A NOTE ON A FULLY ORDERED RING

Kwangil Koh

A ring R (associative ring) is said to be <u>fully ordered</u> provided that R is a linearly ordered set under a relation $\leq$ such that for any a, b and c in R, $a \leq b$ implies that $a + c \leq b + c$ and if $c > 0$ then $ca \leq cb$ and $ac \leq bc$. We say a subset K of R is convex provided that if $a, b \in K$ such that $a \leq b$ then the interval $[a, b]$ is a subset of K. Obviously an additive subgroup K of R is convex if and only if $b \in K$ and $b > 0$ implies $[0, b] \subseteq K$. We observe that if K_1, K_2 are convex subsets of R such that K_1 and K_2 are additive subgroups of R respectively then either $K_1 \subseteq K_2$ or $K_2 \subseteq K_1$. In this note, we will prove that a <u>fully ordered semi-prime ring</u> is an <u>integral domain</u>. This generalizes a result of Birkhoff-Pierce that a fully ordered semi-simple ring with the minimum condition for right ideals is a division ring (See [1: p. 115]).

LEMMA. Let R be a fully ordered ring and K be a convex subset of R which is also a right ideal of R. Then for any $a \in R$, the right ideal $(K : a) = \{r \in R \mid ar \in K\}$ is convex. Similarly, if K is a left ideal of R which is convex then the left ideal $(K : a)_L = \{r \in R \mid ra \in K\}$ is convex.

<u>Proof</u>. We observe that $(K : a) = (K : -a)$. Let $b \in (K : a)$, $b > 0$ and $x \in R$ such that $0 < x < b$. Then $0 \leq |a| x \leq |a| b$ where $|a| = a$ if $a \geq 0$ and $|a| = -a$ if $a < 0$. Since $|a| b \in K$ and K is convex, $|a| x \in K$. Hence $x \in (K : a)$. A similar argument shows that $(K : a)_L$ is also convex when K is a left ideal.

THEOREM. A fully ordered semi-prime ring R is an integral domain.

<u>Proof</u>. Suppose there exist non-zero elements x and y in R such that $xy = 0$. We may assume $x > 0$ and $y > 0$. If $x \geq y > 0$ then $0 = xy \geq y^2 \geq 0$ implies that $y^2 = 0$.

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Similarly if $y \geqq x > 0$ then $0 = xy \geqq x^2 \geqq 0$ implies that $x^2 = 0$. Let us assume $x^2 = 0$. Then $x \varepsilon ((0) : x)$ and $x \varepsilon ((0) : x)_L$. By LEMMA, $((0) : x)_L$ and $((0) : x)$ are convex sets which are additive subgroups of R. Hence either $((0) : x)_L \subseteq ((0) : x)$ or $((0) : x) \subseteq ((0) : x)_L$. If $((0) : x)_L \subseteq ((0) : x)$ then $((0) : x)_L \cdot R \subseteq ((0) : x)$ and $(0) = xR((0) : x)_L \cdot R \supseteq xRxR$. This is impossible since R is a semi-prime ring and so contains no non-zero nilpotent right ideals. On the other hand if $((0) : x) \subseteq ((0) : x)_L$ then $R((0) : x) \subseteq ((0) : x)_L$ and $(0) = R((0) : x)Rx \supseteq RxRx$. This is again impossible. Thus R must be an integral domain.

REFERENCES

1. L. Fuchs, Partially Ordered Algebraic Systems, Pergammon Press (1963).

North Carolina State University
Raleigh