $[-1,1]$. By the spectral theorem

$$I = F_1(\mathcal{H})\mathcal{H}^\ell + F_2(\mathcal{H}). \tag{2.2}$$

The operator $(I+\mathcal{H})^{-\ell}$ maps $L^2(\lambda^\varphi)$ to the bounded continuous functions by [**14**, Theorem B.3.3]. Thus, the same is true of the operators $F_j(\mathcal{H})$, because $F_j(\mathcal{H}) = (I+\mathcal{H})^{-\ell}(I+\mathcal{H})^\ell F_j(\mathcal{H})$, and $(I+\mathcal{H})^\ell F_j(\mathcal{H})$ is bounded on $L^2(\lambda^\varphi)$ since

$$|\!|\!| F_j(\mathcal{H}) |\!|\!|_{L^2(\lambda^\varphi)} \leqslant C \sup_{\lambda\geqslant 0}(1+|\lambda|)^\ell |F_j(\lambda)| \leqslant C.$$

Hence, by the Dunford–Pettis theorem, the operator $F_j(\mathcal{H})$ has a kernel $k_j(x,y)$ such that

$$\sup_x \|k_j(x,\cdot)\|_{L^2(\lambda)} = |\!|\!| F_j(\mathcal{H}) |\!|\!|_{L^2(\lambda);L^\infty(\lambda)} \leqslant C. \tag{2.3}$$

Moreover, $\operatorname{supp} k_j(x,\cdot) \subset \overline{B(x,1)}$ because $\hat{F}_j$, $j=1,2$, is supported in $[-1,1]$ and $\mathcal{H}$ has the finite speed of propagation property.

https://doi.org/10.1017/S0013091506001234 Published online by Cambridge University Press

Let f be a function in $L^2(\lambda)$. By (2.2),

$$\begin{aligned} m(\mathcal{H})f(x) &= F_1(\mathcal{H})\mathcal{H}^\ell m(\mathcal{H})f(x) + F_2(\mathcal{H})m(\mathcal{H})f(x) \\ &= \int k_1(x,y)\mathcal{H}^\ell m(\mathcal{H})f(y)\,\mathrm{d}\lambda^\varphi(y) + \int k_2(x,y)m(\mathcal{H})f(y)\,\mathrm{d}\lambda^\varphi(y) \\ &= \langle \mathcal{H}^\ell m(\mathcal{H})f, \bar{k}_1(x,\cdot)\rangle + \langle m(\mathcal{H})f, \overline{k_2}(x,\cdot)\rangle, \end{aligned}$$

where $\langle\cdot\,,\cdot\rangle$ denotes the inner product in $L^2(\lambda)$. Taking the adjoints of the operators $\mathcal{H}^\ell m(\mathcal{H})$ and $m(\mathcal{H})$ and using the fact that the spectral projections of $\mathcal{H}$ commute with complex conjugation, we see that the kernel of $m(\mathcal{H})$ is

$$\overline{\bar{m}(\mathcal{H})\mathcal{H}^\ell\bar{k}_1(x,\cdot)} + \overline{\bar{m}(\mathcal{H})\bar{k}_1(x,\cdot)} = m(\mathcal{H})\mathcal{H}^\ell k_1(x,\cdot) + m(\mathcal{H})k_1(x,\cdot).$$

This concludes the proof of the proposition. □

The last result of this section concerns estimates of the eigenfunctions and of the spectral projections of $\mathcal{H}$. We shall denote by Π the projection-valued measure associated with the spectral resolution of the identity $\{\mathcal{P}_\lambda\}_{\lambda\in\mathbb{R}}$ of $\mathcal{L}^\varphi$. Similarly, we shall denote by $\{\tilde{\mathcal{P}}_\lambda\}_{\lambda\in\mathbb{R}}$ the spectral resolution of the identity of $\mathcal{H}$ and by $\tilde{\Pi}$ the corresponding projection-valued measure on $\mathbb{R}$. Thus, if a and b are numbers such that $a<b$, $\tilde{\Pi}_{(a,b]}$ is just $\tilde{\mathcal{P}}_b - \tilde{\mathcal{P}}_a$. For simplicity, we shall write $\tilde{\Pi}_a$ instead of $\tilde{\Pi}_{\{a\}}$.

Lemma 2.2. *Let E be an eigenvalue of $\mathcal{H}$ such that $E < b_{\mathrm{e}}$. Then, for every $A < \sqrt{b_{\mathrm{e}} - E}$, there exists a constant C such that*

(i) *if ψ is an eigenfunction of $\mathcal{H}$ with eigenvalue E, then*

$$|\psi(x)| \leqslant C\mathrm{e}^{-A|x|} \quad \text{for all } x\in\mathbb{R}^d;$$

(ii) *if f is a function in $L^2(\lambda)$ with support in $B(z,1)$, then*

$$\biggl(\int_{|x-z|\geqslant r} |\tilde{\Pi}_E f(x)|^2\,\mathrm{d}\lambda(x)\biggr)^{1/2} \leqslant C\|f\|_{L^2(\lambda)}\mathrm{e}^{-Ar}r^{(d-1)/2} \quad \text{for all } r\in\mathbb{R}^+.$$

Proof. Statement (i) is a classical result for Schrödinger operators with potential bounded from below [**1**].

To prove (ii), let ψ be an eigenfunction of $\mathcal{H}$ with eigenvalue E. Then by (i) and Schwarz's inequality,

$$\begin{aligned} |\langle f,\psi\rangle\psi(x)| &= \biggl|\int \psi(x)\overline{\psi(y)}f(y)\,\mathrm{d}\lambda(y)\biggr| \\ &\leqslant C\int_{|y-z|\leqslant 1} \mathrm{e}^{-A|x-y|}|f(y)|\,\mathrm{d}\lambda(y) \\ &\leqslant C\|f\|_{L^2(\lambda)}\biggl(\int_{|y-z|\leqslant 1}\mathrm{e}^{-2A|x-y|}\,\mathrm{d}\lambda(y)\biggr)^{1/2} \\ &\leqslant C\|f\|_{L^2(\lambda)}\mathrm{e}^{-A|x-z|}. \end{aligned}$$

Thus,

$$\biggl(\int_{|x-z|\geqslant r} |\langle f,\psi\rangle\psi(x)|^2 \,\mathrm{d}\lambda(x)\biggr)^{1/2} \leqslant C\|f\|_{L^2(\lambda)}\biggl(\int_{|x-z|\geqslant r} \mathrm{e}^{-2A|x-z|}\,\mathrm{d}\lambda(x)\biggr)^{1/2}$$
$$\leqslant \|f\|_{L^2(\lambda)}\mathrm{e}^{-Ar}r^{(d-1)/2}.$$

To conclude the proof of (ii) it suffices to observe that the integral kernel of the projection $\tilde{\Pi}_E$ is

$$\sum_{j=1}^{d_E} \psi_j(x)\overline{\psi_j(y)},$$

where $\{\psi_j : j = 1,\dots,d_E\}$ is an orthonormal basis of the E-eigenspace. □

3. Functional calculus on parabolic regions

In this section we shall prove a multiplier theorem for functions which are bounded and holomorphic and satisfy Mihlin-type conditions in a parabolic region. For each w in $\mathbb{R}^+$ we consider the strip

$$\Sigma_w = \{z \in \mathbb{C} : |\mathrm{Im}\, z| < w\}.$$

Denote by Φ the function $\Phi(z) = z^2 + b_{\mathrm{e}}$. The function Φ maps the strip Σ_w onto the parabolic region $\boldsymbol{P}_w$, where

$$\boldsymbol{P}_w = \biggl\{x + \mathrm{i}y \in \mathbb{C} : x > \frac{y^2}{4w^2} - w^2\biggr\} + b_{\mathrm{e}}.$$

Suppose that m is a function defined on $\boldsymbol{P}_w$. Denote by m_Φ the function defined on Σ_w by

$$m_\Phi(z) = (m \circ \Phi)(z) \quad \text{for all } z \in \Sigma_w.$$

The space of bounded holomorphic functions in a region Ω of the complex plane will be denoted by $H^\infty(\Omega)$. For each $w > 0$ and for each non-negative integer L we denote by X^L_w the space of all functions f in $H^\infty(\boldsymbol{P}_w)$ for which there exists a constant C such that

$$(1 + |\zeta|)^\ell |D^\ell f(\zeta)| \leqslant C \quad \text{for all } \zeta \in \boldsymbol{P}_w \text{ and all } \ell \in \{0, 1,\dots, L\}, \tag{3.1}$$

which is endowed with the norm

$$\|f\|_{X^L_w} = \inf\{C : (3.1) \text{ holds}\}.$$

Theorem 3.1. *Assume that the function φ satisfies (1.2) and (1.3). Suppose that p is in $[1,\infty)$, that $w > \beta|1/p-1/2|$ and that L is an integer greater than or equal to $d/2+5$. Assume that m is a function defined on $\sigma(\mathcal{L}^\varphi) \cup \boldsymbol{P}_w$ whose restriction to $\boldsymbol{P}_w$ is in X^L_w. The following hold.*

(i) *If p is equal to 1, then $m(\mathcal{L}^\varphi)$ extends to an operator of weak type 1 and of strong type q for all q in $(1,\infty)$, and there exists a constant C such that, for every f in $L^1(\lambda^\varphi)$,*

$$\lambda^\varphi(\{x \in \mathbb{R}^d : m(\mathcal{L}^\varphi)f(x) > t\}) \leqslant C(\|m\|_\infty + \|m\|_{X^L_w})\frac{\|f\|_{L^1(\lambda)^\varphi}}{t} \quad \text{for all } t \in \mathbb{R}^+.$$

https://doi.org/10.1017/S0013091506001234 Published online by Cambridge University Press

(ii) *If p is in $(1,\infty)$, then m is a spectral multiplier of $L^q(\lambda^\varphi)$ for $\mathcal{L}^\varphi$ for all q in $[p,p']$ and there exists a constant C such that*

$$\|m(\mathcal{L}^\varphi)\|_{L^q(\lambda^\varphi)} \leqslant C(\|m\|_\infty + \|m\|_{X_w^L}).$$

Roughly speaking, to prove Theorem 3.1 we would like to use the functional calculus

$$m(\mathcal{L}^\varphi) = m_\Phi(\sqrt{\mathcal{L}^\varphi - b_{\mathrm{e}}}) = (2\pi)^{-1/2}\int_0^\infty \hat{m}_\Phi(t)\cos(t\sqrt{\mathcal{L}^\varphi - b_{\mathrm{e}}})\,\mathrm{d}t, \tag{3.2}$$

where $\hat{m}_\Phi$ denotes the Fourier transform of the restriction of m_Φ to the real axis. Hence, our approach to the proof of Theorem 3.1 follows closely that of Taylor for the Laplace–Beltrami operator $\mathcal{L}$ on a Riemannian manifold [**16**]. Note, however, that Taylor's functional calculus is based on the wave propagator of the non-negative operator $\mathcal{L}-b$, where b is the bottom of the spectrum of $\mathcal{L}$. Thus, the operator $\cos(t\sqrt{\mathcal{L}-b})$ is uniformly bounded on L^2. In our case $\|\cos(t\sqrt{\mathcal{L}^\varphi - b_{\mathrm{e}}})\|_{L^2(\lambda^\varphi)} = \cosh(t\sqrt{b_{\mathrm{e}} - b})$ grows exponentially as $t\to\infty$ and this growth is not matched by the decay of $\hat{m}_\Phi(t)$ when m is in X_w^L, as we shall see in Lemma 3.2 below. Thus, in general, the integral in (3.2) does not converge in the norm of bounded operators on $L^2(\lambda^\varphi)$ and we shall have to estimate separately the contribution to $m(\mathcal{L}^\varphi)$ from the eigenvalues of $\mathcal{L}^\varphi$ below $b_{\mathrm{e}} - w^2$.

We begin by estimating the decay of the Fourier transform $\hat{m}_\Phi$, when m is in X_w^L.

Lemma 3.2. *Suppose that m is in X_w^L for some w in $\mathbb{R}^+$ and some integer $L \geqslant 2$. Then we can write $m_\Phi = m_{\Phi,a} + m_{\Phi,b}$, where*

(i) *$\hat{m}_{\Phi,a}$ has support in $[-1,1]$ and*

$$|D^k(m_{\Phi,a})(\lambda)| \leqslant C\|m\|_{X_w^L}(1+|\lambda|)^{-k} \quad \textit{for all } \lambda\in\mathbb{R},\ k\in\{0,1,\dots,L\}, \tag{3.3}$$

(ii) *$\hat{m}_{\Phi,b}(t) = 0$ for $|t|\leqslant 1/2$ and*

$$|D^k\hat{m}_{\Phi,b}(t)| \leqslant C\|m\|_{X_w^L}(1+|t|)^{-L}\mathrm{e}^{-w|t|} \quad \textit{for all } t\in\mathbb{R},\ k\in\{0,1,\dots,L-1\}. \tag{3.4}$$

Proof. Let ω be an even smooth function supported in $[-1,1]$ such that $\omega = 1$ in $[-1/2,1/2]$, and define $m_{\Phi,a} = m_\Phi \star \check{\omega}$, where $\star$ denotes convolution on the real line and $\check{\omega}$ is the inverse Fourier transform of ω. Then $\hat{m}_{\Phi,a}$ is supported in $[-1,1]$ and estimate (3.3) is a straightforward consequence of the fact that the restriction of m_Φ to the real line satisfies the estimates

$$(1+|\lambda|)^\ell |D^\ell m_{\Phi(\lambda)}| \leqslant C\|m\|_{X_w^L} \quad \text{for all } \lambda\in\Sigma_w \text{ and all } \ell\in\{0,1,\dots,L\},$$

and $\check{\omega}$ is a Schwartz function. The proof of (ii) follows from the fact that $\hat{m}_{\Phi,b} = \hat{m}_\Phi(1-\omega)$ and the estimate

$$|D^k\hat{m}_\Phi(t)| \leqslant C\|m\|_{X_w^L}|t|^{-L}\mathrm{e}^{-w|t|} \quad \text{for all } t\in\mathbb{R}\setminus\{0\},\ k\in\{0,1,\dots,L-1\},$$

proved in [**10**, Lemma 5.4]. □

https://doi.org/10.1017/S0013091506001234 Published online by Cambridge University Press

Now, if m is a function defined on $\sigma(\mathcal{L}^\varphi) \cup \boldsymbol{P}_w$ whose restriction to $\boldsymbol{P}_w$ is in X_w^L for some $L \geqslant 3$ and $\lambda \geqslant b_\mathrm{e} - w^2$, we may write

$$\begin{aligned} m(\lambda) &= m_\Phi(\sqrt{\lambda - b_\mathrm{e}}) \\ &= (m_{\Phi,a})(\sqrt{\lambda - b_\mathrm{e}}) + (2\pi)^{-1/2} \int_{1/2}^\infty \hat{m}_{\Phi,b}(t) \cos(t\sqrt{\lambda - b_\mathrm{e}})\,\mathrm{d}t. \end{aligned} \tag{3.5}$$

Note that $m_{\Phi,a}(\sqrt{\lambda - b_\mathrm{e}})$ is well defined for all complex λ because $m_{\Phi,a}$ is an even entire function, and that the integral is absolutely convergent by (3.4).

Recall that we denote by $\varPi$ the projection-valued measure on $\mathbb{R}$ associated with the spectral resolution of the identity $\{\mathcal{P}_\lambda\}_{\lambda \in \mathbb{R}}$.

Denote by I_1 and I_2 the intervals $[0, b_\mathrm{e} - w^2]$ and $(b_\mathrm{e} - w^2, \infty)$, respectively, and let $E_1, E_2, \dots, E_r$ be the eigenvalues of $\mathcal{L}^\varphi$ in I_1. Define the operator $M(\mathcal{L}^\varphi)$ by setting

$$M(\mathcal{L}^\varphi) = (2\pi)^{-1/2} \int_{1/2}^\infty \hat{m}_{\Phi,b}(t) \cos(t\sqrt{\mathcal{L}^\varphi - b_\mathrm{e}})\varPi_{I_2}\,\mathrm{d}t. \tag{3.6}$$

Note that, since $|||\cos(t\sqrt{\mathcal{L}^\varphi - b_\mathrm{e}})\varPi_{I_2}|||_{L^2(\lambda^\varphi)} \leqslant \cosh(|t|w)$, the integral converges and defines a bounded operator on $L^2(\lambda^\varphi)$ by (3.4). By the spectral theorem and (3.5),

$$\begin{aligned} m(\mathcal{L}^\varphi) &= m(\mathcal{L}^\varphi)\varPi_{I_1} + m(\mathcal{L}^\varphi)\varPi_{I_2} \\ &= \sum_{j=1}^r m(E_j)\varPi_{E_j} + \int_{b_\mathrm{e}-w^2}^\infty m(\lambda)\,\mathrm{d}\mathcal{P}_\lambda \\ &= \sum_{j=1}^r m(E_j)\varPi_{E_j} + m_{\Phi,a}(\sqrt{\mathcal{L}^\varphi - b_\mathrm{e}})\varPi_{I_2} + M(\mathcal{L}^\varphi). \end{aligned} \tag{3.7}$$

We shall estimate the norm on $L^p(\lambda^\varphi)$ of each summand separately. This will be done in Propositions 3.3, 3.4 and 3.5 below. The proof of Theorem 3.1 will be a straightforward consequence of these three propositions.

Proposition 3.3. *Suppose that $w > \beta|1/p - 1/2|$ and let $E_1, E_2, \dots, E_r$ be the eigenvalues $\leqslant b_\mathrm{e} - w^2$ of $\mathcal{L}^\varphi$. Then, for every q in $[p, p']$,*

$$\Big|\Big|\Big| \sum_{j=1}^r m(E_j)\varPi_{E_j} \Big|\Big|\Big|_{L^q(\lambda^\varphi)} \leqslant C\|m\|_\infty.$$

Proof. Let ψ be an eigenfunction of $\mathcal{L}^\varphi$ with eigenvalue $E \leqslant b_\mathrm{e} - w^2$. Then $\psi = \varphi^{-1/2}\tilde{\psi}$, where $\tilde{\psi}$ is an eigenfunction of $\mathcal{H}$ with the same eigenvalue. By Lemma 2.2 and (1.2), for every $A < \sqrt{b_\mathrm{e} - E}$ there exists a constant C such that

$$\begin{aligned} \|\psi\|^q_{L^q(\lambda^\varphi)} &\leqslant C \int_{\mathbb{R}^d} \varphi(x)^{1-q/2} \mathrm{e}^{-Aq|x|}\,\mathrm{d}\lambda(x) \\ &\leqslant C \int_{\mathbb{R}^d} \mathrm{e}^{-\varepsilon q|x|}\,\mathrm{d}\lambda(x), \end{aligned}$$

https://doi.org/10.1017/S0013091506001234 Published online by Cambridge University Press

where $\varepsilon = A - \beta|1/q - 1/2|$ is positive for all q in $[p, p']$ when A is sufficiently close to $\sqrt{b_{\mathrm{e}} - E}$. Thus, ψ is in $L^q(\lambda^\varphi)$ for all q in $[p, p']$.

To conclude the proof, it suffices to observe that the integral kernel of the projection Π_E is

$$\sum_{j=1}^{d_E} \psi_j(x)\overline{\psi_j(y)},$$

where $\{\psi_j : j = 1, \dots, d_E\}$ is an orthonormal basis of the E-eigenspace. Hence,

$$|\!|\!| \Pi_E |\!|\!|_{L^q(\lambda^\varphi)} \leqslant \sum_j \|\psi_j\|_{L^q(\lambda^\varphi)} \|\psi_j\|_{L^{q'}(\lambda^\varphi)},$$

by Hölder's inequality. □

Proposition 3.4. *Suppose that L is an integer greater than $d/2 + 2$ and that w is a positive number. Denote by I_2 the interval $(b_{\mathrm{e}} - w^2, \infty)$. Let m be a function defined on $\sigma(\mathcal{L}^\varphi) \cup \boldsymbol{P}_w$ whose restriction to $\boldsymbol{P}_w$ is in X_w^L.*

(i) *If $w > \beta/2$, then the operator $m_{\Phi,a}(\sqrt{\mathcal{L}^\varphi - b_{\mathrm{e}}})\Pi_{I_2}$ is of weak type 1 and there exists a positive constant C such that, for all f in $L^1(\lambda^\varphi)$,*

$$\lambda^\varphi(\{x \in \mathbb{R}^d : |m_{\Phi,a}(\sqrt{\mathcal{L}^\varphi - b_{\mathrm{e}}})\Pi_{I_2} f(x)| > t\}) \leqslant C\|m\|_{X_w^L} \frac{\|f\|_{L^1(\lambda^\varphi)}}{t} \quad \forall t \in \mathbb{R}^+.$$

(ii) *If $w > \beta|1/p - 1/2|$ for some p in $(1, \infty)$, then $m_{\Phi,a}(\sqrt{\mathcal{L}^\varphi - b_{\mathrm{e}}})\Pi_{I_2}$ is bounded on $L^p(\lambda^\varphi)$ and*

$$|\!|\!| m_{\Phi,a}(\sqrt{\mathcal{L}^\varphi - b_{\mathrm{e}}})\Pi_{I_2} |\!|\!|_{L^p(\lambda^\varphi)} \leqslant C\|m\|_{X_w^L}.$$

Proof. We shall prove only the weak-type estimate (i), because the proof of (ii) is similar. By Proposition 3.3, the projection

$$\Pi_{I_2} = I - \sum_{j=1}^{r} \Pi_{E_j}$$

is bounded on $L^1(\lambda^\varphi)$. Hence, it is sufficient to prove that the operator $m_{\Phi,a}(\sqrt{\mathcal{L}^\varphi - b_{\mathrm{e}}})$ is of weak type 1. To simplify notation for the duration of this proof we shall write $\mathcal{A}$ instead of $m_{\Phi,a}(\sqrt{\mathcal{L}^\varphi - b_{\mathrm{e}}})$ and $\mathcal{B}$ instead of $m_{\Phi,a}(\sqrt{\mathcal{H} - b_{\mathrm{e}}})$. Recall that $\mathcal{U}_2^{-1}\mathcal{A}\mathcal{U}_2 = \mathcal{B}$ because $\mathcal{U}_2^{-1}\mathcal{L}^\varphi\mathcal{U}_2 = \mathcal{H}$. First we show that the operator $\mathcal{B}$ satisfies the following weak-type estimate: there exists a constant C such that, for all f in $L^1(\lambda)$,

$$\lambda(\{x \in \mathbb{R}^d : |\mathcal{B}f(x)| > t\}) \leqslant C\|m\|_{X_w^L} \frac{\|f\|_{L^1(\lambda)}}{t} \quad \text{for all } t \in \mathbb{R}^+. \tag{3.8}$$

Observe that, by assumption (1.3), the potential V^φ is bounded from below by $-\kappa$. Therefore, the Schrödinger operator $\mathcal{H}_{V^\varphi+\kappa}$, formally defined by $-\Delta + V^\varphi + \kappa$, has non-negative potential. Since the function $m_{\Phi,a}$ is even and entire, the function F defined by

$$F(\zeta) = m_{\Phi,a}(\sqrt{\zeta - b_{\mathrm{e}} - k}) \quad \text{for all } \zeta \in \mathbb{C} \setminus (-\infty, \kappa + b_{\mathrm{e}}]$$

https://doi.org/10.1017/S0013091506001234 Published online by Cambridge University Press

extends to an entire function, still denoted by F. Note that

$$\mathcal{B} = F(\mathcal{H}_{V^\varphi+\kappa}).$$

By (3.3), the function F satisfies

$$|D^k F(\lambda)| \leqslant C\|m\|_{X^L_w}(1+\lambda)^{-k} \quad \text{for all } \lambda \in [b_{\mathrm{e}}+\kappa+1,\infty) \text{ and all } k \in \{0,\dots,L\}.$$

Since F is entire, the same estimate holds for all λ in $[0,\infty)$. Now [**8**, Theorem 1] implies the desired weak-type inequality (3.8).

Next we show that (3.8) implies the required weak-type 1 estimate for $\mathcal{A} = m_{\Phi,a}(\sqrt{\mathcal{L}^\varphi - b_{\mathrm{e}}})$. Let $\{B_j\}$ be a sequence of balls of radius 1 which cover $\mathbb{R}^d$ and such that, for some constant N,

$$\sum_j \mathbf{1}_{2B_j} \leqslant N,$$

where $2B_j$ denotes the ball with the same centre of B_j and twice the radius. By assumption (1.2) there exists a constant C such that

$$C^{-1} \leqslant \frac{\varphi(x)}{\varphi(y)} \leqslant C \quad \text{for all } x,y \in 2B_j \text{ and all } j \in \mathbb{N}. \tag{3.9}$$

In other words, the density φ is essentially constant in each of the balls B_j, uniformly in j. By Lemma 3.2 the Fourier transform of $m_{\Phi,a}$ has support in $[-1,1]$. Since the operator $\mathcal{L}^\varphi - b_{\mathrm{e}}$ has the finite propagation speed property, the Schwartz kernel of the operator $\mathcal{A}$ is supported in the set $\{(x,y) \in \mathbb{R}^d \times \mathbb{R}^d : |x-y| \leqslant 1\}$. Thus, if f is a smooth function with compact support, the values of $\mathcal{A}f$ on B_j depend only on the values of f on $2B_j$, i.e.

$$\mathbf{1}_{B_j}\mathcal{A}f = \mathbf{1}_{B_j}\mathcal{A}(f\mathbf{1}_{2B_j}).$$

Hence,

$$\begin{aligned}\lambda^\varphi(\{|\mathcal{A}f(x)| > t\}) &\leqslant \sum_j \lambda^\varphi(\{x \in B_j : |\mathcal{A}f(x)| > t\}) \\ &= \sum_j \lambda^\varphi(\{x \in B_j : |\mathcal{A}(f\mathbf{1}_{2B_j})(x)| > t\}).\end{aligned} \tag{3.10}$$

Now, if x_j denotes any point in B_j, then

$$\begin{aligned}\lambda^\varphi(\{x \in B_j : |\mathcal{A}(f\mathbf{1}_{2B_j})(x)| > t\}) &= \int_{B_j \cap \{|\mathcal{A}(f\mathbf{1}_{2B_j})|>t\}} \varphi(x)\,\mathrm{d}\lambda(x) \\ &\leqslant C\varphi(x_j)\lambda(\{x \in B_j : |\mathcal{A}(f\mathbf{1}_{2B_j})(x)| > t\})\end{aligned} \tag{3.11}$$

by (3.9). Observe that, since $\mathcal{A} = \mathcal{U}_2\mathcal{B}\mathcal{U}_2^{-1}$,

$$\begin{aligned}\lambda(\{x \in B_j : |\mathcal{A}(f\mathbf{1}_{2B_j})(x)| > t\}) &= \lambda(\{x \in B_j : |\mathcal{B}(f\mathbf{1}_{2B_j}\varphi^{1/2})(x)| > t\varphi(x)^{1/2}\}) \\ &\leqslant \lambda(\{x \in B_j : |\mathcal{B}(f\mathbf{1}_{2B_j}\varphi^{1/2})(x)| > Ct\varphi(x_j)^{1/2}\}).\end{aligned}$$

Now, (3.8) implies that, for all t in $\mathbb{R}^+$ and f in $L^1(\lambda)$,

$$\begin{aligned}\lambda(\{x \in B_j : |\mathcal{A}(f\mathbf{1}_{2B_j})(x)| > t\}) &\leqslant C\|m\|_{X^L_w} \frac{\|f\mathbf{1}_{2B_j}\varphi^{1/2}\|_{L^1(\lambda)}}{t\sqrt{\varphi(x_j)}} \\ &\leqslant C\|m\|_{X^L_w} \frac{\|f\mathbf{1}_{2B_j}\|_{L^1(\lambda)}}{t}. \end{aligned} \tag{3.12}$$

By combining (3.10), (3.11) and (3.12) we obtain that

$$\begin{aligned}\lambda^\varphi(\{|\mathcal{A}f| > t\}) &\leqslant C\frac{\|m\|_{X^L_w}}{t}\sum_j \varphi(x_j)\|f\mathbf{1}_{2B_j}\|_{L^1(\lambda)} \\ &\leqslant C\frac{\|m\|_{X^L_w}}{t}\|f\|_{L^1(\lambda^\varphi)}.\end{aligned}$$

This concludes the proof of the proposition. □

To conclude the proof of the multiplier theorem in the parabolic region $\boldsymbol{P}_w$, it remains to estimate the norm on $L^p(\lambda^\varphi)$ of the operator

$$M(\mathcal{L}^\varphi) = (2\pi)^{-1/2}\int_{1/2}^\infty \hat{m}_{\Phi,b}(t)\cos(t\sqrt{\mathcal{L}^\varphi - b_{\mathrm{e}}})\mathit{\Pi}_{I_2}\,\mathrm{d}t.$$

Proposition 3.5. *Assume that $1 \leqslant p < 2$ and let w, L and m be as in Theorem 3.1. Then the operator $M(\mathcal{L}^\varphi)$ is bounded on $L^q(\lambda^\varphi)$ for every q in $[p, p']$ and there exists a constant C such that*

$$\|M(\mathcal{L}^\varphi)\|_{L^q(\lambda^\varphi)} \leqslant C\|m\|_{X^L_w}.$$

Proof. By standard duality and interpolation arguments it is sufficient to prove that $M(\mathcal{L}^\varphi)$ is bounded on $L^p(\lambda^\varphi)$. Let $\mathcal{U}_p : L^p(\lambda) \to L^p(\lambda^\varphi)$ be the isometry defined by

$$\mathcal{U}_p f = \varphi^{-1/p} f.$$

Then the operator $M(\mathcal{L}^\varphi)$ is bounded on $L^p(\lambda^\varphi)$ if and only if

$$\mathcal{U}_p^{-1}M(\mathcal{L}^\varphi)\mathcal{U}_p = \mathcal{U}_p^{-1}\mathcal{U}_2 M(\mathcal{H})\mathcal{U}_2^{-1}\mathcal{U}_p$$

is bounded on $L^p(\lambda)$. Moreover, the norm of the operator $M(\mathcal{L}^\varphi)$ on $L^p(\lambda^\varphi)$ coincides with the norm of $\mathcal{U}_p^{-1}\mathcal{U}_2 M(\mathcal{H})\mathcal{U}_2^{-1}\mathcal{U}_p$ on $L^p(\lambda)$.

Note that

$$M(\mathcal{H}) = (2\pi)^{-1/2}\int_{1/2}^\infty \hat{m}_{\Phi,b}(t)\cos(t\sqrt{\mathcal{H} - b_{\mathrm{e}}})\tilde{\mathit{\Pi}}_{I_2}\,\mathrm{d}t, \tag{3.13}$$

where $\tilde{\mathit{\Pi}}$ is the projection-valued measure associated with the self-adjoint operator $\mathcal{H}$.

Let ℓ be the least even integer greater than $\frac{1}{4}d$. Then

$$(I + \mathcal{H})^\ell M(\mathcal{H}) = (2\pi)^{-1/2}\int_{1/2}^\infty (1 + b_{\mathrm{e}} - D_t^2)^\ell \hat{m}_{\Phi,b}(t)\cos(t\sqrt{\mathcal{H} - b_{\mathrm{e}}})\tilde{\mathit{\Pi}}_{I_2}\,\mathrm{d}t.$$

https://doi.org/10.1017/S0013091506001234 Published online by Cambridge University Press

Since $\|\cos(t\sqrt{\mathcal{H}-b_{\mathrm{e}}})\tilde{\Pi}_{I_2}\|_{L^2(\lambda)} \leqslant C\mathrm{e}^{wt}$ and $L > \frac{1}{2}d+5$, the estimates (3.4) imply that the operator $(I+\mathcal{H})^{\ell}M(\mathcal{H})$ is bounded on $L^2(\lambda)$. Thus, by Proposition 2.1, the operator $M(\mathcal{H})$ is an integral operator whose kernel with respect to the Lebesgue measure is the function

$$k_M(x,y) = \mathcal{H}^{\ell}M(\mathcal{H})k_1(x,\cdot) + M(\mathcal{H})k_2(x,\cdot), \tag{3.14}$$

where $k_j(x,\cdot)$, $j=1,2$, are two functions in $L^2(\lambda)$ with support in $\overline{B(x,1)}$ such that $\sup_x \|k_j(x,\cdot)\|_{L^2(\lambda)} < \infty$. To prove that $\mathcal{U}_p^{-1}\mathcal{U}_2 M(\mathcal{H})\mathcal{U}_2^{-1}\mathcal{U}_p$ is bounded on $L^p(\lambda)$ it suffices to show that it is bounded on $L^1(\lambda)$ and on $L^{\infty}(\lambda)$, i.e. that its kernel

$$\left(\frac{\phi(x)}{\phi(y)}\right)^{1/p-1/2} k_M(x,y)$$

satisfies the estimates

$$\sup_y \int \left(\frac{\phi(x)}{\phi(y)}\right)^{1/p-1/2} |k_M(x,y)|\,\mathrm{d}\lambda(x) < \infty,$$

$$\sup_x \int \left(\frac{\phi(x)}{\phi(y)}\right)^{1/p-1/2} |k_M(x,y)|\,\mathrm{d}\lambda(y) < \infty.$$

Note that $\varphi(x)/\varphi(y)$ and $\varphi(y)/\varphi(x)$ are both bounded by $C\mathrm{e}^{\beta|x-y|}$ by (1.2). Since the adjoint $M(\mathcal{H})^*$ is the operator obtained by replacing m by $m^{\sharp}(z) = \overline{m(\bar{z})}$ in (3.13) and the map $m \mapsto m^{\sharp}$ is an isometry of X_w^L, it suffices to prove that

$$\sup_x \int \mathrm{e}^{\beta(1/p-1/2)|x-y|}|k_M(x,y)|\,\mathrm{d}\lambda(y) < \infty.$$

To estimate this integral we divide $\mathbb{R}^d$ into shells

$$A_x(j) = \{y \in \mathbb{R}^d : j \leqslant |x-y| < j+1\}$$

and we observe that

$$\begin{aligned}\int \mathrm{e}^{\beta(1/p-1/2)|x-y|}|k_M(x,y)|\,\mathrm{d}\lambda(y) &= \sum_{j=0}^{\infty}\int_{A_x(j)} \mathrm{e}^{\beta(1/p-1/2)|x-y|}|k_M(x,y)|\,\mathrm{d}\lambda(y) \\ &\leqslant C\sum_{j=0}^{\infty} \mathrm{e}^{\beta(1/p-1/2)j}\|k_M(x,\cdot)\|_{L^2(A_x(j))}, \end{aligned} \tag{3.15}$$

by Schwarz's inequality. By (3.14),

$$\|k_M(x,\cdot)\|_{L^2(A_x(j))} \leqslant \|\mathcal{H}^{\ell}M(\mathcal{H})k_1(x,\cdot)\|_{L^2(A_x(j))} + \|M(\mathcal{H})k_2(x,\cdot)\|_{L^2(A_x(j))}. \tag{3.16}$$

By (3.13) we have

$$M(\mathcal{H})k_2(x,\cdot) = (2\pi)^{-1/2}\int_{1/2}^{\infty} \hat{m}_{\Phi,b}(t)\cos(t\sqrt{\mathcal{H}-b_{\mathrm{e}}})\tilde{\Pi}_{I_2}k_2(x,\cdot)\,\mathrm{d}t, \tag{3.17}$$

$$\mathcal{H}^{\ell}M(\mathcal{H})k_1(x,\cdot) = (2\pi)^{-1/2}\int_{1/2}^{\infty} (b_{\mathrm{e}} - D_t^2)^{\ell}\hat{m}_{\Phi,b}(t)\cos(t\sqrt{\mathcal{H}-b_{\mathrm{e}}})\tilde{\Pi}_{I_2}k_1(x,\cdot)\,\mathrm{d}t. \tag{3.18}$$

https://doi.org/10.1017/S0013091506001234 Published online by Cambridge University Press

Moreover, by (3.4),

$$|\hat{m}_{\Phi,b}(t)| + |(b_{\rm e} - D_t^2)^{\ell} \hat{m}_{\Phi,b}(t)| \leqslant C\|m\|_{X_w^L}(1+|t|)^{-L}{\rm e}^{-w|t|} \quad \text{for all } t \in \mathbb{R}. \tag{3.19}$$

We begin by estimating the second summand in the right-hand side of (3.16). As in [**16**] we estimate the L^2-norm on $A_x(j)$ using the finite propagation speed property of $\mathcal{H} - b_{\rm e}$. Note, however, that the operator $\cos(t\sqrt{\mathcal{H} - b_{\rm e}})\tilde{\Pi}_{I_2}$ in (3.17) is the wave propagator of $(\mathcal{H} - b_{\rm e})\tilde{\Pi}_{I_2}$, which does not have finite speed of propagation. Thus, to be able to exploit the property we write

$$\cos(t\sqrt{\mathcal{H} - b_{\rm e}})\tilde{\Pi}_{I_2} = \cos(t\sqrt{\mathcal{H} - b_{\rm e}}) - \sum_{i=1}^{r} \cos(t\sqrt{E_i - b_{\rm e}})\tilde{\Pi}_{E_i}, \tag{3.20}$$

where $E_1 < E_2 < \cdots < E_r \leqslant b_{\rm e} - w^2$ are the eigenvalues of $\mathcal{H}$ to the left of $I_2 = (b_{\rm e} - w^2, \infty)$ and we split the integral in (3.17) into the sum of two integrals over the intervals $[1/2, j-1]$ and $[j-1, \infty)$. Using (3.20) and finite propagation speed, we get

$$\begin{aligned}
&\int_{1/2}^{j-1} \cdots \,{\rm d}t|_{A_x(j)} \\
&\quad = \int_{1/2}^{j-1} \hat{m}_{\Phi,b}(t)\biggl[\cos(t\sqrt{\mathcal{H} - b_{\rm e}}) - \sum_{i=1}^{r} \cos(t\sqrt{E_i - b_{\rm e}})\tilde{\Pi}_{E_i}\biggr] k_2(x,\cdot)\,{\rm d}t|_{A_x(j)} \\
&\quad = -\int_{1/2}^{j-1} \hat{m}_{\Phi,b}(t) \sum_{i=1}^{r} \cos(t\sqrt{E_i - b_{\rm e}})\tilde{\Pi}_{E_i} k_2(x,\cdot)\,{\rm d}t|_{A_x(j)}.
\end{aligned}$$

Hence, by (3.19),

$$\begin{aligned}
&\biggl\|\int_{1/2}^{j-1} \cdots \,{\rm d}t\biggr\|_{L^2(A_x(j))} \\
&\qquad \leqslant C\|m\|_{X_w^L} \sum_{i=1}^{r} \int_{1/2}^{j-1} \frac{{\rm e}^{-wt}}{(1+t)^L} \cosh(t\sqrt{b_{\rm e} - E_i})\,{\rm d}t \|\tilde{\Pi}_{E_i} k_2(x,\cdot)\|_{L^2(A_x(j))} \\
&\qquad \leqslant C\|m\|_{X_w^L} \sum_{i=1}^{r} \int_{1/2}^{j-1} \exp((\sqrt{b_{\rm e} - E_i} - w)t)\,{\rm d}t \|\tilde{\Pi}_{E_i} k_2(x,\cdot)\|_{L^2(A_x(j))} \\
&\qquad \leqslant C\|m\|_{X_w^L} \sum_{i=1}^{r} \exp((\sqrt{b_{\rm e} - E_i} - w)j) \|\tilde{\Pi}_{E_i} k_2(x,\cdot)\|_{L^2(A_x(j))}.
\end{aligned}$$

Now, choose $\epsilon > 0$ such that $w > \beta(1/p - 1/2) + 2\epsilon$. By Lemma 2.2 (ii) there exists a constant C such that

$$\begin{aligned}
\|\tilde{\Pi}_{E_i} k_2(x,\cdot)\|_{L^2(A_x(j))} &\leqslant C\|k_2(x,\cdot)\|_{L^2(\lambda)} \exp(-(\sqrt{b_{\rm e} - E_i} - \epsilon)j) \\
&\leqslant C\exp(-(\sqrt{b_{\rm e} - E_i} - \epsilon)j) \quad \text{for all } i = 1, \ldots, r,
\end{aligned}$$

where the second inequality follows from Proposition 2.1 (ii). Thus,

$$\begin{aligned}\bigg\|\int_{1/2}^{j-1}\cdots\,\mathrm{d}t\bigg\|_{L^2(A_x(j))} &\leqslant C\|m\|_{X_w^L}\mathrm{e}^{-(w-\epsilon)j}\\ &\leqslant C\|m\|_{X_w^L}\mathrm{e}^{-(\beta(1/p-1/2)+\epsilon)j}. \end{aligned}\tag{3.21}$$

Next we estimate the norm in $L^2(A_x(j))$ of the integral over the interval $[j-1,\infty)$. Let I_3 denote the interval $(b_\mathrm{e}-\epsilon,\infty)$ and let $E_{r+1}<E_{r+2}<\cdots<E_{r+s}$ be the eigenvalues of $\mathcal{H}$ in $I_2\setminus I_3$. Then

$$\cos(t\sqrt{\mathcal{H}-b_\mathrm{e}})\tilde{\Pi}_{I_2}=\cos(t\sqrt{\mathcal{H}-b_\mathrm{e}})\tilde{\Pi}_{I_3}+\sum_{i=1}^{s}\cos(t\sqrt{E_{r+i}-b_\mathrm{e}})\tilde{\Pi}_{E_{r+i}},\tag{3.22}$$

and

$$\|\cos(t\sqrt{\mathcal{H}-b_\mathrm{e}})\tilde{\Pi}_{I_3}\|_{L^2(\lambda)}\leqslant \mathrm{e}^{\epsilon t}\quad\text{for all } t\geqslant 0.$$

Thus,

$$\begin{aligned}&\bigg\|\int_{j-1}^{\infty}\cdots\,\mathrm{d}t\bigg\|_{L^2(A_x(j))}\\ &\quad\leqslant\int_{j-1}^{\infty}|\hat{m}_{\Phi,b}(t)|\|\cos(t\sqrt{\mathcal{H}-b_\mathrm{e}})\tilde{\Pi}_{I_3}\|_{L^2(\lambda)}\,\mathrm{d}t\|k_2(x,\cdot)\|_{L^2(\lambda)}\\ &\quad\quad+\int_{j-1}^{\infty}|\hat{m}_{\Phi,b}(t)|\sum_{i=1}^{s}\cosh(t\sqrt{b_\mathrm{e}-E_{r+i}})\,\mathrm{d}t\|\tilde{\Pi}_{E_{r+i}}k_2(x,\cdot)\|_{L^2(A_x(j))}\\ &\quad\leqslant C\|m\|_{X_w^L}\int_{j-1}^{\infty}\mathrm{e}^{-wt}\mathrm{e}^{\epsilon t}\,\mathrm{d}t\\ &\quad\quad+C\|m\|_{X_w^L}\sum_{i=1}^{s}\int_{j-1}^{\infty}\exp((\sqrt{b_\mathrm{e}-E_{r+i}}-w)t)\,\mathrm{d}t\exp(-(\sqrt{b_\mathrm{e}-E_{r+i}}-\epsilon)j)\\ &\quad\leqslant C\|m\|_{X_w^L}\mathrm{e}^{-(w-\epsilon)j}\\ &\quad\leqslant C\|m\|_{X_w^L}\mathrm{e}^{-(\beta(1/p-1/2)+\epsilon)j},\end{aligned}\tag{3.23}$$

where, as before, we have used (3.19) to estimate $|\hat{m}_{\Phi,b}(t)|$ and Lemma 2.2 (ii) to estimate the norm of $\tilde{\Pi}_{E_{r+i}}k_2(x,\cdot)$ in $L^2(A_x(j))$.

Now, by (3.17) and estimates (3.21) and (3.23),

$$\|M(\mathcal{H})k_2(x,\cdot)\|_{L^2(A_x(j))}\leqslant C\|m\|_{X_w^L}\mathrm{e}^{-(\beta(1/p-1/2)+\epsilon)j}.\tag{3.24}$$

A similar argument shows that the same estimate holds for $\|\mathcal{H}^\ell M(\mathcal{H})k_1(x,\cdot)\|_{L^2(A_x(j))}$. Thus, by (3.15) and (3.16),

$$\int \mathrm{e}^{\beta(1/p-1/2)|x-y|}|k_M(x,y)|\,\mathrm{d}\lambda(y)\leqslant C\|m\|_{X_w^L}\sum_{j=1}^{\infty}\mathrm{e}^{-\epsilon j}\leqslant C\|m\|_{X_w^L}.$$

This concludes the proof of the proposition. □

https://doi.org/10.1017/S0013091506001234 Published online by Cambridge University Press

4. Functional calculus in pencil-like regions

In order to state our main result we need more notation. Suppose that w is in $\mathbb{R}^+$, that θ is in $(0,\pi)$ and that L is a non-negative integer. We denote by $Z^L_{w,\theta}$ the space of all f in $H^\infty(\boldsymbol{R}_{w,\theta})$ such that the quantity $N(f)$, defined by

$$N(f) = \max_{\ell\in\{0,1,\dots,L\}} \sup_{\zeta\in\boldsymbol{R}_{w,\theta}:|\zeta|\geqslant 1} |\zeta|^\ell |D^\ell f(\zeta)|, \tag{4.1}$$

is finite. We endow $Z^L_{w,\theta}$ with the norm

$$\|f\|_{Z^L_{w,\theta}} = \|f\|_\infty + N(f).$$

Remark 4.1. Observe that if f is in $H^\infty(\boldsymbol{R}_{w',\theta'})$ and there exists a number $r > 0$ such that

$$\max_{\ell\in\{0,1,\dots,L\}} \sup_{\zeta\in\boldsymbol{R}_{w',\theta'}:|\zeta|\geqslant r} |\zeta|^\ell |D^\ell f(\zeta)| < \infty,$$

then f is in $Z^L_{w,\theta}$ whenever $w < w'$ and $\theta < \theta'$.

Theorem 4.2. *Assume that the function φ satisfies (1.2) and (1.3). Suppose that p is in $(1,\infty)$, that $w' > \beta|1/p-1/2|$, that L is an integer $> d/2+5$, that $b_{\mathrm{e}} > 0$ and that $\theta' > \phi^*_p$. If m is a function defined on $\sigma(\mathcal{L}^\varphi)\cup\boldsymbol{R}_{w,'\theta'}$ and holomorphic in $\boldsymbol{R}_{w,'\theta'}$ whose restriction to $\boldsymbol{R}_{w,\theta}$ is in $Z^L_{w,\theta}$, for some θ in (ϕ^*_p,θ') and w in $(\beta|1/p-1/2|, w')$, then m is a spectral multiplier of $L^p(\lambda^\varphi)$ for $\mathcal{L}^\varphi$. Furthermore, there exists a constant C, independent of m, such that*

$$|||m(\mathcal{L}^\varphi)|||_{L^p(\lambda^\varphi)} \leqslant C(\|m\|_\infty + \|m\|_{Z^L_{w,\theta}}).$$

We do not know whether functions in $Z^L_{\beta|1/p-1/2|,\phi^*_p}$ are $L^p(\lambda^\varphi)$ spectral multipliers of $\mathcal{L}^\varphi$ for some large integer L: we need to assume that m belongs to $Z^L_{w,\theta}$ for some $w > \beta|1/p-1/2|$ and $\theta > \phi^*_p$. By slightly decreasing both w and θ we may always assume that m is in $Z^L_{w,\theta}$ and is holomorphic and bounded in a region bigger than $\boldsymbol{R}_{w,\theta}$. It is not restrictive to assume that such a bigger region is $\boldsymbol{R}_{w',\theta'}$ for some $w' > \beta|1/p-1/2|$ and $\theta' > \phi^*_p$.

It may be worth stating explicitly the following corollary.

Corollary 4.3. *Assume that the function φ satisfies (1.2) and (1.3). Suppose that p is in $(1,\infty)$ and that $\theta > \phi^*_p$. Assume also that $b_{\mathrm{e}} > 0$. If m is defined on $\{0\}\cup\boldsymbol{S}_\theta$ and belongs to $H^\infty(\boldsymbol{S}_\theta)$, then m is an $L^p(\lambda^\varphi)$ spectral multiplier of $\mathcal{L}^\varphi$. Furthermore, there exists a constant C, independent of m, such that*

$$|||m(\mathcal{L}^\varphi)|||_{L^p(\lambda^\varphi)} \leqslant C\|m\|_\infty.$$

Proof. It is straightforward to check that the restriction of m to $\boldsymbol{R}_{w,\theta'}$ is in $Z^L_{w,\theta'}$ for any w in $\mathbb{R}^+$, any integer L, and for all $\theta' < \theta$. Furthermore, its norm in $Z^L_{w,\theta'}$ is dominated by $C\|m\|_{H^\infty(\boldsymbol{S}_\theta)}$. Therefore, Theorem 4.2 applies whenever $\phi^*_p < \theta'$, and the required result follows. □

https://doi.org/10.1017/S0013091506001234 Published online by Cambridge University Press

Remark 4.4. For each u in $\mathbb{R}$, denote by m_u the function defined on the closed right half-plane as follows:

$$m_u(0) = 0 \quad \text{and} \quad m_u(z) = z^{\mathrm{i}u} \quad \text{for all } z \in \bar{\boldsymbol{S}}_{\pi/2} \setminus \{0\}.$$

Corollary 4.3 applied to m_u gives the following estimate: for every $\theta > \phi_p^*$ there exists a constant C, independent of u, such that

$$|||m_u(\mathcal{L}^\varphi)|||_{L^p(\lambda^\varphi)} \leqslant C\mathrm{e}^{\theta|u|} \quad \text{for all } u \in \mathbb{R}.$$

In the case where $\log\varphi$ is an admissible weight in the sense of [**9**, § 4], the following estimate from below holds [**9**, Theorem 5.2 (i)]:

$$C\mathrm{e}^{\phi_p^*|u|} \leqslant |||m_u(\mathcal{L}^\varphi)|||_{L^p(\lambda^\varphi)} \quad \text{for all } u \in \mathbb{R},$$

proving that the upper estimate above is almost optimal.

The idea of the proof of Theorem 4.2 is simple. Given m as in the statement of the theorem, we write $m = m_0 + m_\infty$, where m_∞ satisfies the hypotheses of Theorem 3.1, and m_0 is holomorphic and bounded outside a certain compact curve in the complex plane whose support lies outside $\boldsymbol{S}_\theta$ for some $\theta > \phi_p^*$. This decomposition is accomplished in Lemma 4.5 below, except for a technical assumption that will be removed later.

Recall that $\Phi(z) = z^2 + b_{\mathrm{e}}$ and that m_Φ denotes the composite function $m \circ \Phi$. We denote by z_- and z_+ the two points in $\partial \boldsymbol{P}_w \cap \partial \boldsymbol{S}_\theta$ with largest real part, and by t_0 the point in $\mathbb{R}^+$ such that $z_- = \Phi(-t_0 + \mathrm{i}w)$ and $z_+ = \Phi(t_0 + \mathrm{i}w)$. Observe that the boundary of $\boldsymbol{R}_{w,\theta}$ is the union of the support of three curves ω_0, ω_- and ω_+. The curve ω_0 runs from z_+ to z_- along the boundary of $\boldsymbol{R}_{w,\theta}$ and ω_+, ω_- are the two arcs of parabola from ∞ to z_+ and from z_- to ∞, respectively. We shall denote by ω_0^* the support of the curve ω_0.

Lemma 4.5. *Suppose that w is positive, that θ is in $(0, \pi/2)$ and that L is a positive integer. Let m be a function in $C^L(\bar{\boldsymbol{R}}_{w,\theta} \setminus \{0\}) \cap H^\infty(\boldsymbol{R}_{w,\theta})$. Assume, further that m vanishes of order at least $L+1$ at z_- and z_+, and that there exists a constant C such that*

$$|D^\ell m_\Phi(x + \mathrm{i}w)| \leqslant C(1 + |x|)^{-\ell}$$

for $|x| \geqslant t_0$ and for all $\ell \in \{0, 1, \dots, L\}$. Define

$$m_0(z) = \frac{1+z}{2\pi\mathrm{i}} \int_{\omega_0} \frac{m(\zeta)}{(\zeta - z)(\zeta + 1)}\,\mathrm{d}\zeta \quad \textit{for all } z \in \mathbb{C} \setminus \omega_0^*.$$

Then $m = m_0 + m_\infty$, where m_∞ is in X_w^L, and there exists a constant C, independent of m, such that

$$\|m_\infty\|_{X_w^L} \leqslant C\|m\|_{Z_{w,\theta}^L}.$$

The proof of Lemma 4.5 is somewhat technical and will be deferred to the next section.

https://doi.org/10.1017/S0013091506001234 Published online by Cambridge University Press

The proof of Theorem 4.2 requires some information (contained in Lemma 4.6 below) concerning the spectra of $\mathcal{L}^\varphi$ and some related operators, which we now define.

It is well known that the operator $-\mathcal{L}^\varphi$ generates a symmetric diffusion semigroup. Therefore, the restriction of $-\mathcal{L}^\varphi$ to $C_c(\mathbb{R}^d)$ is a closable operator on $L^p(\lambda^\varphi)$ for every p in $[1,\infty)$ and its closure, which we denote by $-\mathcal{L}^\varphi_p$, is the generator of the semigroup on $L^p(\lambda^\varphi)$. Note that $\mathcal{L}^\varphi_2 = \mathcal{L}^\varphi$ and that all the semigroups $\{e^{-t\mathcal{L}^\varphi_p} : t \geqslant 0\}$ are consistent, i.e. they agree on $C_c(\mathbb{R}^d)$. We shall denote by $\sigma(\mathcal{L}^\varphi_p)$, $\sigma_d(\mathcal{L}^\varphi_p)$ and $\rho(\mathcal{L}^\varphi_p)$ the spectrum, the discrete spectrum and the resolvent of $\mathcal{L}^\varphi_p$, respectively.

Lemma 4.6. *Suppose that p is in $[1,\infty)$, and denote by β_p the number $\beta|1/p-1/2|$. Then $\sigma(\mathcal{L}^\varphi_p)$ is contained in $\bar{\boldsymbol{S}}_{\phi^*_p}$ and, for every θ in (ϕ^*_p,π),*

$$\sup_{\zeta\in\mathbb{C}\setminus\boldsymbol{S}_\theta} |\zeta| |||(\zeta\mathcal{I}-\mathcal{L}^\varphi_p)^{-1}|||_{L^p(\lambda^\varphi)} < \infty. \tag{4.2}$$

Moreover, for p in $(1,\infty)$, the following hold:

(i) *if $0\in\rho(\mathcal{L}^\varphi)$, then $0\in\rho(\mathcal{L}^\varphi_p)$;*

(ii) *if $0\in\sigma_d(\mathcal{L}^\varphi)$, then $0\in\sigma_d(\mathcal{L}^\varphi_p)$ and the spectral projection Π_0 onto the kernel of $\mathcal{L}^\varphi$ is bounded on $L^p(\lambda^\varphi)$;*

(iii) *the spectrum of $\mathcal{L}^\varphi_p$ is contained in $\sigma_d(\mathcal{L}^\varphi)\cup(\bar{\boldsymbol{S}}_{\phi^*_p}\cap\bar{\boldsymbol{P}}_{\beta_p})$. Moreover, if λ is in the discrete spectrum of $\mathcal{L}^\varphi$ and λ is not in $\bar{\boldsymbol{P}}_{\beta_p}$, then the spectral projection Π_λ is bounded on $L^p(\lambda^\varphi)$ and λ belongs to the discrete spectrum of $\mathcal{L}^\varphi_p$.*

Proof. By [**12**] the semigroup generated by $-\mathcal{L}^\varphi_p$ on $L^p(\lambda^\varphi)$ is bounded and holomorphic in the sector $S_{\pi/2-\phi^*_p}$. Hence, the spectrum of $\mathcal{L}^\varphi_p$ is contained in the closed sector $\bar{\boldsymbol{S}}_{\phi^*_p}$ and estimate (4.2) holds.

Now we prove (i). Suppose first that p is in $(1,2)$. Since 0 is in the resolvent set of $\mathcal{L}^\varphi$, the bottom b of $\sigma(\mathcal{L}^\varphi)$ is positive. By spectral theory $|||e^{-t\mathcal{L}^\varphi}|||_{L^2(\lambda^\varphi)} \leqslant e^{-bt}$ for all positive t. Recall that $|||e^{-t\mathcal{L}^\varphi_1}|||_{L^1(\lambda^\varphi)} \leqslant 1$ for all positive t. Since the semigroups generated by $\mathcal{L}^\varphi_1$ on $L^1(\lambda^\varphi)$ and by $\mathcal{L}^\varphi$ on $L^2(\lambda^\varphi)$ are consistent, an interpolation argument shows that $|||e^{-t\mathcal{L}^\varphi_p}|||_{L^p(\lambda^\varphi)} \leqslant e^{-2b(1-1/p)t}$ for all positive t. Therefore, $\sigma(\mathcal{L}^\varphi_p)$ is contained in the half-plane $\{z : \operatorname{Re} z \geqslant 2b(1-1/p)\}$ by abstract semigroup theory. This proves (i) for $1<p<2$.

The result for $p>2$ follows by symmetry and duality.

Next we prove (ii). Assume that 0 is in the discrete spectrum of $\mathcal{L}^\varphi$. By (1.1) the kernel of $\mathcal{L}^\varphi$ is the space $\mathbb{C}\mathbf{1}$ of constant functions. Thus, λ^φ is a finite measure and the spectral projection Π_0 on the kernel of $\mathcal{L}^\varphi$ is bounded on $L^p(\lambda^\varphi)$ for all p in $[1,\infty)$. Define $L^p_0(\lambda^\varphi) = (\mathcal{I}-\Pi_0)L^p(\lambda^\varphi)$. Clearly, $L^p_0(\lambda^\varphi)$ is the closed subspace of all functions f in $L^p(\lambda^\varphi)$ such that $\int f\,d\lambda^\varphi = 0$ and $L^p(\lambda^\varphi) = \mathbb{C}\mathbf{1}\oplus L^p_0(\lambda^\varphi)$. Note that both subspaces are $\mathcal{L}^\varphi_p$-invariant. Denote by $\mathcal{L}^\varphi_{p,0}$ the restriction of $\mathcal{L}^\varphi_p$ to $L^p_0(\lambda^\varphi)$. Then $\mathcal{L}^\varphi_p = 0\oplus\mathcal{L}^\varphi_{p,0}$ and therefore $\sigma(\mathcal{L}^\varphi_p) = \{0\}\cup\sigma(\mathcal{L}^\varphi_{p,0})$. Next we show that $0\in\rho(\mathcal{L}^\varphi_{2,0})$. Indeed, it is well known that all the isolated points in the spectrum of a self-adjoint operator on a Hilbert space

https://doi.org/10.1017/S0013091506001234 Published online by Cambridge University Press

are semisimple eigenvalues. Therefore, 0 is a semisimple eigenvalue for the operator $\mathcal{L}^\varphi$ on $L^2(\lambda^\varphi)$. This means that

$$L^2(\lambda^\varphi) = \mathrm{Ker}\,(\mathcal{L}^\varphi) \oplus \mathrm{Ran}\,(\mathcal{L}^\varphi)\,.$$

Since obviously $\mathcal{L}^\varphi_{2,0}$ is injective and $L^2_0(\lambda^\varphi) = \mathrm{Ran}\,(\mathcal{L}^\varphi) = \mathrm{Ran}\,\big(\mathcal{L}^\varphi_{2,0}\big)$, 0 belongs to the resolvent set of $\mathcal{L}^\varphi_{2,0}$, whence the bottom of the spectrum b_0 of $\mathcal{L}^\varphi_{2,0}$ is positive. Thus,

$$|||\exp(-t\mathcal{L}^\varphi_{2,0})|||_{L^2_0(\lambda^\varphi)} \leqslant \mathrm{e}^{-b_0 t}.$$

Since

$$|||\exp(-t\mathcal{L}^\varphi_{1,0})|||_{L^1_0(\lambda^\varphi)} \leqslant C,$$

the same interpolation argument used in the proof of (i) shows that 0 is in $\rho(\mathcal{L}^\varphi_{p,0})$ for every p in $(1,\infty)$. Thus, 0 is an isolated eigenvalue of $\mathcal{L}^\varphi_p$. Hence, $\mathrm{Ker}\,\big(\mathcal{L}^\varphi_p\big) \subset \varPi_0(L^p(\lambda^\varphi))$, and 0 has finite multiplicity.

Finally, we prove (iii). We already know that the spectrum of $\mathcal{L}^\varphi_p$ is contained in $\bar{\boldsymbol{S}}_{\phi^*_p}$. Thus, it remains to show that if λ does not belong to $\sigma_{\mathrm{d}}(\mathcal{L}^\varphi) \cup (\bar{\boldsymbol{P}}_{\beta_p})$, then λ is in the resolvent set of $\mathcal{L}^\varphi_p$.

Consider the function defined by $r_\lambda(z) = (\lambda - z)^{-1}$. Then $r_\lambda(\mathcal{L}^\varphi) = (\lambda\mathcal{I} - \mathcal{L}^\varphi)^{-1}$ by the spectral theorem. Observe that r_λ is in X^L_w for some $w > \beta_p = \beta|1/p - 1/2|$ and for all positive integers L. Then Theorem 3.1 implies that for every p in $(1,\infty)$ the operator $r_\lambda(\mathcal{L}^\varphi)$ extends from $L^2(\lambda^\varphi) \cap L^p(\lambda^\varphi)$ to a bounded operator on $L^p(\lambda^\varphi)$. Thus, λ is also in the resolvent set of $\mathcal{L}^\varphi_p$ by [**2**, Proposition 2.3], and $(\lambda - \mathcal{L}^\varphi_p)^{-1} = r_\lambda(\mathcal{L}^\varphi)$ on $L^2(\lambda^\varphi) \cap L^p(\lambda^\varphi)$, as required.

Next, assume that λ is a point in the discrete spectrum of $\mathcal{L}^\varphi$ which lies outside $\bar{\boldsymbol{P}}_{\beta_p}$. Since $\sigma(\mathcal{L}^\varphi_p) \subset \sigma_d(\mathcal{L}^\varphi) \cup \bar{\boldsymbol{P}}_{\beta_p}$, there is a neighbourhood N of λ such that $N \setminus \{\lambda\}$ is contained in $\rho(\mathcal{L}^\varphi) \cap \rho(\mathcal{L}^\varphi_p)$. Let γ be a closed simple curve which surrounds λ in N counterclockwise and is bounded away from $\bar{\boldsymbol{P}}_{\beta_p}$. Then, by functional calculus,

$$\varPi_\lambda = \frac{1}{2\pi\mathrm{i}} \int_\gamma (z\mathcal{I} - \mathcal{L}^\varphi)^{-1}\,\mathrm{d}z.$$

Since the support γ^* of γ is compact and contained in $\rho(\mathcal{L}^\varphi_p)$,

$$\sup_{z\in\gamma^*} |||(z\mathcal{I} - \mathcal{L}^\varphi)^{-1}|||_{L^p(\lambda^\varphi)} < \infty.$$

Hence, the spectral projection $\varPi_\lambda$ extends to a bounded operator on $L^p(\lambda^\varphi)$. Therefore, the finite-dimensional space $\varPi_\lambda(L^2(\lambda^\varphi) \cap L^p(\lambda^\varphi))$ is dense both in $\varPi_\lambda L^2(\lambda^\varphi)$ and in $\varPi_\lambda L^p(\lambda^\varphi)$. This shows that $\varPi_\lambda L^p(\lambda^\varphi) = \varPi_\lambda L^2(\lambda^\varphi)$. Thus, λ is an isolated eigenvalue of finite multiplicity of $\mathcal{L}^\varphi_p$. □

Proof of Theorem 4.2. We observe that m is holomorphic in a neighbourhood of $\bar{\boldsymbol{R}}_{w,\theta} \setminus \{0\}$ and satisfies Mihlin-type conditions of the form (4.1) for all ζ in $\bar{\boldsymbol{R}}_{w,\theta}$ such that $|\zeta| \geqslant t_0/2$.

We claim that it is not restrictive to assume that m vanishes of order at least $L+1$ at $z_\pm$.

Indeed, we may write $m = gh$, where

$$h(z) = \left(\frac{z+1}{z-z_+}\right)^{L+1}\left(\frac{z+1}{z-z_-}\right)^{L+1}$$

and $g = m/h$. Clearly, g vanishes of order at least $L+1$ at $z_\pm$ and satisfies the same assumptions as m. By the spectral theorem,

$$h(\mathcal{L}^\varphi) = (1+\mathcal{L}^\varphi)^{L+1}(1+\mathcal{L}^\varphi)^{L+1}(\mathcal{L}^\varphi - z_+)^{-(L+1)}(\mathcal{L}^\varphi - z_-)^{-(L+1)}.$$

By Lemma 4.6 the spectrum of $\mathcal{L}_p^\varphi$ is contained in the closed sector $\bar{\boldsymbol{S}}_{\phi_p^*}$. Since the points $z_\pm$ are on the boundary of the sector $\boldsymbol{S}_\theta$ and $\theta > \phi_p^*$, they are in $\rho(\mathcal{L}_p^\varphi)$, and hence the operator $h(\mathcal{L}^\varphi)$ is bounded on $L^p(\lambda^\varphi)$. Therefore, it suffices to show that the operator $g(\mathcal{L}^\varphi)$ is bounded on $L^p(\lambda^\varphi)$. This proves the claim.

Henceforth, we assume that the function m vanishes of order $L+1$ at $z_\pm$. We need to consider two cases, depending on whether $b_{\mathrm{e}} - w^2$ is positive or not.

Case 1 ($\boldsymbol{b_{\mathrm{e}} - w^2 > 0}$). In this case the support ω_0^* of the path ω_0 is the union of two segments on $\partial\boldsymbol{S}_\theta$ and a compact arc of parabola on $\partial\boldsymbol{P}_w$. By Lemma 4.6 (iii), ω_0^* is bounded away from $\sigma(\mathcal{L}_p^\varphi)$ unless a point of $\sigma_{\mathrm{d}}(\mathcal{L}^\varphi)$ lies on ω_0^*, in which case we may always modify ω_0^* by choosing a slightly smaller w, so that the modified ω_0^* is bounded away from $\sigma(\mathcal{L}_p^\varphi)$. Hence,

$$\sup_{\zeta\in\omega_0^*} \|(\zeta\mathcal{I} - \mathcal{L}^\varphi)^{-1}\|_{L^p(\lambda^\varphi)} < \infty. \tag{4.3}$$

By Lemma 4.5 we may write $m(\mathcal{L}^\varphi) = m_0(\mathcal{L}^\varphi) + m_\infty(\mathcal{L}^\varphi)$, where

$$m_0(z) = \frac{1+z}{2\pi\mathrm{i}}\int_{\omega_0} \frac{m(\zeta)}{\zeta+1}(\zeta - z)^{-1}\,\mathrm{d}\zeta,$$

and m_∞ is in X_w^L. The operator $m_\infty(\mathcal{L}^\varphi)$ is bounded on $L^p(\lambda^\varphi)$ by Theorem 3.1. Using the identity $(1+z)(\zeta-z)^{-1} = -1 + (\zeta+1)(\zeta-z)^{-1}$, we see that

$$m_0(z) = c_m + \frac{1}{2\pi\mathrm{i}}\int_{\omega_0}\frac{m(\zeta)}{\zeta - z}\,\mathrm{d}\zeta, \tag{4.4}$$

where

$$c_m = -\frac{1}{2\pi\mathrm{i}}\int_{\omega_0} m(\zeta)(1+\zeta)^{-1}\,\mathrm{d}\zeta.$$

Now (4.3) implies that

$$\|m_0(\mathcal{L}^\varphi)\|_{L^p(\lambda^\varphi)} \leqslant C\|m\|_\infty.$$

This concludes the proof of the theorem when $b_{\mathrm{e}} - w^2 > 0$.

https://doi.org/10.1017/S0013091506001234 Published online by Cambridge University Press

Case 2 ($b_e - w^2 \leqslant 0$). In this case ω_0^* is the union of two segments on ∂S_θ joining at 0.

If 0 is in the resolvent set of $\mathcal{L}^\varphi$, then 0 is also in the resolvent set of $\mathcal{L}_p^\varphi$ by Lemma 4.6 (i) and again ω_0^* is bounded away from $\sigma(\mathcal{L}_p^\varphi)$. Hence, we obtain the desired conclusion by arguing as in the previous case.

Now suppose that 0 is in $\sigma(\mathcal{L}^\varphi)$. First we show that there exists a constant C such that

$$\|m(\mathcal{L}^\varphi + \varepsilon\mathcal{I})\|_{L^p(\lambda^\varphi)} \leqslant C(\|m\|_\infty + \|m\|_{Z^L_{w,\theta}}) \quad \text{for all } \varepsilon \in (0,1]. \tag{4.5}$$

According to Lemma 4.5 we may write

$$m(\mathcal{L}^\varphi + \varepsilon\mathcal{I}) = m_0(\mathcal{L}^\varphi + \varepsilon\mathcal{I}) + m_\infty(\mathcal{L}^\varphi + \varepsilon\mathcal{I}).$$

Define $(\tau_\varepsilon m_\infty)(\lambda) = m_\infty(\lambda + \varepsilon)$. It is straightforward to check that there exists a constant C such that $\|\tau_\varepsilon m_\infty\|_{X^L_w} \leqslant C\|m_\infty\|_{X^L_w}$ for every $\varepsilon \in (0,1]$. Therefore, by Theorem 3.1 and Lemma 4.5,

$$\|m_\infty(\mathcal{L}^\varphi + \varepsilon\mathcal{I})\|_{L^p(\lambda^\varphi)} \leqslant C\|m_\infty\|_{Z^L_{w,\theta}} \quad \text{for all } \varepsilon \in (0,1]. \tag{4.6}$$

It remains to estimate $\|m_0(\mathcal{L}^\varphi + \varepsilon\mathcal{I})\|_{L^p(\lambda^\varphi)}$. Denote by ω_ε the path defined by $\omega_\varepsilon(t) = \omega_0(t) - \varepsilon$. Since b_e is positive by assumption, and 0 is in the spectrum of $\mathcal{L}^\varphi$, 0 is in the discrete spectrum of $\mathcal{L}^\varphi$. Hence, by Lemma 4.6 (iii), the point 0 is in the discrete spectrum of $\mathcal{L}_p^\varphi$. Therefore, there exist $\eta > 0$ such that $\sigma(\mathcal{L}_p^\varphi) \setminus \{0\}$ is contained in $\{z \in \mathbb{C} : |\arg z| \leqslant \phi_p^*, |z| \geqslant \eta\}$. It is straightforward to check that for every ε in $(0,1]$ there exists a path γ_ε from $z_- - \varepsilon$ to $z_+ - \varepsilon$ whose support γ_ε^* lies entirely in $\rho(\mathcal{L}_p^\varphi)$ and is bounded away from $\sigma(\mathcal{L}_p^\varphi)$ uniformly in ε (we may, and will, also assume that $\omega_\varepsilon + \gamma_\varepsilon$ is homotopic to a circle that surrounds the origin). Hence, there exists a constant C such that

$$\max_{\zeta \in \gamma_\varepsilon^*} \|(\zeta\mathcal{I} - \mathcal{L}_p^\varphi)^{-1}\|_{L^p(\lambda^\varphi)} \leqslant C \quad \text{for all } \varepsilon \in (0,1]. \tag{4.7}$$

By (4.4),

$$\begin{aligned} m_0(z+\varepsilon) &= c_m + \frac{1}{2\pi \mathrm{i}} \int_{\omega_0} \frac{m(\zeta)}{\zeta - z - \varepsilon}\, \mathrm{d}\zeta \\ &= c_m + \frac{1}{2\pi \mathrm{i}} \int_{\omega_\varepsilon} \frac{m(\zeta+\varepsilon)}{\zeta - z}\, \mathrm{d}\zeta \\ &= c_m + \frac{1}{2\pi \mathrm{i}} \int_{\omega_\varepsilon + \gamma_\varepsilon} \frac{m(\zeta+\varepsilon)}{\zeta - z}\, \mathrm{d}\zeta - \frac{1}{2\pi \mathrm{i}} \int_{\gamma_\varepsilon} \frac{m(\zeta+\varepsilon)}{\zeta - z}\, \mathrm{d}\zeta \end{aligned}$$

for all z in the spectrum of $\mathcal{L}^\varphi$. Now, by spectral theory,

$$m_0(\mathcal{L}^\varphi + \varepsilon\mathcal{I}) = c_m\mathcal{I} + n_\varepsilon(\mathcal{L}^\varphi) - \frac{1}{2\pi \mathrm{i}} \int_{\gamma_\varepsilon} m(\zeta+\varepsilon)(\zeta\mathcal{I} - \mathcal{L}^\varphi)\, \mathrm{d}\zeta,$$

where the function n_ε is defined by

$$n_\varepsilon(z) = \frac{1}{2\pi \mathrm{i}} \int_{\omega_\varepsilon + \gamma_\varepsilon} m(\zeta+\varepsilon)(\zeta - z)^{-1}\, \mathrm{d}\zeta.$$

Note that $n_\varepsilon(z)$ is equal to $m(z+\varepsilon)$ if z is inside the curve $\omega_\varepsilon+\gamma_\varepsilon$ and to 0 if z is outside $\omega_\varepsilon+\gamma_\varepsilon$. Therefore, $n_\varepsilon(\mathcal{L}^\varphi)=m(\varepsilon)\varPi_0$, and the boundedness of $\varPi_0$ on $L^p(\lambda^\varphi)$ implies that

$$\sup_{\varepsilon\in(0,1]}|\!|\!|n_\varepsilon(\mathcal{L}^\varphi)|\!|\!|_{L^p(\lambda^\varphi)} \leqslant \|m\|_\infty.$$

Then, using (4.7), we obtain that

$$|\!|\!|m_0(\mathcal{L}^\varphi+\varepsilon\mathcal{I})|\!|\!|_{L^p(\lambda^\varphi)} \leqslant C\|m\|_\infty \quad \text{for all } \varepsilon\in(0,1],$$

which, combined with (4.6), proves (4.5).

Now, (4.5) implies that

$$|\!|\!|(\mathcal{I}-\varPi_0)m(\mathcal{L}^\varphi+\varepsilon\mathcal{I})|\!|\!|_{L^p(\lambda^\varphi)} \leqslant C(\|m\|_\infty+\|m\|_{Z^L_{w,\theta}}) \quad \text{for all } \varepsilon\in(0,1]. \tag{4.8}$$

Observe that, for every $\varepsilon>0$,

$$m(\mathcal{L}^\varphi+\varepsilon\mathcal{I})f = m(\varepsilon)\varPi_0 f+\int_{0^+}^\infty m(\lambda+\varepsilon)\,\mathrm{d}\mathcal{P}_\lambda f \quad \text{for all } f\in L^2(\lambda^\varphi).$$

Thus, if $\varepsilon\to0^+$,

$$\begin{aligned}(\mathcal{I}-\varPi_0)m(\mathcal{L}^\varphi+\varepsilon\mathcal{I})f &= \int_{0^+}^\infty m(\lambda+\varepsilon)\,\mathrm{d}\mathcal{P}_\lambda f \to \int_{0^+}^\infty m(\lambda)\,\mathrm{d}\mathcal{P}_\lambda f\\ &= m(\mathcal{L}^\varphi)f-m(0)\varPi_0 f \quad \text{for all } f\in L^2(\lambda^\varphi).\end{aligned}$$

Therefore, (4.8) and Fatou's lemma imply that

$$|\!|\!|m(\mathcal{L}^\varphi)-m(0)\varPi_0|\!|\!|_{L^p(\lambda^\varphi)} \leqslant C(\|m\|_\infty+\|m\|_{Z^L_{w,\theta}})$$

and finally that

$$|\!|\!|m(\mathcal{L}^\varphi)|\!|\!|_{L^p(\lambda^\varphi)} \leqslant C(\|m\|_\infty+\|m\|_{Z^L_{w,\theta}}),$$

as required to complete the proof of the theorem. □

5. Proof of Lemma 4.5

Let γ denote a (possibly unbounded) piecewise smooth curve in $\mathbb{C}$ and let γ^* be its support. Suppose that -1 is not in γ^* and that if γ is unbounded,

$$\frac{|\gamma'(t)|}{1+|\gamma(t)|^2} \leqslant C(1+|t|)^{-1-\epsilon}$$

for some $\epsilon>0$. For G in $L^\infty(\gamma^*)$ consider the function F defined by

$$F(z)=\frac{z+1}{2\pi\mathrm{i}}\int_\gamma \frac{G(\zeta)}{\zeta-z}\frac{\mathrm{d}\zeta}{\zeta+1} \quad \text{for all } z\in\mathbb{C}\setminus\gamma^*.$$

Clearly, F is analytic in $\mathbb{C} \setminus \gamma^*$, in view of Morera's theorem. Note that the derivatives of F are given by

$$F^{(j)}(z) = \frac{1}{2\pi \mathrm{i}} \int_\gamma G(\zeta) \partial_z^j \biggl(\frac{z+1}{\zeta - z}\biggr) \frac{\mathrm{d}\zeta}{\zeta + 1} \quad \text{for all } z \in \mathbb{C} \setminus \gamma^*. \tag{5.1}$$

Note that

$$\frac{z+1}{\zeta - z} = -1 + \frac{\zeta + 1}{\zeta - z},$$

so that, for every j in $\mathbb{N} \setminus \{0\}$, there exists a constant C such that

$$\biggl|\partial_z^j \biggl(\frac{z+1}{\zeta - z}\biggr)\biggr| \leqslant C \frac{|\zeta + 1|}{|\zeta - z|^{j+1}}. \tag{5.2}$$

For z in a compact set $K \subset \mathbb{C} \setminus \gamma^*$, we have bounds

$$|F^{(j)}(z)| \leqslant C \|G\|_\infty \quad \text{for all } j \in \mathbb{N},$$

where C depends on j and K. Under the additional hypothesis that G has compact support in γ^*, we also have that

$$|F^{(j)}(z)| \leqslant C \|G\|_\infty \quad \text{and} \quad |F^{(j)}(z)| \leqslant C \frac{\|G\|_\infty}{|z|^{j+1}} \quad \text{for all } j \in \mathbb{N} \setminus \{0\} \tag{5.3}$$

for z bounded away from the support of G.

Next, we give a simple lemma saying that if G is small near the endpoints of γ^*, then F and some of its derivatives have non-tangential limits at the endpoints. We need more notation. Let γ_0 and γ_1 denote the endpoints of γ. We assume that $\gamma_0 \in \mathbb{C}$ and that $\gamma_1 = \infty$ if γ^* is unbounded. For every δ in $(0, 1)$, define the region $\Gamma_\delta(\gamma)$ of non-tangential approach to the endpoints of γ by

$$\Gamma_\delta(\gamma) = \{z \in \mathbb{C} : \operatorname{dist}(z, \gamma^*) \geqslant \delta \min\{|z - \gamma_0|, |z - \gamma_1|\}\}.$$

Lemma 5.1. *Suppose that G is a function of class C^{L+1} on $L^\infty(\gamma^*)$ that vanishes of order at least $L+1$ at the endpoints of γ. If γ is bounded, then, for every ℓ in $\{0, \dots, L\}$, j in $\{0, 1\}$ and δ in $(0, 1)$,*

$$\lim_{z \to \gamma_j, z \in \Gamma_\delta(\gamma)} F^{(\ell)}(z) = \frac{1}{2\pi \mathrm{i}} \int_\gamma G(\zeta) \partial_z^\ell \biggl(\frac{z+1}{\zeta - z}\biggr)\biggr|_{z = \gamma_j} \frac{\mathrm{d}\zeta}{\zeta + 1}. \tag{5.4}$$

Furthermore,

$$\sup_{z \in \Gamma_\delta(\gamma)} |F^{(\ell)}(z)| \leqslant C(\|G\|_\infty + \|G^{(L+1)}\|_\infty).$$

If $\gamma_1 = \infty$, the same conclusion holds for the limit in γ_0, assuming that G vanishes of order at least $L+1$ at γ_0.

https://doi.org/10.1017/S0013091506001234 Published online by Cambridge University Press

Proof. We shall prove the lemma only in the case where $\gamma_1 = \infty$. The proof when the endpoints are both finite requires only minor changes. Observe that the integral in (5.4) is convergent because of (5.2). Using (5.2), the definition of $\Gamma_\delta(\gamma)$ and the assumption on G, it is easy to see that if z is in $\Gamma_\delta(\gamma)$, and j is in $\{0, \dots, L\}$, then the integrand in (5.1) is dominated by a $C\|G^{(L+1)}\|_\infty$ when $|\zeta - \gamma_0| < 1$, by $C\|G\|_\infty(1 + |\zeta|)^{-2}$ when $j = 0$ and $|\zeta - \gamma_0| \geqslant 1$, and by $C\|G\|_\infty(1 + |\zeta|)^{-(j+1)}$ when $j > 0$ and $|\zeta - \gamma_0| \geqslant 1$. The desired convergence and estimate follow by the dominated convergence theorem. □

Proof of Lemma 4.5. Applying Cauchy's theorem to $m(z)/(z + 1)$, we see that

$$m(z) = \frac{z+1}{2\pi \mathrm{i}} \int_{\omega_+ + \omega_0 + \omega_-} \frac{m(\zeta)}{\zeta - z} \frac{\mathrm{d}\zeta}{\zeta + 1} \quad \text{for all } z \in \boldsymbol{R}_{w,\theta}.$$

Define the function m_∞ on $\mathbb{C}$ by setting

$$m_\infty(z) = \begin{cases} \dfrac{z+1}{2\pi \mathrm{i}} \displaystyle\int_{\omega_+ + \omega_-} \frac{m(\zeta)}{\zeta - z} \frac{\mathrm{d}\zeta}{\zeta + 1} & \text{if } z \in \mathbb{C} \setminus (\omega_+^* \cup \omega_-^*), \\ m(z) - m_0(z) & \text{if } z \in \omega_+^* \cup \omega_-^* \setminus \{z_+, z_-\}, \\ \displaystyle\lim_{z \to z_+,\, z \in \Gamma_\delta(\omega_+)} m_\infty(z) & \text{if } z = z_+, \\ \displaystyle\lim_{z \to z_-,\, z \in \Gamma_\delta(\omega_-)} m_\infty(z) & \text{if } z = z_-. \end{cases}$$

Note that the limits exist and do not depend on δ by Lemma 5.1. Thus, $m(z) = m_0(z) + m_\infty(z)$ for all z in $\bar{\boldsymbol{R}}_{w,\theta} \setminus \omega_0$.

Note that m_∞ is holomorphic on $\mathbb{C} \setminus (\omega_+^* \cup \omega_-^*)$, and hence in $\boldsymbol{P}_w$, because it is defined as a Cauchy integral over $\omega_+ + \omega_-$. Furthermore, m_∞ is bounded in $\boldsymbol{P}_w$. Indeed, observe that m is bounded on $\boldsymbol{R}_{w,\theta}$ by assumption, and that m_0 is bounded outside any neighbourhood of ω_0^* by (5.3), Since $m_\infty = m - m_0$, we only need to consider m_∞ near the points z_+ and z_-. By symmetry, we only need to consider z_+. Lemma 5.1 shows that m_∞ is bounded near z_+, except possibly in the complement of a region $\Gamma_\delta(\omega_+)$ of non-tangential approach to z_+. But in that complement m_∞ is the difference of m, which is bounded by assumption, and m_0, which is bounded again by Lemma 5.1. Thus, m_∞ is bounded on $\boldsymbol{P}_w$ near z_+.

To prove that m_∞ is in X_w^L it suffices to show that

$$|D^\ell (m_\infty)_\Phi (x \pm \mathrm{i}w)| \leqslant C \|m\|_{Z^L_{w,\theta}} (1 + |x|)^{-\ell} \quad \text{for all } x \in \mathbb{R} \text{ and all } \ell \in \{0, 1, \dots, L\}. \tag{5.5}$$

Note that m_Φ satisfies similar estimates for $|x| > t_0$, by assumption. Since $m_\infty = m - m_0$ and, by (5.3), m_0 satisfies the estimates

$$|m_0^{(j)}(z)| \leqslant C\|G\|_\infty \quad \text{and} \quad |m_0^{(j)}(z)| \leqslant C\|G\|_\infty |z|^{-(j+1)} \quad \text{for all } j \in \mathbb{N} \setminus \{0\},$$

away from ω_0^*, it is easy to see that $(m_\infty)_\Phi$ satisfies (5.5) for $|x| > t_0 + \epsilon$ for every $\epsilon > 0$, with a constant C which may depend on ϵ. Since m_∞ is holomorphic on the intersection of the boundary of $\boldsymbol{P}_w$ with the half-plane $\operatorname{Re} z < \operatorname{Re} z_+$, it is easy to see that (5.5) also holds for $|x| < t_0 - \epsilon$ for all $\epsilon > 0$. Therefore, it remains only to prove that the derivatives of order less than or equal to L of the restriction of m_∞ to the boundary of $\boldsymbol{P}_w$ are continuous near z_+ and z_-. Once more, by symmetry, we only need to consider z_+. Let $\partial_-^\ell m_\infty(z_+)$ and $\partial_+^\ell m_0(z_+)$ denote respectively the left and right derivatives of m_∞ and of m_0 at z_+ along the curve $\partial \boldsymbol{P}_w$, i.e.

$$\partial_-^\ell m_\infty(z_+) = \lim_{z \to z_+} \partial^\ell m_\infty(z),$$

where z is in $\partial \boldsymbol{P}_w$ and $\operatorname{Re} z < \operatorname{Re} z_+$, and

$$\partial_+^\ell m_0(z_+) = \lim_{z \to z_+} \partial^\ell m_0(z),$$

where z is in $\partial \boldsymbol{P}_w$ and $\operatorname{Re} z > \operatorname{Re} z_+$. Lemma 5.1 implies that the restriction of m_∞ to $\{z \in \partial \boldsymbol{P}_w : \operatorname{Re} z \leqslant \operatorname{Re} z_+\}$ is of class C^L even at the point z_+. Moreover, the values of m_∞ and its derivatives up to order L along $\{z \in \partial \boldsymbol{P}_w : \operatorname{Re} z \leqslant \operatorname{Re} z_+\}$ at z_+ may be obtained as limits of their values in $\boldsymbol{R}_{w,\theta} \cap \Gamma_\delta(\omega_+)$. Similarly, the values at z_0 of m_0 and its derivatives up to the order L, computed along the arc of parabola $\{z \in \partial \boldsymbol{P}_w : \operatorname{Re} z \geqslant \operatorname{Re} z_+\}$, coincide with the limits of their values in $\boldsymbol{R}_{w,\theta} \cap \Gamma_\delta(\omega_0)$. Hence, in these two cases we may use the same points from $\boldsymbol{E} = \boldsymbol{R}_{w,\theta} \cap \Gamma_\delta(\omega_0) \cap \Gamma_\delta(\omega_+)$ when taking the limits. But $m = m_0 + m_\infty$ in $\boldsymbol{E}$ and m, and all its derivatives up to order $L+1$ vanish as $z \to z_+$ from within $\boldsymbol{E}$. This shows that the left derivative $\partial_-^\ell m_\infty(z_+)$ of m_∞ coincides with the right derivative $-\partial_+^\ell m_0(z_+)$ of $-m_0$ at z_+, for $\ell = 0, \dots, L$. Since $m_\infty = m - m_0$ on $\{z \in \partial \boldsymbol{P}_w : \operatorname{Re} z > \operatorname{Re} z_+\}$ and $\partial^\ell m(z_+) = 0$ for $\ell = 0, \dots, L$, this proves that m_∞ is L times continuously differentiable at z_+. The proof of the lemma is complete. □

Acknowledgements. This work was partly supported by the Progetto Cofinanziato MIUR 'Analisi Armonica' and the Gruppo Nazionale INdAM per l'Analisi Matematica, la Probabilità e le loro Applicazioni. The authors thank the referee for careful reading and constructive comments which helped to improve an earlier version of the proof of Theorem 3.1.

References

1. S. Agmon, *Lectures on exponential decay of solutions of second order elliptic operator* (Princeton University Press, 1982).
2. W. Arendt, Gaussian estimates and interpolation of the spectrum in L^p, *Diff. Integr. Eqns* **7** (1994), 1153–1168.
3. R. Brooks, A relation between growth and the spectrum of the Laplacian, *Math. Z.* **178** (1981), 501–508.
4. A. Carbonaro, Functional calculus for some perturbations of the Ornstein–Uhlenbeck operator, *Math. Z.* doi:10.1007/s00209-008-0375-9.
5. M. Cowling, Harmonic analysis on semigroups, *Annals Math.* **111** (1983), 267–283.
6. E. B. Davies, *Heat kernels and spectral theory* (Cambridge University Press, 1989).
7. J. Garcia-Cuerva, G. Mauceri, S. Meda, P. Sjögren and J. L. Torrea, Functional calculus for the Ornstein–Uhlenbeck operator, *J. Funct. Analysis* **183** (2001), 413–450.
8. W. Hebisch, A multiplier theorem for Schrödinger operators, *Coll. Math.* **60** (1990), 659–664.

9. W. Hebisch, G. Mauceri and S. Meda, Holomorphy of spectral multipliers of the Ornstein–Uhlenbeck operator, *J. Funct. Analysis* **210** (2004), 101–124.
10. W. Hebisch, G. Mauceri and S. Meda, Spectral multipliers for sub-Laplacians with drift on Lie groups, *Math. Z.* **251** (2005), 899–927.
11. P. C. Kunstmann and Z. Strkalji, H^∞-calculus for submarkovian generators, *Proc. Am. Math. Soc.* **131** (2003), 2081–2088.
12. V. A. Liskevich and M. A. Perelmuter, Analyticity of submarkovian semigroups, *Proc. Am. Math. Soc.* **123** (1995), 1097–1104.
13. E. Sasso, Functional calculus for the Laguerre operator, *Math. Z.* **249** (2005), 683–711.
14. B. Simon, Schrödinger semigroups, *Bull. Am. Math. Soc.* **7** (1982), 447–526.
15. C. D. Sogge, *Lectures on nonlinear wave equations*, Monographs in Analysis, Volume II (International Press, Somerville, MA, 1995).
16. M. E. Taylor, L^p-estimates on functions of the Laplace operator, *Duke Math. J.* **58** (1989), 773–793.

https://doi.org/10.1017/S0013091506001234 Published online by Cambridge University Press