Bull. Aust. Math. Soc. **91** (2015), 155–166
doi:10.1017/S0004972714000859

A NOTE ON THE MORSE–NOVIKOV COHOMOLOGY OF BLOW-UPS OF LOCALLY CONFORMAL KÄHLER MANIFOLDS

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(Received 28 May 2014; accepted 11 September 2014; first published online 14 October 2014)

Abstract

We prove a blow-up formula for Morse–Novikov cohomology on a compact locally conformal Kähler manifold.

2010 *Mathematics subject classification*: primary 53C55; secondary 53C56.

Keywords and phrases: locally conformal Kähler manifold, Morse–Novikov cohomology, blow-up formula.

1. Introduction

Let X be a complex manifold with a Hermitian metric h. We denote the Kähler form of h by ω. If there exists a closed 1-form θ such that $d\omega = \theta \wedge \omega$, then ω is called a locally conformal Kähler structure, abbreviated LCK structure, on X with Lee form θ. In particular, a locally conformal Kähler structure yields a Kähler metric on the universal covering of X. Equivalently, the complex manifold X is an LCK manifold if it has a Kähler covering with the monodromy acting on this covering by the holomorphic homotheties. In particular, every Kähler manifold is a special LCK manifold with zero Lee form. However, there exist many interesting examples of non-Kähler manifolds which admit LCK structures. For instance, although the Hopf manifold $S^1 \times S^{2n+1}$ cannot admit any Kähler metric, it admits a canonical LCK metric (see [2, Ch. 3]).

Given an LCK manifold (X, ω, θ), we may define an operator d_θ as follows:

$$d_\theta(\alpha) = d\alpha - \theta \wedge \alpha \quad \forall \alpha \in \Omega^*(X).$$

Since $(d_\theta)^2 = 0$, we have a θ-twisted de Rham complex $(\Omega^*(X), d_\theta)$, which is called the *Morse–Novikov complex*. The associated cohomology, denoted by $H^*_\theta(X)$, is called the *Morse–Novikov cohomology*. Generally speaking, the Morse–Novikov cohomology is a generalisation of de Rham cohomology.

It is well known that the blow-up is a useful operation in complex geometry. In particular, the blow-up of a Kähler manifold (at a point or along a complex submanifold) is also Kählerian. In the LCK case a natural problem is to consider the blow-up of an LCK manifold. Tricerri [5] and Vuletescu [7] proved that the blow-up of an LCK manifold at a point is LCK; however, whether the blow-up of an LCK manifold along a submanifold is also LCK is not an immediate result. In 2013, Ornea *et al.* [4] showed that the blow-up of an LCK manifold along a submanifold is LCK if and only if the submanifold is globally conformally equivalent to a Kähler submanifold. The main purpose of this paper is to show that the Morse–Novikov cohomology of the blow-up of an LCK manifold is determined by the Morse–Novikov cohomology of the original LCK manifold and the de Rham cohomology of the exceptional divisor, that is, we prove a blow-up formula for the Morse–Novikov cohomology as follows.

Theorem 1.1 (Theorem 3.1). *Let (X, ω, θ) be a compact locally conformal Kähler manifold of dimension $2n$. Assume that $Z \subset X$ is a compact induced globally conformal Kähler submanifold. Then*

$$H^k_\theta(X) \oplus \left(\bigoplus_{i=0}^{r-2} H^{k-2i-2}_{dR}(Z) \right) \cong H^k_{\tilde{\theta}}(\tilde{X}_Z),$$

where $r = \operatorname{codim}_{\mathbb{C}} Z$ and $\tau : \tilde{X}_Z \to X$ is the blow-up of X along Z.

This paper is organised as follows. We devote Section 2 to preliminaries of Morse–Novikov cohomology and the blow-up of an LCK manifold along a submanifold. Then, in Section 3, we give the proof of the main theorem.

2. Preliminaries

2.1. Locally conformal Kähler manifolds and Morse–Novikov cohomology. Let (X, h) be a Hermitian manifold with Kähler form ω. We say that h is a locally conformally Kähler metric if there exist an open covering of X, denoted by $\{U_i\}_{i\in\Lambda}$, and a family of smooth functions

$$\{f_i : U_i \to \mathbb{R}^1\}_{i\in\Lambda}$$

such that

$$h_i := \exp(-f_i) \cdot h\,|_{U_i}$$

is Kählerian on every open subset U_i.

Notice that the Kähler form of h_i is $\omega_i = \exp(-f_i) \cdot \omega|_{U_i}$. Therefore, we have $d\omega_i = 0$, that is,

$$\begin{aligned} 0 &= d(\exp(-f_i) \cdot \omega) \\ &= \exp(-f_i)(-df_i \wedge \omega) + \exp(-f_i)\, d\omega \\ &= \exp(-f_i)(d\omega - df_i \wedge \omega). \end{aligned}$$

It follows that

$$d\omega - df_i \wedge \omega = 0 \quad (\text{on } U_i).$$

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For any pair of open subsets U_i and U_j such that $U_{ij} = U_i \cap U_j \neq \emptyset$,

$$df_i \wedge \omega = d\omega \quad (\text{on } U_{ij}) \tag{2.1}$$

and

$$df_j \wedge \omega = d\omega \quad (\text{on } U_{ij}). \tag{2.2}$$

Furthermore, from (2.1) and (2.2),

$$(df_i - df_j) \wedge \omega = 0 \quad (\text{on } U_{ij}). \tag{2.3}$$

Note that ω is nondegenerate; hence, from (2.3),

$$df_i = df_j \quad (\text{on } U_{ij}).$$

This implies that $\{(df_i, U_i)\}$ defines a globally closed 1-form θ, which is called the *Lee form* of an LCK metric, and, furthermore, $\lambda_{ij} = f_i - f_j$ is constant on U_{ij}. Equivalently, we have the following result.

Definition 2.1. A Hermitian manifold $(X; h)$ is an LCK manifold if and only if there exists a closed 1-form (Lee form) θ such that

$$d\omega = \theta \wedge \omega,$$

where ω is the K*ä*hler form of h.

Let $\Omega^*(X)$ be the space of smooth forms on X. We may define a differential operator by

$$\begin{aligned} d_\theta : \Omega^*(X) &\to \Omega^{*+1}(X) \\ \alpha &\mapsto d\alpha - \theta \wedge \alpha. \end{aligned}$$

Furthermore, we have the θ-twisted complex

$$\cdots \xrightarrow{d_\theta} \Omega^{k-1}(X) \xrightarrow{d_\theta} \Omega^k(X) \xrightarrow{d_\theta} \cdots.$$

The complex $(\Omega^*(X), d_\theta)$ is called the *Morse–Novikov complex*, and the associated cohomology group

$$H^*_\theta(X) = H^*(\Omega^*(X); d_\theta)$$

is called the *Morse–Novikov cohomology*.

Suppose that $\mathbb{R}$ is the sheaf of locally constant real functions over X and $\mathbb{R}^*$ is the sheaf of locally constant nonzero real functions over X. Consider the exponential homomorphism of sheaves

$$\exp : \mathbb{R} \to \mathbb{R}^*. \tag{2.4}$$

From (2.4), we get a homomorphism of Čech cohomology of sheaves

$$\exp^* : H^1(X, \mathbb{R}) \to H^1(X, \mathbb{R}^*).$$

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Note that $H^1(X,\mathbb{R}) \cong H^1_{dR}(X)$ and there is a one-to-one correspondence between the set of real line bundles, over X, up to isomorphism, and the Čech cohomology group $H^1(X,\mathbb{R}^*)$. Therefore, the Lee form determines a real line bundle L with the locally constant transition functions $\{(\exp(-\lambda_{ij}), U_{ij})\}$, which is called the *weight line bundle* of the LCK structure. In particular, L is a flat line bundle over X with the flat connection D_L induced by the LCK structure by the formula $D_L = d - \theta$. Furthermore, the connection D_L determines a covariant differential

$$d_L : \Gamma(X, \wedge^{\bullet} T^*X \otimes L) \to \Gamma(X, \wedge^{\bullet+1} T^*X \otimes L).$$

The flatness of the connection D_L implies that $d_L^2 = 0$; therefore, we get a generalised de Rham complex $(\Omega^*(X; L), d_L)$ with the cohomology group $H^*_{dR}(X, L)$ and we have the following result.

Proposition 2.2. *The Morse–Novikov cohomology group $H^k_\theta(X)$ is isomorphic to the cohomology group $H^k_{dR}(X, L)$ for every k.*

Proof. Note that there is a one-to-one correspondence between isomorphism classes of smooth line bundles equipped with a flat connection and isomorphism classes of local systems of one-dimensional vector spaces (see [6, Proposition 9.11]). On one hand, assume that $\mathcal{L}$ is the local system on X corresponding to the weight line bundle L; then the cohomology of the local system $H^*(X, \mathcal{L})$ is isomorphic to $H^*_{dR}(X, L)$. On the other hand, $H^*(X, \mathcal{L})$ is naturally identified with the Morse–Novikov cohomology $H^*_\theta(X)$ (see [3, Proposition 3.1]). This implies that $H^*_\theta(X)$ is isomorphic to $H^*_{dR}(X, L)$. □

2.2. Blow-ups of locally conformal Kähler manifolds. In this section we summarise the results of Ornea *et al.* [4] on the construction of an LCK structure on the blow-up along a submanifold.

Let (X, ω, θ) be an LCK manifold of complex dimension n. Suppose that $Z \subset X$ is a complex submanifold with complex codimension r.

Definition 2.3. We say that $Z \subset X$ admits an *induced globally conformal Kähler* (i.g.c.K.) structure if the pullback of the Lee form $i^*\theta \in \Omega^1(Z)$ is exact; more precisely, $0 = [i^*\theta] \in H^1_{dR}(Z)$, where $i : Z \hookrightarrow X$ is the inclusion.

From now on we assume that X is compact and Z is a compact complex submanifold with dimension $\dim_{\mathbb{C}} Z \geq 1$. Then the blow-up of X along Z, denoted by $\tau : \tilde{X}_Z \to X$, is a compact complex manifold with dimension $\dim_{\mathbb{C}} \tilde{X}_Z = n$.

Let $E = \tau^{-1}(Z)$ be the exceptional divisor of $\tilde{X}_Z$. We are now in a position to construct the LCK metric on $\tilde{X}_Z$ when Z is an induced globally conformal Kähler submanifold of X. First we need the following lemma [4, Lemma 3.4], which is a well-known fact in Kähler geometry.

Lemma 2.4. *Suppose that (U, ω) is a Kähler manifold and $Z \subset U$ is a compact submanifold. Let $\tau : \tilde{U} \to U$ be the blow-up of U along Z. Then, for any open neighbourhood V of Z, there exists a Kähler metric $\tilde{\omega}$ on $\tilde{U}$ such that*

$$\tilde{\omega}|_{\tilde{U}-\tilde{V}} = \tau^*(\omega|_{U-V}),$$

where $\tilde{V} = \tau^{-1}(V)$.

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Since Z is an induced globally conformal Kähler submanifold, the restriction of the Lee form $\theta|_Z$ is exact. Choose an open neighbourhood U of Z such that the inclusion $i : Z \hookrightarrow U$ induces an isomorphism on the first de Rham cohomology,

$$i^* : H^1_{dR}(U) \xrightarrow{\cong} H^1_{dR}(Z).$$

Via a conformal rescaling of the Hermitian metric, we may assume that $\theta|_U = 0$. It follows that $\omega|_U$ is a Kähler metric on U. Let $\tilde{U} = \tau^{-1}(U)$. Then $\tilde{U}$ is an open neighbourhood of the exceptional divisor E in $\tilde{X}_Z$. Choose a smaller open neighbourhood V of Z in U. According to Lemma 2.4, we get a Kähler metric, denoted by $\tilde{\omega}$, on $\tilde{U}$ such that $\tilde{\omega}$ is equal to $\tau^*(\omega|_U)$ outside of $\tilde{V} = \tau^{-1}(V)$. Therefore, we may glue $\tilde{\omega}$ to $\tau^*\omega$ to get an LCK metric on $\tilde{X}_Z$.

In particular, Ornea *et al.* proved the following properties.

(1) [4, Theorem 2.8] If $Z \subset X$ is an induced globally conformal Kähler submanifold, then the blow-up $\tilde{X}_Z$ admits a locally conformal Kähler metric with Lee form $\tilde{\theta} = \tau^*\theta$.

(2) [4, Theorem 2.9] If $\tilde{X}_Z$ admits a locally conformal Kähler metric, then the exceptional divisor $E \subset \tilde{X}_Z$ is an induced globally conformal Kähler submanifold.

(3) [4, Corollary 2.11] If $\tilde{X}_Z$ admits a locally conformal Kähler metric and, furthermore, $\dim_{\mathbb{C}} Z > 1$, then $Z \subset X$ is an induced globally conformal Kähler submanifold.

Given any form $\alpha \in \Omega^k(X)$ such that $d_\theta(\alpha) = 0$, the pullback $\tau^*\alpha$ is a k-form on $\tilde{X}_Z$. Note that

$$\begin{aligned}
d_{\tilde{\theta}}(\tau^*\alpha) &= d(\tau^*\alpha) - \tilde{\theta} \wedge \tau^*\alpha \\
&= \tau^*(d\alpha) - \tau^*\theta \wedge \tau^*\alpha \quad (\tilde{\theta} = \tau^*\theta) \\
&= \tau^*(d\alpha - \theta \wedge \alpha) \\
&= \tau^*(d_\theta(\alpha)) \\
&= 0.
\end{aligned}$$

Therefore, τ induces a homomorphism between Morse–Novikov cohomology groups

$$\tau^* : H^k_\theta(X) \to H^k_{\tilde{\theta}}(\tilde{X}_Z).$$

Furthermore, we have the following result.

PROPOSITION 2.5. *The map $\tau^* : H^k_\theta(X) \to H^k_{\tilde{\theta}}(\tilde{X}_Z)$ is injective for every k.*

PROOF. Let $\tilde{L}$ be the weight line bundle over $\tilde{X}_Z$, which is determined by the Lee form $\tilde{\theta}$. By definition, the locally constant transition functions of $\tilde{L}$ are $\{(\tau^*(\exp(-\lambda_{ij})), \tau^{-1}(U_{ij}))\}$. It follows that $\tilde{L}$ is the pullback of L, that is, $\tilde{L} = \tau^*L$. The map τ also induces a homomorphism

$$\tau^* : H^k_{dR}(X, L) \to H^*_{dR}(\tilde{X}_Z, \tilde{L}). \tag{2.5}$$

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From Proposition 2.2, we have $H^k_\theta(X) \cong H^k_{dR}(X, L)$ and $H^k_{\tilde\theta}(\tilde X_Z) \cong H^k_{dR}(\tilde X_Z, \tilde L)$. To prove Proposition 2.5, it is sufficient to show that (2.5) is injective for every k. Assume that L^* is the dual bundle of L. Note that there exists a naturally defined flat connection D_{L^*} such that for any differential forms with values in vector bundles $\alpha \in \Gamma(X, \wedge^\bullet T^*X \otimes L)$ and $\beta \in \Gamma(X, \wedge^\bullet T^*X \otimes L^*)$,

$$d(\alpha \wedge \beta) = (d_L\alpha) \wedge \beta + (-1)^{\deg \alpha}\alpha \wedge d_{L^*}\beta,$$

where d_{L^*} is the covariant differential of D_{L^*}. Similarly, we have the generalised de Rham cohomology $H^*_{dR}(X, L^*)$. Using the harmonic theory of elliptic operators, we may show that the pairing

$$\int : H^k_{dR}(X, L) \times H^{2n-k}_{dR}(X, L^*) \to \mathbb{R}^1$$
$$(\alpha, \beta) \mapsto \int_X \alpha \wedge \beta$$

is nondegenerate.

The injectivity of (2.5) is proved by contradiction. Assume that the assertion does not hold, so that there exists a nonzero $\alpha \in H^k_{dR}(X, L)$ such that $\tau^*\alpha = 0$. Since α represents a nonzero class in $H^k_{dR}(X, L)$, it follows that there exists an L^*-valued $(2n-k)$-form $\hat\alpha \in H^{2n-k}_{dR}(X, L^*)$ such that $\alpha \wedge \hat\alpha$ is the generator of $H^{2n}_{dR}(X)$ and $\int_X \alpha \wedge \hat\alpha = 1$. On the one hand, because $\tau^*\alpha = 0$,

$$\begin{aligned} 0 &= \int_{\tilde X_Z} \tau^*\alpha \wedge \tau^*\hat\alpha \\ &= \int_{\tilde X_Z} \tau^*(\alpha \wedge \hat\alpha). \end{aligned}$$

Let $\deg(\tau)$ be the degree of the smooth map $\tau : \tilde X_Z \to X$. Since X and $\tilde X_Z$ are closed and oriented manifolds and $\alpha \wedge \hat\alpha$ is the generator of $H^{2n}_{dR}(X)$, from the definition of degree,

$$\begin{aligned} \deg(\tau) &= \int_{\tilde X_Z} \tau^*(\alpha \wedge \hat\alpha) \\ &= 0. \end{aligned}$$

On the other hand, the degree of τ is equal to the number of points, counted with multiplicity ± 1, in the inverse image of any regular point in X. We may choose a point $x \in X$ such that $x \notin Z$. Since $\tau|_{X-Z}$ is a diffeomorphism, x is a regular point and the inverse image of x contains only one point. Therefore, by definition we obtain $\deg(\tau) = 1$ and this leads to a contradiction. □

3. Blow-up formula of Morse–Novikov cohomology

Assume that (X, ω, θ) is a locally conformal Kähler manifold. Let $Z \subset X$ be an induced globally conformal Kähler submanifold, that is, the restriction of the

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Lee form $\theta|_Z$ is exact. By a conformal rescaling of the LCK metric, we may assume that $\theta|_Z = 0$. In fact, an induced globally conformal Kähler submanifold is a Kähler submanifold. In this section we will prove the following result.

THEOREM 3.1. *Let (X, ω, θ) be a compact locally conformal Kähler manifold. Assume that $Z \subset X$ is a compact Kähler submanifold such that $\theta|_Z = 0$. Then*

$$H^k_\theta(X) \oplus \left(\bigoplus_{i=0}^{r-2} H^{k-2i-2}_{dR}(Z) \right) \cong H^k_{\tilde{\theta}}(\tilde{X}_Z),$$

where $r = \operatorname{codim}_{\mathbb{C}} Z$ and $\tau : \tilde{X}_Z \to X$ is the blow-up of X along Z.

The key idea of the proof is to construct the Mayer–Vietoris sequence for Morse–Novikov cohomology.

We may choose a tubular neighbourhood V of Z in X such that the inclusion $i : Z \hookrightarrow X$ induces an isomorphism of first de Rham cohomology groups

$$i^* : H^1_{dR}(V) \xrightarrow{\cong} H^1_{dR}(Z).$$

Through a conformal rescaling of the Hermitian metric h, we may assume that $\theta|_V = 0$. In particular, we may choose V small enough such that its inverse image $\tilde{V} = \tau^{-1}(V)$ is also a tubular neighbourhood of the exceptional divisor E in $\tilde{X}_Z$. Let U be the open subset $X - Z$. Then $\{U, V\}$ forms an open covering of X. Denote the intersection of U and V by W. Similarly, let $\tilde{U} = \tilde{X}_Z - E$ and $\tilde{W} = \tilde{U} \cap \tilde{V}$. Then $\{\tilde{U}, \tilde{V}\}$ forms an open covering of $\tilde{X}_Z$. Furthermore, by definition we get two diffeomorphisms

$$\tau|_{\tilde{U}} : \tilde{U} \to U \tag{3.1}$$

and

$$\tau|_{\tilde{W}} : \tilde{W} \to W. \tag{3.2}$$

Using the Mayer–Vietoris sequence with compact support (see [1, Proposition 2.7]), we obtain two short exact sequences

$$0 \longrightarrow \Omega^*_c(W) \longrightarrow \Omega^*_c(U) \oplus \Omega^*_c(V) \longrightarrow \Omega^*_c(X) \longrightarrow 0 \tag{3.3}$$

and

$$0 \longrightarrow \Omega^*_c(\tilde{W}) \longrightarrow \Omega^*_c(\tilde{U}) \oplus \Omega^*_c(\tilde{V}) \longrightarrow \Omega^*_c(\tilde{X}_Z) \longrightarrow 0. \tag{3.4}$$

Note that Ω^*_c is a contravariant functor under proper maps; in particular, the maps $\tau|_{\tilde{U}}$, $\tau|_{\tilde{V}}$, $\tau|_{\tilde{W}}$ and τ are proper. It follows that the following diagram of Mayer–Vietoris sequences of forms with compact support is well defined.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \Omega^k_c(W) & \longrightarrow & \Omega^k_c(U) \oplus \Omega^k_c(V) & \longrightarrow & \Omega^k_c(X) & \longrightarrow & 0 \\
 & & \big\downarrow{\scriptstyle (\tau|_{\tilde{W}})^*} & & \big\downarrow{\scriptstyle (\tau|_{\tilde{U}})^* \oplus (\tau|_{\tilde{V}})^*} & & \big\downarrow{\scriptstyle \tau^*} & & \\
0 & \longrightarrow & \Omega^k_c(\tilde{W}) & \longrightarrow & \Omega^k_c(\tilde{U}) \oplus \Omega^k_c(\tilde{V}) & \longrightarrow & \Omega^k_c(\tilde{X}_Z) & \longrightarrow & 0
\end{array}$$

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Denote the Morse–Novikov cohomology of X with compact support by $H^*_{c,\theta}(X)$; then the sequences (3.3) and (3.4) induce two long exact sequences as follows:

$$\cdots \longrightarrow H^k_{c,\theta}(W) \longrightarrow H^k_{c,\theta}(U) \oplus H^k_{c,\theta}(V) \longrightarrow H^k_{c,\theta}(X) \longrightarrow H^{k+1}_{c,\theta}(W) \longrightarrow \cdots$$

and

$$\cdots \longrightarrow H^k_{c,\tilde{\theta}}(\tilde{W}) \longrightarrow H^k_{c,\tilde{\theta}}(\tilde{U}) \oplus H^k_{c,\tilde{\theta}}(\tilde{V}) \longrightarrow H^k_{c,\tilde{\theta}}(\tilde{X}_Z) \longrightarrow H^{k+1}_{c,\tilde{\theta}}(\tilde{W}) \longrightarrow \cdots.$$

Since $\tilde{\theta} = \tau^*\theta$, the map τ induces the following commutative diagram of the long exact sequences:

$$\begin{array}{ccccccccc}
\cdots \longrightarrow & H^k_{c,\theta}(W) & \xrightarrow{f} & H^k_{c,\theta}(U) \oplus H^k_{c,\theta}(V) & \xrightarrow{g} & H^k_{c,\theta}(X) & \xrightarrow{h} & H^{k+1}_{c,\theta}(W) & \longrightarrow \cdots \\
& \downarrow{\scriptstyle (\tau|_{\tilde{W}})^*} & & \downarrow{\scriptstyle (\tau|_{\tilde{U}})^*\oplus(\tau|_{\tilde{V}})^*} & & \downarrow{\scriptstyle \tau^*} & & \downarrow{\scriptstyle (\tau|_{\tilde{W}})^*} & \\
\cdots \longrightarrow & H^k_{c,\tilde{\theta}}(\tilde{W}) & \xrightarrow{\tilde{f}} & H^k_{c,\tilde{\theta}}(\tilde{U}) \oplus H^k_{c,\tilde{\theta}}(\tilde{V}) & \xrightarrow{\tilde{g}} & H^k_{c,\tilde{\theta}}(\tilde{X}_Z) & \xrightarrow{\tilde{h}} & H^{k+1}_{c,\tilde{\theta}}(\tilde{W}) & \longrightarrow \cdots
\end{array} \tag{3.5}$$

Note that (3.1) and (3.2) are diffeomorphic and, therefore, the induced homomorphisms $(\tau|_{\tilde{U}})^*$ and $(\tau|_{\tilde{W}})^*$ are isomorphisms. Furthermore, since X and $\tilde{X}_Z$ are compact,

$$H^k_{c,\theta}(X) = H^k_\theta(X) \tag{3.6}$$

and

$$H^k_{c,\tilde{\theta}}(\tilde{X}_Z) = H^k_{\tilde{\theta}}(\tilde{X}_Z). \tag{3.7}$$

Consider the Morse–Novikov cohomology $H^k_{c,\theta}(V)$. Since $\theta|_V = 0$,

$$H^k_{c,\theta}(V) = H^k_c(V). \tag{3.8}$$

Similarly,

$$H^k_{c,\tilde{\theta}}(\tilde{V}) = H^k_c(\tilde{V}). \tag{3.9}$$

From (3.6)–(3.9), the commutative diagram (3.5) is equivalent to

$$\begin{array}{ccccccccc}
\cdots \longrightarrow & H^k_{c,\theta}(W) & \xrightarrow{f} & H^k_{c,\theta}(U) \oplus H^k_c(V) & \xrightarrow{g} & H^k_\theta(X) & \xrightarrow{h} & H^{k+1}_{c,\theta}(W) & \longrightarrow \cdots \\
& \downarrow{\scriptstyle \cong} & & \downarrow{\scriptstyle (\tau|_{\tilde{U}})^*\oplus(\tau|_{\tilde{V}})^*} & & \downarrow{\scriptstyle \tau^*} & & \downarrow{\scriptstyle \cong} & \\
\cdots \longrightarrow & H^k_{c,\tilde{\theta}}(\tilde{W}) & \xrightarrow{\tilde{f}} & H^k_{c,\tilde{\theta}}(\tilde{U}) \oplus H^k_c(\tilde{V}) & \xrightarrow{\tilde{g}} & H^k_{\tilde{\theta}}(\tilde{X}_Z) & \xrightarrow{\tilde{h}} & H^{k+1}_{c,\tilde{\theta}}(\tilde{W}) & \longrightarrow \cdots
\end{array} \tag{3.10}$$

The next step in the proof is to verify the following proposition.

PROPOSITION 3.2. *The homomorphism $(\tau|_{\tilde{U}})^* \oplus (\tau\,|_{\tilde{V}})^*$ is monomorphic.*

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Proof. Since $(\tau|_{\tilde{U}})^*$ is isomorphic, we only need to verify that $(\tau|_{\tilde{V}})^*$ is monomorphic. Note that V and $\tilde{V}$ are tubular neighbourhoods of Z and E, respectively. Moreover, Z and E are compact. According to Poincaré duality (see [1, Proposition 6.13]), we have $H_c^k(V) \cong H_{dR}^{k-2r}(Z)$ and $H_c^k(\tilde{V}) \cong H_{dR}^{k-2}(E)$. By definition, the exceptional divisor E is the projectivisation of the normal bundle of Z, namely, $E = \mathbf{P}(N_{Z/X})$. Let $\rho : S \to E$ be the universal subbundle and denote the first Chern class of the dual bundle S^* by $t \in H_{dR}^2(E)$. Then, by the Leray–Hirsch theorem (see [1, Theorem 5.11]), the de Rham cohomology $H_{dR}^*(E)$ is a free module over $H_{dR}^*(Z)$ with basis $\{1, t, \ldots, t^{r-1}\}$. More precisely, we can consider $(\tau|_{\tilde{V}})^*$ as a morphism τ_E^*, which is denoted by

$$\begin{aligned} \tau_E^* : H_{dR}^{k-2r}(Z) &\to H_{dR}^{k-2}(E) \\ \alpha &\mapsto t^{r-1} \wedge (\tau|_E)^*(\alpha). \end{aligned}$$

By definition, the injectivity of τ_E^* is straightforward. Therefore, $(\tau|_{\tilde{V}})^*$ is a monomorphism. □

To prove Theorem 3.1, we need the following general proposition and we give its proof at the end of this section for completeness.

Proposition 3.3. *Given a commutative diagram of abelian groups such that the horizontal rows are exact*

$$\begin{array}{ccccccccccc} \cdots & \longrightarrow & A_1 & \xrightarrow{f} & A_2 & \xrightarrow{g} & A_3 & \xrightarrow{h} & A_4 & \longrightarrow & \cdots \\ & & \downarrow{\scriptstyle i_1} & & \downarrow{\scriptstyle i_2} & & \downarrow{\scriptstyle i_3} & & \downarrow{\scriptstyle i_4} & & \\ \cdots & \longrightarrow & B_1 & \xrightarrow{k} & B_2 & \xrightarrow{l} & B_3 & \xrightarrow{m} & B_4 & \longrightarrow & \cdots \end{array}$$

and where i_1 and i_4 are isomorphic and i_2 and i_3 are monomorphic, then there is a natural isomorphism

$$\operatorname{coker} i_2 \cong \operatorname{coker} i_3.$$

Proof of Theorem 3.1. Consider the commutative diagram (3.10). According to Propositions 2.5, 3.2 and 3.3, we get an isomorphism

$$\operatorname{coker}(\tau|_{\tilde{U}})^* \oplus \operatorname{coker}(\tau|_{\tilde{V}})^* \xrightarrow{\cong} \operatorname{coker} \tau^*. \tag{3.11}$$

Note that

$$\begin{aligned} \operatorname{coker}(\tau|_{\tilde{U}})^* \oplus \operatorname{coker}(\tau|_{\tilde{V}})^* &= (H_{c,\tilde{\theta}}^k(\tilde{U}) \oplus H_c^k(\tilde{V}))/(\operatorname{Im}(\tau|_{\tilde{U}})^* \oplus \operatorname{Im}(\tau|_{\tilde{V}})^*) \\ &\cong H_c^k(\tilde{V})/\operatorname{Im}(\tau|_{\tilde{V}})^* \quad ((\tau|_{\tilde{U}})^* \text{ is isomorphic}) \\ &= \operatorname{coker}(\tau|_{\tilde{V}})^* \\ &= \operatorname{coker} \tau_E^*. \end{aligned}$$

Therefore, the isomorphism (3.11) is equivalent to

$$\operatorname{coker} \tau_E^* \xrightarrow{\cong} \operatorname{coker} \tau^*.$$

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Now let us consider $\operatorname{coker}\tau_E^*$. Recall that $H_{dR}^*(E)$ is a free module over $H_{dR}^*(Z)$ with the basis $\{1, t, \ldots, t^{r-1}\}$. Therefore,

$$H_{dR}^{k-2}(E) = \bigoplus_{i=0}^{r-1}(t^i \wedge (\tau|_E)^* H_{dR}^{k-2i-2}(Z)).$$

By definition of τ_E^*,

$$\begin{aligned}\operatorname{coker}\tau_E^* &= H^{k-2}(E)/\operatorname{Im}\tau_E^* \\ &= \bigoplus_{i=0}^{r-2}(t^i \wedge (\tau|_E)^* H_{dR}^{k-2i-2}(Z)) \\ &\cong \bigoplus_{i=0}^{r-2}(H_{dR}^{k-2i-2}(Z)).\end{aligned}$$

Finally,

$$\begin{aligned}H_{\tilde{\theta}}^k(\tilde{X}_Z) &\cong \operatorname{Im}\tau^* \oplus \operatorname{coker}\tau_E^* \\ &\cong H_\theta^k(X) \oplus \left(\bigoplus_{i=0}^{r-2} H_{dR}^{k-2i-2}(Z)\right) \quad (\tau^* \text{ is injective}).\end{aligned}$$

This completes the proof. □

Proof of Proposition 3.3. Consider the diagram

$$\begin{array}{ccccccccccc} \cdots & \longrightarrow & A_1 & \xrightarrow{f} & A_2 & \xrightarrow{g} & A_3 & \xrightarrow{h} & A_4 & \longrightarrow & \cdots \\ & & \downarrow{\scriptstyle i_1} & & \downarrow{\scriptstyle i_2} & & \downarrow{\scriptstyle i_3} & & \downarrow{\scriptstyle i_4} & & \\ \cdots & \longrightarrow & B_1 & \xrightarrow{k} & B_2 & \xrightarrow{l} & B_3 & \xrightarrow{m} & B_4 & \longrightarrow & \cdots \end{array} \tag{3.12}$$

According to the exactness, we have $\operatorname{Im} k = \ker l$ and $\operatorname{Im} f = \ker g$. From the commutativity of the first square of (3.12), we get $i_2(\operatorname{Im} f) = k(\operatorname{Im} i_1)$. Note that i_1 is an isomorphism; therefore, $\operatorname{Im} i_1 = B_1$. Furthermore, we have $\operatorname{Im} k = i_2(\operatorname{Im} f) \subset i_2(A_2)$, that is, $\operatorname{Im} k \subset \operatorname{Im} i_2$. Consider the decomposition $B_2 = i_2(A_2) \oplus C$. Note that

$$\begin{aligned}\ker(l|_C) &= \ker\ l \cap C \\ &= \operatorname{Im} k \cap C \quad (\ker\ l = \operatorname{Im} k) \\ &= 0 \quad (\operatorname{Im} k \subset i_2(A_2)).\end{aligned}$$

Hence, the restriction $l|_C$ is injective. By the commutativity of the second square of (3.12), we get $i_3(\operatorname{Im} g) = l(\operatorname{Im} i_2)$. This implies that $l(\operatorname{Im} i_2) \subset i_3(A_3)$ and, furthermore, that there exists a well-defined homomorphism

$$\bar{l} : \operatorname{coker} i_2 = B_2/\operatorname{Im} i_2 \to \operatorname{coker} i_3 = B_3/\operatorname{Im} i_3.$$

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First we verify that $\bar{l}$ is injective. Equivalently, we need to show that for any $b_2 \in B_2$, if $l(b_2) \in \operatorname{Im} i_3$, then $b_2 \in \operatorname{Im} i_2$. Assume that $l(b_2) = i_3(a_3)$ for some $a_3 \in A_3$. Because of the exactness, $m(l(b_2)) = 0$ and

$$\begin{aligned} 0 &= m(l(b_2)) \\ &= m(i_3(a_3)) \quad (i_3(a_3) = l(b_2)) \\ &= i_4(h(a_3)) \quad (m \circ i_3 = i_4 \circ h). \end{aligned}$$

Since i_4 is isomorphic, we get $h(a_3) = 0$, that is, $a_3 \in \ker h = \operatorname{Im} g$. Therefore, $a_3 = g(a_2)$ for some $a_2 \in A_2$. It follows that

$$\begin{aligned} l(b_2) &= i_3(a_3) \\ &= i_3(g(a_2)) \quad (a_3 = g(a_2)) \\ &= l(i_2(a_2)) \quad (i_3 \circ g = l \circ i_2). \end{aligned}$$

Hence, $l(b_2 - i_2(a_2)) = 0$, that is, $b_2 - i_2(a_2) \in \ker l = \operatorname{Im} k \subset \operatorname{Im} i_2$. Therefore, there exists $a_2' \in A_2$ such that $b_2 - i_2(a_2) = i_2(a_2')$ and it follows that $b_2 = i_2(a_2 + a_2') \in \operatorname{Im} i_2$. Hence, $\bar{l}$ is injective.

Finally, we need to show that $\bar{l}$ is surjective. Consider the next square in the diagram (3.12)

$$\begin{array}{ccccccc} A_2 & \xrightarrow{g} & A_3 & \xrightarrow{h} & A_4 & \xrightarrow{f'} & A_2' \\ \downarrow{\scriptstyle i_2} & & \downarrow{\scriptstyle i_3} & & \downarrow{\scriptstyle i_4}\,\cong & & \downarrow{\scriptstyle i_2'} \\ B_2 & \xrightarrow{l} & B_3 & \xrightarrow{m} & B_4 & \xrightarrow{k'} & B_2' \end{array}$$

Let $b_3 \in B_3$, $b_4 = m(b_3)$ and $a_4 = i_4^{-1}(b_4)$. Consider $i_2'(f'(a_4)) \in B_2'$. Since i_2' is monomorphic, $f'(a_4) \neq 0$ if and only if $i_2'(f'(a_4)) \neq 0$. According to commutativity and exactness,

$$\begin{aligned} i_2'(f'(a_4)) &= k'(i_4(a_4)) \quad (i_2' \circ f' = k' \circ i_4) \\ &= k'(m(b_3)) \quad (m(b_3) = i_4(a_4)) \\ &= 0 \quad (k' \circ m = 0). \end{aligned}$$

It follows that $f'(a_4) = 0$, that is, $a_4 \in \ker f' = \operatorname{Im} h$. Therefore, there exists $a_3 \in A_3$ such that $a_4 = h(a_3)$. Furthermore,

$$\begin{aligned} m(b_3) &= i_4(a_4) \\ &= i_4(h(a_3)) \quad (a_4 = h(a_3)) \\ &= m(i_3(a_3)) \quad (i_4 \circ h = m \circ i_3). \end{aligned}$$

This implies that $m(b_3 - i_3(a_3)) = 0$, that is, $b_3 - i_3(a_3) \in \ker m = \operatorname{Im} l$. This completes the proof. □

https://doi.org/10.1017/S0004972714000859 Published online by Cambridge University Press

References

[1] R. Bott and L. Tu, *Differential Forms in Algebraic Topology*, Graduate Texts in Mathematics, 82 (Springer, Berlin, Heidelberg, 1982).
[2] S. Dragomir and L. Ornea, *Locally Conformal Kähler Geometry*, Progress in Mathematics, 155 (Birkhäuser, Boston, Basel, 1998).
[3] L. Ornea and M. Verbitsky, 'Morse–Novikov cohomology of locally conformally Kähler manifolds', *J. Geom. Phys.* **59** (2009), 295–305.
[4] L. Ornea, M. Verbitsky and V. Vuletescu, 'Blow-ups of locally conformally Kähler manifolds', *Int. Math. Res. Not.* **2013**(12) (2013), 2809–2821.
[5] F. Tricerri, 'Some examples of locally conformal Kähler manifolds', *Rend. Semin. Mat. Univ. Politec. Torino* **40** (1982), 81–92.
[6] C. Voisin, *Hodge Theory and Complex Algebraic Geometry I* (Cambridge University Press, New York, 2002).
[7] V. Vuletescu, 'Blowing-up points of locally conformal Kähler manifolds', *Bull. Math. Soc. Sci. Math. Roumanie* **52** (2009), 387–390.

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