MATRICES ASSOCIATED WITH FRACTIONAL HANKEL AND FOURIER TRANSFORMATIONS

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1. *Introduction.* Several writers (**4**), (**6**), (**7**), (**9**) have used orthogonal expansions in discussing properties of Fourier transformations, and Kober (**3**) has used such expansions to derive fractional Fourier and Hankel transformations. In 1950 Barrucand (**1**) noted a reciprocity holding between the coefficients in the expansions in Laguerre polynomials of pairs of functions which are transforms with respect to the kernel $J_0(2x^{\frac{1}{2}})$.

In the present paper I extend Barrucand's result to kernels $J_\alpha(2x^{\frac{1}{2}})$, $R(\alpha) > -1$, and to Fourier sine and cosine kernels. I also discuss the relationship between fractional powers of unit matrices and fractional transformations, and I show how this method gives an alternative approach to the fractional Hankel transformations of Kober.

2. *Formalities.* Let us suppose that the sets of functions

$$\{\phi_n(x)\}, \quad \{\psi_n(x)\} \quad (n = 0, 1, 2, \ldots),$$

are normalised and biorthogonal over $(0, \infty)$, that $f(x)$ and $g(x)$ are transforms with respect to a Fourier kernel $K(x)$, and that

$$f(x) \sim \sum_{n=0}^{\infty} a_n \phi_n(x), \qquad \text{.........(1)}$$

and

$$g(x) \sim \sum_{n=0}^{\infty} b_n \phi_n(x)$$

in the sense that* the coefficients a_n and b_n are given by

$$a_n = \int_0^\infty f(x)\,\psi_n(x)\,dx, \quad b_n = \int_0^\infty g(x)\psi_n(x)\,dx.$$

Further, suppose that $\Psi_n(x)$ is the transform of $\psi_n(x)$ with respect to the kernel $K(x)$ and that it can be expanded in terms of the functions $\psi_n(x)$. That is

$$\Psi_n(x) = \int_0^\infty \psi_n(t) K(xt)\,dt = \sum_{m=0}^{\infty} k_{n,m}\,\psi_m(x),$$

say. By the Parseval theorem for these transforms we have, formally,

$$\begin{aligned} a_n &= \int_0^\infty f(x)\psi_n(x)\,dx = \int_0^\infty g(x)\Psi_n(x)\,dx \\ &= \int_0^\infty g(x)\left\{\sum_{m=0}^{\infty} k_{n,m}\,\psi_m(x)\right\}dx \\ &= \sum_{m=0}^{\infty} k_{n,m} \int_0^\infty g(x)\psi_m(x)\,dx \\ &= \sum_{m=0}^{\infty} k_{n,m}\, b_m. \qquad \text{.........(2)} \end{aligned}$$

* The sign $\sim$ is used with this meaning throughout the paper.

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Simiarly

$$b_n = \sum_{m=0}^{\infty} k_{n,m}\, a_m \qquad \text{.......(3)}$$

If K, A, B denote the infinite matrix and the column vectors

$$\begin{bmatrix} k_{00} & k_{01} & k_{02} \dots \\ k_{10} & k_{11} & k_{12} \dots \\ k_{20} & k_{21} & k_{22} \dots \\ \\ \dots\dots\dots\dots \end{bmatrix}, \quad \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ \cdot \\ \cdot \\ \cdot \end{bmatrix}, \quad \begin{bmatrix} b_0 \\ b_1 \\ b_2 \\ \cdot \\ \cdot \\ \cdot \end{bmatrix}$$

respectively, then (2) and (3) become $B = KA$ and $A = KB$. Hence $K^2A = A$ for all column vectors A arising from the series (1), and thus $K^2 = I$, where I is the unit infinite matrix. Thus the Fourier kernel $K(x)$ may be regarded as corresponding to the square root K of I.

In the same way other fractional powers of the unit infinite matrix I correspond to formal transformations which are fractional powers of the transformation with respect to the Fourier kernel $K(x)$. In particular, the fractional transformations of Kober correspond to the diagonal matrix

$$\begin{bmatrix} 1 & 0 & 0 & \dots \\ 0 & e^{2\pi ik} & 0 & \dots \\ 0 & 0 & e^{4\pi ik} & \dots \\ \dots & \dots & \dots & \dots \end{bmatrix}. \qquad \text{.......(4)}$$

3. *Powers of Unit Matrices*

DEFINITION 1. *If I_n is the unit $n \times n$ matrix and $I(x)$ is an $n \times n$ matrix whose elements are functions of a parameter x defined for all real x, then $I(x)$ is said to be an evaluation of I_n^x, the xth power of I_n, if*

(i) $I(x) \,.\, I(y) = I(x+y)$ *for all real x and y,*

(ii) $I(0) = I(1) = I_n$.

With this terminology we have:

THEOREM 1. *If $c_1, c_2, \dots, c_{n-1}$ is any set of $n-1$ constants and $I(x)$ is the lower semi-matrix defined by*

$$I(x) = \begin{bmatrix} a_{11} & a_{12} \dots & a_{1n} \\ a_{21} & a_{22} \dots & a_{2n} \\ \dots & \dots & \dots \\ a_{n1} & a_{n2} \dots & a_{nn} \end{bmatrix},$$

where $a_{rs} = 0$ if $r < s$, and

$$a_{rs} = \frac{1}{(r-s)!} e^{2\pi ix(s-1)} (1 - e^{2\pi ix})^{r-s} \prod_{m=s}^{r-1} c_m \qquad \text{.......(5)}$$

if $r \geqslant s$, then $I(x)$ is an evaluation of I_n^x.

Proof. By the multiplication rule for matrices the element in the rth row and sth column of the product $I(x).I(y)$ is

$$\sum_{p=s}^{r} \frac{e^{2\pi i x(p-1)+2\pi i y(s-1)}}{(r-p)!\,(p-s)!} c_s c_{s+1} \ldots c_{p-1} c_p c_{p+1} \ldots (1-e^{2\pi i x})^{r-p}(1-e^{2\pi i y})^{p-s}$$

$$= \frac{1}{(r-s)!}\left\{\prod_{m=s}^{r-1} c_m\right\} e^{2\pi i(x+y)(s-1)} \sum_{p=s}^{r} \binom{r-s}{r-p} (1-e^{2\pi i x})^{r-p}(e^{2\pi i x}-e^{2\pi i x+2\pi i y})^{p-s}$$

$$= \frac{1}{(r-s)!}\left\{\prod_{m=s}^{r-1} c_m\right\} e^{2\pi i(x+y)(s-1)}\{1-e^{2\pi i(x+y)}\}^{r-s} \quad \ldots\ldots(6)$$

if $r \geqslant s$ and is zero if $r<s$. That is, condition (i) is satisfied. Condition (ii) follows immediately on substituting $x=0$ and $x=1$ in (5).

The result of Theorem 1 extends immediately to infinite lower semi-matrices, since the product of two lower semi-matrices is a lower semi-matrix and the elements of the rth row of the product depend only on the elements of the first r rows of the two factor matrices.

In particular, if we put $c_m = \frac{1}{2}(\alpha+m)$, $x=\frac{1}{2}$, then it follows that the infinite lower semi-matrix

$$\begin{bmatrix} \binom{\alpha}{0} & 0 & 0 & \ldots \\ \binom{\alpha+1}{1} & -\binom{\alpha+1}{0} & 0 & \ldots \\ \binom{\alpha+2}{2} & -\binom{\alpha+2}{1} & \binom{\alpha+2}{0} & \ldots \\ \ldots & \ldots & \ldots & \ldots \end{bmatrix}$$

is a square root of the unit infinite matrix I.

Hence if $a_0, a_1, a_2, \ldots$ is any sequence of numbers and

$$b_n = \sum_{m=0}^{n} (-1)^m \binom{n+\alpha}{n-m} a_m, \quad \ldots\ldots(7)$$

then*

$$a_n = \sum_{m=0}^{n} (-1)^m \binom{n+\alpha}{n-m} b_m. \quad \ldots\ldots(7)$$

4. *Application to Hankel transforms.* Suppose that $f(x)$ and $g(x)$ are transforms with respect to the kernel† $J_\alpha(2x^{\frac{1}{2}})$, $R(\alpha) > -1$, and that they have formal developments in Laguerre polynomials of the form

$$f(x) \sim x^{\frac{1}{2}\alpha} \sum_{n=0}^{\infty} a_n L_n^{(\alpha)}(x),$$

$$g(x) \sim x^{\frac{1}{2}\alpha} \sum_{n=0}^{\infty} b_n L_n^{(\alpha)}(x). \quad \ldots\ldots(8)$$

That is (see (**5**))

$$a_n = \frac{n!}{\Gamma(n+\alpha+1)} \int_0^\infty e^{-x} x^{\frac{1}{2}\alpha} L_n^{(\alpha)}(x) f(x)\,dx,$$

$$b_n = \frac{n!}{\Gamma(n+\alpha+1)} \int_0^\infty e^{-x} x^{\frac{1}{2}\alpha} L_n^{(\alpha)}(x) g(x)\,dx. \quad \ldots\ldots(9)$$

* The case $\alpha=0$ gives the "reciprocal sequences" of Barrucand (**1**), corresponding to the Euler semi-matrix (**2**).

†In this case $x^{\frac{1}{2}}f(\frac{1}{2}x^2)$ and $x^{\frac{1}{2}}g(\frac{1}{2}x^2)$ are transforms with respect to the usual Hankel kernel $x^{\frac{1}{2}}J_\alpha(x)$.

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Now the functions $n!\, e^{-x}x^{\frac{1}{2}\alpha}L_n^{(\alpha)}(x)$, $e^{-x}x^{n+\frac{1}{2}\alpha}$ are transforms ((**5**), 5·4·1) with respect to the kernel $J_\alpha(2x^{\frac{1}{2}})$. Hence, by the Parseval theorem for these transforms, (9) becomes

$$a_n = \frac{1}{\Gamma(n+\alpha+1)} \int_0^\infty e^{-x}x^{n+\frac{1}{2}\alpha} g(x)\, dx. \qquad \text{(10)}$$

Further ((**5**), 5·1·6)

$$L_n^{(\alpha)}(x) = \sum_{m=0}^{n} (-1)^m \binom{n+\alpha}{n-m} \frac{x^m}{m!}.$$

Hence, by the reciprocity (7),

$$\frac{x^n}{n!} = \sum_{m=0}^{n} (-1)^m \binom{n+\alpha}{n-m} L_m^{(\alpha)}(x). \qquad \text{(11)}$$

Substituting (11) in (10), we obtain

$$\begin{aligned} a_n &= \frac{n!}{\Gamma(n+\alpha+1)} \sum_{m=0}^{n} (-1)^m \binom{n+\alpha}{n-m} \int_0^\infty e^{-x}x^{\frac{1}{2}\alpha}L_m^{(\alpha)}(x) g(x)\, dx \\ &= \frac{n!}{\Gamma(n+\alpha+1)} \sum_{m=0}^{n} (-1)^m \binom{n+\alpha}{n-m} \frac{\Gamma(m+\alpha+1)}{m!} b_m \\ &= \sum_{m=0}^{n} (-1)^m \binom{n}{m} b_m. \end{aligned}$$

Now it is not necessary for this argument that the series (8) should converge ; the integrals (9) converge and the use of the Parseval theorem is justified if $f(x)$ belongs to $L^2(0,\infty)$. Thus we have :

THEOREM 2. *If $f(x)$ belongs to $L^2(0,\infty)$ and $g(x)$ is its transform with respect to the kernel $J_\alpha(2x^{\frac{1}{2}})$, $R(\alpha) > -1$, and a_n and b_n are defined by* (9), *then*

$$b_n = \sum_{m=0}^{n} (-1)^m \binom{n}{m} a_m, \qquad \text{(12)}$$

and

$$a_n = \sum_{m=0}^{n} (-1)^m \binom{n}{m} b_m.$$

5. *The cases of Fourier transforms.* The particular cases $\alpha = \pm\frac{1}{2}$ of Theorem 2 are equivalent to the cases of Fourier sine and cosine transforms, respectively, and the Laguerre polynomials can then be expressed in terms of Hermite polynomials ((**5**), 5·6·1). If we put

$$F(x) = x^{\frac{1}{2}} f(\tfrac{1}{2}x^2), \quad G(x) = x^{\frac{1}{2}} g(\tfrac{1}{2}x^2),$$

Theorem 2 becomes :

THEOREM 3. *If $F(x)$ belongs to $L^2(0,\infty)$, and either (i) $G(x)$ is its cosine transform and $k=0$, or (ii) $G(x)$ is its sine transform and $k=1$, and $F(x)$ and $G(x)$ have the formal expansions*

$$F(x) \sim \sum_{n=0}^{\infty} A_n H_{2n+k}(x/\sqrt{2}), \quad G(x) \sim \sum_{n=0}^{\infty} B_n H_{2n+k}(x/\sqrt{2}),$$

in the sense that

$$A_n = \frac{2^{\frac{1}{2}-2n-k}}{(2n+k)!} \int_0^\infty e^{-\frac{1}{4}x^2} H_{2n+k}(x/\sqrt{2}) F(x)\, dx,$$

$$B_n = \frac{2^{\frac{1}{2}-2n-k}}{(2n+k)!} \int_0^\infty e^{-\frac{1}{4}x^2} H_{2n+k}(x/\sqrt{2}) G(x)\, dx,$$

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then
$$B_n = (-1)^n \sum_{m=0}^{n} \frac{2^{2m-2n}}{(n-m)!} A_m,$$

and
$$A_n = (-1)^n \sum_{m=0}^{n} \frac{2^{2m-2n}}{(n-m)!} B_m.$$

6. *An example.* If we put $a_n = (\frac{1}{2} + \frac{1}{2}a)^n$ in (8), then ((5) 5·1·9)

$$f(x) = x^{\frac{1}{2}\alpha} \sum_{n=0}^{\infty} (\tfrac{1}{2} + \tfrac{1}{2}a)^n L_n^{(\alpha)}(x)$$
$$= x^{\frac{1}{2}\alpha} (\tfrac{1}{2} - \tfrac{1}{2}a)^{-\alpha-1} \exp\left(-x \frac{1+a}{1-a}\right), \qquad (13)$$

for $-3 < a < 1$. Further, (12) gives

$$b_n = \sum_{m=0}^{n} (-1)^m \binom{n}{m} (\tfrac{1}{2} + \tfrac{1}{2}a)^m = (\tfrac{1}{2} - \tfrac{1}{2}a)^n.$$

Hence
$$g(x) = x^{\frac{1}{2}\alpha} (\tfrac{1}{2} + \tfrac{1}{2}a)^{-\alpha-1} \exp\left(-x \frac{1-a}{1+a}\right).$$

Also, it follows from an extension of Weber's integral (8) that

$$g(x) = \int_0^\infty f(t) J_\alpha(2x^{\frac{1}{2}}t^{\frac{1}{2}})\, dt,$$

in accordance with Theorem 2.

7. *Fractional transforms.* Kober (3) has discussed a class of transformations which can be regarded as fractional powers of the ordinary Hankel transformation. If we re-arrange some of Kober's results so that they pertain to the kernel $J_\alpha(2x^{\frac{1}{2}})$ instead of the usual Hankel kernel $x^{\frac{1}{2}} J_\alpha(x)$ then we obtain :

THEOREM 4. *If $f(x)$ belongs to $L^2(0, \infty)$, α and k are real, $\alpha > -1$, and*

$$f(x) \sim e^{-x} x^{\frac{1}{2}\alpha} \sum_{n=0}^{\infty} a_n L_n^{(\alpha)}(2x), \qquad (14)$$

in the sense that

$$a_n = \frac{n!}{2^{\alpha+1}\Gamma(n+\alpha+1)} \int_0^\infty e^{-t} t^{\frac{1}{2}\alpha} L_n^{(\alpha)}(2t) f(t)\, dt,$$

then there exists a family of transformations T_k with the following properties :

(i) $T_k f(x) \sim e^{-x} x^{\frac{1}{2}\alpha} \sum_{n=0}^{\infty} a_n e^{2\pi i n k} L_n^{(\alpha)}(2x)$,

(ii) $T_k T_l f(x) = T_{k+l} f(x)$, $\quad T_{k+1} f(x) = T_k f(x)$,

(iii) $T_{\frac{1}{2}} f(x) = \underset{T\to\infty}{\text{l.i.m.}} \int_0^T f(t) J_\alpha(2x^{\frac{1}{2}}t^{\frac{1}{2}})\, dt$,

(iv) *If k is not an integer and*

$c_k = |\operatorname{cosec} \pi k| \exp \pi i (1+\alpha)(\frac{1}{2} - k + [k])$,

then

$$T_k f(x) = c_k \underset{T\to\infty}{\text{l.i.m.}} \int_0^T f(t) J_\alpha(2x^{\frac{1}{2}}t^{\frac{1}{2}} |\operatorname{cosec} \pi k|) e^{i(x+t)\cot \pi k}\, dt.$$

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In the present scheme these fractional transforms correspond to the evaluation I^k of the kth power of the unit infinite matrix which is given by the diagonal matrix (4).

Now the fractional transformation of the sequence $\{a_n\}$ corresponding by Theorem 1 to the kth power of the transformation (12) is

$$b_n' = \sum_{m=0}^{n} e^{2\pi imk}(\tfrac{1}{2} - \tfrac{1}{2}e^{2\pi ik})^{n-m}\binom{n}{m} a_m. \qquad \text{.....(15)}$$

Hence we can derive a transformation T_k' which is also a kth power of the transformation with respect to the kernel $J_\alpha(2x^{\frac{1}{2}})$ if we replace the b_n in Theorem 2 by the b_n' given by (15). In particular, if $f(x)$ is the function (13) considered in § 6, then (15) gives

$$b_n' = \sum_{m=0}^{n} e^{2\pi imk}(\tfrac{1}{2} - \tfrac{1}{2}e^{2\pi ik})^{n-m}\binom{n}{m}(\tfrac{1}{2} + \tfrac{1}{2}a)^m$$
$$= (\tfrac{1}{2} + \tfrac{1}{2}ae^{2\pi ik})^n.$$

Hence, in this case,

$$T_k' f(x) = x^{\frac{1}{2}\alpha} \sum_{n=0}^{\infty} b_n' L_n^{(\alpha)}(x)$$
$$= (\tfrac{1}{2} - \tfrac{1}{2}ae^{2\pi ik})^{-\alpha-1} x^{\frac{1}{2}\alpha} \exp\left(-x\frac{1+ae^{2\pi ik}}{1-ae^{2\pi ik}}\right). \qquad \text{.....(16)}$$

Thus as k varies (16) gives functions which are kth power transforms of the function (13) with respect to the kernel $J_\alpha(2x^{\frac{1}{2}})$. We can use this family of fractional transforms to obtain fractional transforms of more general functions by the following device.

Definition 2. *If $f(x)$ and $g(x)$ belong to $L^2(0, \infty)$, and $F_a(x)$, $G_a(x)$ are sets of functions belonging to $L^2(0, \infty)$ and defined for some set of values of a, then we say that $f(x)$ is related to $g(x)$ as the $F_a(x)$ are related to the $G_a(x)$ if*

$$\int_0^\infty f(x) G_a(x)\, dx = \int_0^\infty g(x) F_a(x)\, dx$$

for all a of the set.

With this terminology we can prove :

Theorem 5. *If $f(x)$ is related to $g(x)$ as the functions* (16) *are related to the functions* (13) *for $|a| < 1$, and*

$$f(x) \sim x^{\frac{1}{2}\alpha} \sum_{n=0}^{\infty} a_n L_n^{(\alpha)}(x)$$

and

$$g(x) \sim x^{\frac{1}{2}\alpha} \sum_{n=0}^{\infty} b_n' L_n^{(\alpha)}(x),$$

then the sequences $\{a_n\}$ and $\{b_n'\}$ satisfy (15).

Proof. By the assumption of the theorem we have

$$(1 - ae^{2\pi ik})^{-\alpha-1} \int_0^\infty f(x) x^{\frac{1}{2}\alpha} \exp\left(-x\frac{1+ae^{2\pi ik}}{1+ae^{2\pi ik}}\right) dx$$
$$= (1-a)^{-\alpha-1} \int_0^\infty g(x) x^{\frac{1}{2}\alpha} \exp\left(-x\frac{1+a}{1-a}\right) dx, \qquad \text{.....(17)}$$

for $|a| < 1$.

Now for $|2a/(1+a)|<1$ the right-hand side is equal, by the expansion (5), 5·1·9, to

$$(1+a)^{-\alpha-1}\int_0^\infty g(x)e^{-x}x^{\frac{1}{2}\alpha}\left\{\sum_{n=0}^{\infty} L_n^{(\alpha)}(x)\left(\frac{2a}{1+a}\right)^n\right\}dx$$

$$=(1+a)^{-\alpha-1}\sum_{n=0}^{\infty}\left(\frac{2a}{1+a}\right)^n\int_0^\infty g(x)e^{-x}x^{\frac{1}{2}\alpha}L_n^{(\alpha)}(x)\,dx$$

$$=(1+a)^{-\alpha-1}\sum_{n=0}^{\infty}\left(\frac{2a}{1+a}\right)^n\frac{\Gamma(n+\alpha+1)}{n!}b_n'. \quad\text{.............................(18)}$$

The term by term integration of the series is justified by absolute convergence since

$$|b_n'|=\frac{n!}{\Gamma(n+\alpha+1)}\left|\int_0^\infty g(x)x^{\frac{1}{2}\alpha}e^{-x}L_n^{(\alpha)}(x)\,dx\right|$$

$$\leqslant\frac{n!}{\Gamma(n+\alpha+1)}\left[\int_0^\infty |g(x)|^2e^{-x}\,dx\right]^{\frac{1}{2}}\left[\int_0^\infty e^{-x}x^{\alpha}\{L_n^{(\alpha)}(x)\}^2\,dx\right]^{\frac{1}{2}}$$

$$=\left[\frac{n!}{\Gamma(n+\alpha+1)}\right]^{\frac{1}{2}}\left[\int_0^\infty |g(x)|^2e^{-x}\,dx\right]^{\frac{1}{2}}$$

$$=O(n^{-\frac{1}{2}\alpha}).$$

Hence the series (18) converges absolutely for $|2a/(1+a)|<1$.

Similarly the left-hand side of (17) is equal to

$$(1+ae^{2\pi ik})^{-\alpha-1}\sum_{n=0}^{\infty}\left(\frac{2ae^{2\pi ik}}{1+ae^{2\pi ik}}\right)^n\frac{\Gamma(n+\alpha+1)}{n!}a_n. \quad\text{.......................(19)}$$

If we now put

$$b_n'=\sum_{m=0}^{n}e^{2\pi imk}(\tfrac{1}{2}-\tfrac{1}{2}e^{2\pi ik})^{n-m}\binom{n}{m}a_m', \quad\text{.............................(20)}$$

then (18) becomes

$$(1+a)^{-\alpha-1}\sum_{n=0}^{\infty}\left(\frac{2a}{1+a}\right)^n\frac{\Gamma(n+\alpha+1)}{n!}\sum_{m=0}^{n}e^{2\pi imk}(\tfrac{1}{2}-\tfrac{1}{2}e^{2\pi ik})^{n-m}\binom{n}{m}a_m'$$

$$=(1+a)^{-\alpha-1}\sum_{m=0}^{\infty}\frac{a_m'}{m!}\Gamma(m+\alpha+1)\left(\frac{2ae^{2\pi ik}}{1+a}\right)^m\sum_{n=m}^{\infty}\binom{n+\alpha}{n-m}\left(\frac{a-ae^{2\pi ik}}{1+a}\right)^{n-m}$$

$$=(1+ae^{2\pi ik})^{-\alpha-1}\sum^{\infty}\frac{a_m'}{m!}\Gamma(m+\alpha+1)\left(\frac{2ae^{2\pi ik}}{1+ae^{2\pi ik}}\right)^m.$$

Equating this to (19) we have

$$\sum_{n=0}^{\infty}\left(\frac{2ae^{2\pi ik}}{1+ae^{2\pi ik}}\right)^n\frac{\Gamma(n+\alpha+1)}{n!}(a_n-a_n')=0,$$

for all relevant a. Hence $a_n'=a_n$ and (20) becomes the required result (15).

8. *Identity with Kober's transforms.* The identity of the fractional transforms of Theorems 4 and 5 is proved if we show that $T_kf(x)=T_k'f(x)$ for the function (13). In this case we have in Theorem 4

$$a_n=\frac{n!}{2^{\alpha+1}\Gamma(n+\alpha+1)}(\tfrac{1}{2}-\tfrac{1}{2}a)^{-\alpha-1}\int_0^\infty e^{-x}x^{\frac{1}{2}\alpha}L_n^{(\alpha)}(2x)x^{\frac{1}{2}\alpha}\exp\left(-x\frac{1+a}{1-a}\right)dx$$

$$=\frac{n!}{2^{\alpha+1}\Gamma(n+\alpha+1)}(1-a)^{-\alpha-1}\int_0^\infty x^{\alpha}L_n^{(\alpha)}(x)\exp\left(-\frac{x}{1-a}\right)dx$$

$$=2^{-\alpha-1}a^n,$$

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on evaluating this integral ((**5**), p. 370, problem 19). Hence (i) of Theorem 4 gives

$$T_k f(x) = 2^{-\alpha-1} e^{-x} x^{\frac{1}{2}\alpha} \sum_{n=0}^{\infty} (ae^{2\pi i k})^n L_n^{(\alpha)}(2x)$$

$$= (\tfrac{1}{2} - \tfrac{1}{2}ae^{2\pi i k})^{-\alpha-1} x^{\alpha} \exp\left(-x\frac{1+ae^{2\pi i k}}{1-ae^{2\pi i k}}\right)$$

$$= T'_k f(x),$$

as required.

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